(1) Milne 3-1

(2) Let $R$ be a subring of the ring of integers $\mathcal{O}_K$ of a number field $K$. Show that the following are equivalent
   (a) The index $[\mathcal{O}_K : R]$ (as abelian groups) is finite
   (b) $R$ contains a basis of $K$ over $\mathbb{Q}$
   (c) The field of fractions of $R$ is $K$.

(3) The rings satisfying the conditions above are called *orders* of $K$. Give an example of a $K$ and a subring of $\mathcal{O}_K$ that is not an order of $K$. If a subring of $\mathcal{O}_K$ is not an order of $\mathcal{O}_K$, show it is still an order of some number field. (What number field is it an order of in your example?)

(4) Give an example that shows that unique factorization of ideals fails in $\mathbb{Z}[\sqrt{-3}]$.

(5) Let $a$ and $b$ be ideals of $\mathcal{O}_K$. Determine the prime factorizations of $a + b$ and $ab$ in terms of those for $a$ and $b$.

(6) Let $K = \mathbb{Q}(\sqrt{-5})$ and we work in $\mathcal{O}_K$. Show (by hand) that $p = (2, 1 + \sqrt{-5})$, and $q_1 = (7, 3 + \sqrt{-5})$, and $q_2 = (7, 3 - \sqrt{-5})$ are prime ideals in $\mathcal{O}_K$. Use sage to factor $(9 + 11\sqrt{-5})$ as a product of prime ideals. Use sage to factor $(p)$ as a product of prime ideals for the first 10 rational primes $p$ (i.e. primes in $\mathbb{Z}$, though the notation $(p)$ refers to the $\mathcal{O}_K$ ideal).

(7) Let $A$ be a domain in which all non-zero ideals admit a unique factorization into prime ideals. Show that $A$ is a Dedekind domain.