(1) Find $\mu(K)$ for each quadratic number field $K$.

(2) Prove, without using Dirichlet’s unit Theorem, that an imaginary quadratic number field has at most finitely many units.

(3) The continued fraction expansion for $\alpha \in \mathbb{R}$ is the writing of $\alpha$ as

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

with each $a_i \in \mathbb{Z}$. To find the $a_i$, first let $[\alpha]$ be the greatest integer less than or equal to $\alpha$, so that $a_0 = [\alpha]$. Let $\beta$ be the reciprocal of the fractional part $\alpha - [\alpha]$, so that from above we have $\beta = a_1 + (1/(a_2 + \cdots))$. Thus $a_1 = [\beta]$. Continue in this manner to obtain the other $a_i$. If we truncate the expression above at the nth step, we obtain a rational number $p_n/q_n$. For instance, $p_0/q_0 = a_0/1$, $p_1/q_1 = a_0 + 1/a_1 = (a_0a_1 + 1)/a_1$. The numbers $p_n$ and $q_n$ are called the convergents of $\alpha$, and are given by the Fibonacci-like recurrences

$$p_{n+1} = a_{n+1}p_n + p_{n-1} \quad q_{n+1} = a_{n+1}q_n + q_{n-1}$$

with initial values $p_0, p_1, q_0, q_1$ as given above. The rational numbers $p_n/q_n$ give successively better approximations of $\alpha$. Now let $\alpha = \sqrt{d}$, where $d > 0$ is squarefree and $d \equiv 2, 3 \pmod{4}$.

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(5) Let $K = \mathbb{Q}(\sqrt{26})$ and let $\epsilon = 5 + \sqrt{26}$. Show

$$(2) = (2, \epsilon + 1)^2 \quad (5) = (5, \epsilon + 1)(5, \epsilon - 1) \quad (\epsilon + 1) = (2, \epsilon + 1)(5, \epsilon + 1).$$

Show that $K$ has class number 2. Verify that $\epsilon$ is the fundamental unit. Deduce that all solutions in integers $x, y$ to the equation $x^2 - 26y^2 = \pm 10$ are given by $x + \sqrt{26}y = \pm \epsilon^n(\epsilon \pm 1)$ for $n \in \mathbb{Z}$.