Definability, automorphisms and enumeration degrees

Mariya I. Soskova

In honor of Ivan Soskov’s 60’th birthday

1Supported by a Marie Curie International Outgoing Fellowship STRIDE (298471) and Sofia University Science Fund project 97/2014
Enumeration reducibility

<table>
<thead>
<tr>
<th>Reducibility</th>
<th>Oracle set $B$</th>
<th>Reduced set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A \leq_T B$</td>
<td>Complete information</td>
<td>Complete information</td>
</tr>
<tr>
<td>$A$ c.e. in $B$</td>
<td>Complete information</td>
<td>Positive information</td>
</tr>
<tr>
<td>$A \leq_e B$</td>
<td>Positive information</td>
<td>Positive information</td>
</tr>
</tbody>
</table>

**Definition (Friedberg, Rogers (59))**

$A \leq_e B$ if there is a c.e. set $W$, such that

$$A = W(B) = \{ x \mid \exists D (\langle x, D \rangle \in W \& D \subseteq B) \}.$$

The structures of the Turing degrees $\mathcal{D}_T$ and the enumeration degrees $\mathcal{D}_e$ are upper semi-lattices with least element and jump operation.
The automorphism problem

Question

*Is there a non-trivial automorphism of $D_T$ or $D_e$?*
The automorphism problem

**Question**

Is there a non-trivial automorphism of $D_T$ or $D_e$?

**Theorem (Slaman, Woodin)**

The rigidity of $D_T$ is equivalent to its biinterpretability with second order arithmetic.
Biinterpretability

Theorem (Simpson, Slaman and Woodin)

The first order theories of $D_T$ and $D_e$ are each computably isomorphic to the theory of Second order arithmetic.
Biinterpretability

Theorem (Simpson, Slaman and Woodin)

The first order theories of $\mathcal{D}_T$ and $\mathcal{D}_e$ are each computably isomorphic to the theory of Second order arithmetic.

$\mathcal{D}_T$ is biinterpretable with second order arithmetic if the relation $\varphi(\vec{p}, x)$ defined by “$\vec{p}$ codes a standard model of arithmetic with a unary predicate for the set $Y$ and $Y$ is of the same degree as $x$” is definable in $\mathcal{D}_T$. 

Mariya I. Soskova

Definability, automorphisms and e-degrees
Biinterpretability

Theorem (Simpson, Slaman and Woodin)

The first order theories of $\mathcal{D}_T$ and $\mathcal{D}_e$ are each computably isomorphic to the theory of Second order arithmetic.

$\mathcal{D}_T$ is biinterpretable with second order arithmetic if the relation $\varphi(\vec{p}, x)$ defined by “$\vec{p}$ codes a standard model of arithmetic with a unary predicate for the set $Y$ and $Y$ is of the same degree as $x$” is definable in $\mathcal{D}_T$.

Theorem (Slaman, Woodin)

There is an element $g \leq 0^{(5)}$ such that $\varphi$ is definable with parameter $g$. 
Biinterpretability

Theorem (Simpson, Slaman and Woodin)

The first order theories of $\mathcal{D}_T$ and $\mathcal{D}_e$ are each computably isomorphic to the theory of Second order arithmetic.

$\mathcal{D}_T$ is biinterpretable with second order arithmetic if the relation $\varphi(\vec{p}, x)$ defined by “$\vec{p}$ codes a standard model of arithmetic with a unary predicate for the set $Y$ and $Y$ is of the same degree as $x$” is definable in $\mathcal{D}_T$.

Theorem (Slaman, Woodin)

There is an element $g \leq 0^{(5)}$ such that $\varphi$ is definable with parameter $g$. The singleton $\{g\}$ is an automorphism base for the structure of the Turing degrees $\mathcal{D}_T$. 
Biinterpretability

**Theorem (Simpson, Slaman and Woodin)**

The first order theories of $D_T$ and $D_e$ are each computably isomorphic to the theory of Second order arithmetic.

$D_T$ is biinterpretable with second order arithmetic if the relation $\varphi(\vec{p}, x)$ defined by “$\vec{p}$ codes a standard model of arithmetic with a unary predicate for the set $Y$ and $Y$ is of the same degree as $x$” is definable in $D_T$.

**Theorem (Slaman, Woodin)**

There is an element $g \leq 0^{(5)}$ such that $\varphi$ is definable with parameter $g$. The singleton $\{g\}$ is an automorphism base for the structure of the Turing degrees $D_T$. $\text{Aut}(D_T)$ is countable and every member has an arithmetically definable presentation.
Biinterpretability

**Theorem (Simpson, Slaman and Woodin)**

The first order theories of $D_T$ and $D_e$ are each computably isomorphic to the theory of Second order arithmetic.

$D_T$ is biinterpretable with second order arithmetic if the relation $\varphi(\vec{p}, x)$ defined by “$\vec{p}$ codes a standard model of arithmetic with a unary predicate for the set $Y$ and $Y$ is of the same degree as $x$” is definable in $D_T$.

**Theorem (Slaman, Woodin)**

There is an element $g \leq 0^{(5)}$ such that $\varphi$ is definable with parameter $g$. The singleton $\{g\}$ is an automorphism base for the structure of the Turing degrees $D_T$.

$\text{Aut}(D_T)$ is countable and every member has an arithmetically definable presentation.

Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters.
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \iff A \oplus \overline{A}$ is c.e. in $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$. 
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

Proposition

$A \leq_T B \iff A \oplus \overline{A}$ is c.e. in $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

**Proposition**

$A \leq_T B \iff A \oplus \overline{A}$ is c.e. in $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$\mathcal{TOT} = \iota(\mathcal{D}_T)$ is the set of total enumeration degrees.
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

**Proposition**

$A \leq_T B \iff A \oplus \overline{A}$ is c.e. in $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \rightarrow \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$\mathcal{T}OT = \iota(\mathcal{D}_T)$ is the set of total enumeration degrees.

$$(\mathcal{D}_T, \leq_T, \lor', 0_T) \cong (\mathcal{T}OT, \leq_e, \lor', 0_e) \subseteq (\mathcal{D}_e, \leq_e, \lor', 0_e)$$
What connects $\mathcal{D}_T$ and $\mathcal{D}_e$

**Proposition**

$A \leq_T B \iff A \oplus \overline{A}$ is c.e. in $B \iff A \oplus \overline{A} \leq_e B \oplus \overline{B}$.

The embedding $\iota : \mathcal{D}_T \to \mathcal{D}_e$, defined by $\iota(d_T(A)) = d_e(A \oplus \overline{A})$, preserves the order, the least upper bound and the jump operation.

$\mathcal{TOT} = \iota(\mathcal{D}_T)$ is the set of total enumeration degrees.

$$(\mathcal{D}_T, \leq_T, \lor', 0_T) \cong (\mathcal{TOT}, \leq_e, \lor', 0_e) \subseteq (\mathcal{D}_e, \leq_e, \lor', 0_e)$$

**Question (Rogers (67))**

*Is the set of total enumeration degrees first order definable in $\mathcal{D}_e$?*
Theorem (Selman)

A is enumeration reducible to B if and only if
\[ \{ x \in \text{TOT} \mid d_e(A) \leq x \} \supseteq \{ x \in \text{TOT} \mid d_e(B) \leq x \} . \]
The total degrees as an automorphism base

Theorem (Selman)

A is enumeration reducible to B if and only if
\[ \{ x \in TOT \mid d_e(A) \leq x \} \supseteq \{ x \in TOT \mid d_e(B) \leq x \}. \]

- The total enumeration degrees are an automorphism base for \( D_e \).
The total degrees as an automorphism base

**Theorem (Selman)**

A is enumeration reducible to B if and only if
\[
\{ x \in TOT \mid d_e(A) \leq x \} \supseteq \{ x \in TOT \mid d_e(B) \leq x \}.
\]

- The total enumeration degrees are an automorphism base for \( D_e \).
- If \( TOT \) is definable then a nontrivial automorphism of \( D_e \) implies a nontrivial automorphism of \( D_T \).
Semi-computable sets

**Definition (Jockusch)**

A is semi-computable if there is a total computable function $s_A$, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

Example: A left cut in a computable linear ordering is a semi-computable set.

In particular for any set $A$, consider $L_A = \{\sigma \in 2^{<\omega} | \sigma \leq A\}$.

Every nonzero Turing degree contains a semi-computable set that is not c.e. or co-c.e.

**Theorem (Arslanov, Cooper, Kalimullin)**

If $A$ is a semi-computable set then for every $X$:

$$(d_e(X) \lor d_e(A)) \land (d_e(X) \lor d_e(A)) = d_e(X).$$

If $X$ is not computable then there is a semi-computable set $A$ with $d_e(X \oplus X) = d_e(A) \lor d_e(A)$. 

Mariya I. Soskova
Definability, automorphisms and e-degrees
Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function $s_A$, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

Example:

- A *left cut* in a computable linear ordering is a semi-computable set.
Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function $s_A$, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

Example:

- A *left cut* in a computable linear ordering is a semi-computable set.
- In particular for any set $A$ consider $L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq A\}$.
Semi-computable sets

**Definition (Jockusch)**

A is semi-computable if there is a total computable function $s_A$, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

**Example:**
- A *left cut* in a computable linear ordering is a semi-computable set.
- In particular for any set $A$ consider $L_A = \{\sigma \in 2^{<\omega} | \sigma \leq A\}$.
- Every nonzero Turing degree contains a semi-computable set that is not c.e. or co-c.e.
Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function $s_A$, such that $s_A(x, y) \in \{x, y\}$ and if $\{x, y\} \cap A \neq \emptyset$ then $s_A(x, y) \in A$.

Example:
- A left cut in a computable linear ordering is a semi-computable set.
- In particular for any set $A$ consider $L_A = \{\sigma \in 2^{<\omega} | \sigma \leq A\}$.
- Every nonzero Turing degree contains a semi-computable set that is not c.e. or co-c.e.

Theorem (Arslanov, Cooper, Kalimullin)

If $A$ is a semi-computable set then for every $X$:

$$(d_e(X) \lor d_e(A)) \land (d_e(X) \lor d_e(\overline{A})) = d_e(X).$$
Semi-computable sets

Definition (Jockusch)

A is semi-computable if there is a total computable function \( s_A \), such that
\[ s_A(x, y) \in \{x, y\} \text{ and if } \{x, y\} \cap A \neq \emptyset \text{ then } s_A(x, y) \in A. \]

Example:
- A left cut in a computable linear ordering is a semi-computable set.
- In particular for any set \( A \) consider \( L_A = \{ \sigma \in 2^{<\omega} | \sigma \leq A \} \).
- Every nonzero Turing degree contains a semi-computable set that is not c.e. or co-c.e.

Theorem (Arslanov, Cooper, Kalimullin)

If \( A \) is a semi-computable set then for every \( X \):
\[
(d_e(X) \lor d_e(A)) \land (d_e(X) \lor d_e(\overline{A})) = d_e(X).
\]

- If \( X \) is not computable then there is a semi-computable set \( A \) with \( d_e(X \oplus \overline{X}) = d_e(A) \lor d_e(\overline{A}) \).
Kalimullin pairs

**Definition (Kalimullin)**

A pair of sets $A, B$ are called a $K$-pair if there is a c.e. set $W$, such that

$A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

---

Mariya I. Soskova

Definability, automorphisms and e-degrees
**Definition (Kalimullin)**

A pair of sets $A, B$ are called a $K$-pair if there is a c.e. set $W$, such that $A \times B \subseteq W$ and $\bar{A} \times \bar{B} \subseteq \bar{W}$.

**Example:**

1. A trivial example is $\{A, U\}$, where $U$ is c.e: $W = \mathbb{N} \times U$. 
Kalimullin pairs

**Definition (Kalimullin)**

A pair of sets $A, B$ are called a $\mathcal{K}$-pair if there is a c.e. set $W$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

**Example:**

1. A trivial example is $\{A, U\}$, where $U$ is c.e: $W = \mathbb{N} \times U$.
2. If $A$ is a semi-computable set, then $\{A, \overline{A}\}$ is a $\mathcal{K}$-pair: $W = \{(m, n) \mid s_A(m, n) = m\}$.

Theorem (Kalimullin)

A pair of sets $A, B$ is a $\mathcal{K}$-pair if and only if their enumeration degrees $a$ and $b$ satisfy:

$$K(a, b) \equiv (\forall x \in \mathcal{D})(a \lor x) \land (b \lor x) = x.$$
Kalimullin pairs

**Definition (Kalimullin)**

A pair of sets $A, B$ are called a $K$-pair if there is a c.e. set $W$, such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

**Example:**

1. A trivial example is $\{A, U\}$, where $U$ is c.e: $W = \mathbb{N} \times U$.
2. If $A$ is a semi-computable set, then $\{A, \overline{A}\}$ is a $K$-pair: $W = \{(m, n) \mid s_A(m, n) = m\}$.

**Theorem (Kalimullin)**

A pair of sets $A, B$ is a $K$-pair if and only if their enumeration degrees $a$ and $b$ satisfy:

$$K(a, b) \iff (\forall x \in D_e)((a \lor x) \land (b \lor x) = x).$$
Definability of the enumeration jump

**Theorem (Kalimullin)**

$0_e'$ is the largest degree which can be represented as the least upper bound of a triple $a, b, c$, such that $K(a, b), K(b, c)$ and $K(c, a)$.
Definability of the enumeration jump

Theorem (Kalimullin)

$0_e'$ is the largest degree which can be represented as the least upper bound of a triple $a, b, c$, such that $\mathcal{K}(a, b), \mathcal{K}(b, c)$ and $\mathcal{K}(c, a)$.

Corollary (Kalimullin)

1. The enumeration jump is first order definable in $\mathcal{D}_e$. 
Definability of the enumeration jump

Theorem (Kalimullin)

0_e' is the largest degree which can be represented as the least upper bound of a triple a, b, c, such that \( \mathcal{K}(a, b), \mathcal{K}(b, c) \) and \( \mathcal{K}(c, a) \).

Corollary (Kalimullin)

1. The enumeration jump is first order definable in \( D_e \).
2. The set of total enumeration degrees above 0_e' is first order definable in \( D_e \).
Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $0'_e$ is first order definable in $D_e(\leq 0'_e)$. 
Definability in the local structure of the enumeration degrees

Theorem (Ganchev, S)

The class of $\mathcal{K}$-pairs below $0'_e$ is first order definable in $\mathcal{D}_e(\leq 0'_e)$.

Theorem (Ganchev, S)

1. The theory of $\mathcal{D}_e(\leq 0'_e)$ is computably isomorphic to the theory of first order arithmetic.
2. The low enumeration degrees are first order definable in $\mathcal{D}_e(\leq 0'_e)$. 
Maximal $\mathcal{K}$-pairs

**Definition**

A $\mathcal{K}$-pair $\{a, b\}$ is maximal if for every $\mathcal{K}$-pair $\{c, d\}$ with $a \leq c$ and $b \leq d$, we have that $a = c$ and $b = d$. 
Maximal $\mathcal{K}$-pairs

**Definition**

A $\mathcal{K}$-pair $\{a, b\}$ is maximal if for every $\mathcal{K}$-pair $\{c, d\}$ with $a \leq c$ and $b \leq d$, we have that $a = c$ and $b = d$.

*Example*: A semi-computable pair is a maximal $\mathcal{K}$-pair. Total enumeration degrees are joins of maximal $\mathcal{K}$-pairs.
Maximal $\mathcal{K}$-pairs

**Definition**

A $\mathcal{K}$-pair $\{a, b\}$ is maximal if for every $\mathcal{K}$-pair $\{c, d\}$ with $a \leq c$ and $b \leq d$, we have that $a = c$ and $b = d$.

**Example**: A semi-computable pair is a maximal $\mathcal{K}$-pair.
Total enumeration degrees are joins of maximal $\mathcal{K}$-pairs.

**Theorem (Ganchev, S)**

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e(\leq 0'_e)$ then there is a semi-computable set $C \leq_e 0'_e$, such that $A \leq_e C$ and $B \leq_e \overline{C}$. 
Maximal $\mathcal{K}$-pairs

**Definition**

A $\mathcal{K}$-pair $\{a, b\}$ is maximal if for every $\mathcal{K}$-pair $\{c, d\}$ with $a \leq c$ and $b \leq d$, we have that $a = c$ and $b = d$.

**Example:** A semi-computable pair is a maximal $\mathcal{K}$-pair. Total enumeration degrees are joins of maximal $\mathcal{K}$-pairs.

**Theorem (Ganchev, S)**

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_{e}(\leq 0'_{e})$ then there is a semi-computable set $C \leq_{e} 0'_{e}$, such that $A \leq_{e} C$ and $B \leq_{e} \overline{C}$.

**Corollary**

In $\mathcal{D}_{e}(\leq 0'_{e})$ a nonzero degree is total if and only if it is the least upper bound of a maximal $\mathcal{K}$-pair.
Theorem (S)

There is an element $g \leq 0 (8)$ such that $D_e$ is biinterpretable with second order arithmetic using parameter $g$.

The singleton $\{g\}$ is an automorphism base for $D_e$.

$\text{Aut}(D_e)$ is countable and every member has an arithmetically definable presentation.

Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters. In particular the total enumeration degrees are definable with parameters in $D_e$. 
Theorem (S)

*There is an element* $g \leq 0^{(8)}$ *such that* $D_e$ *is biinterpretable with second order arithmetic using parameter* $g$. 

Aut ($D_e$) *is countable and every member has an arithmetically definable presentation. Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters. In particular the total enumeration degrees are definable with parameters in* $D_e$. 

Mariya I. Soskova  
Definability, automorphisms and e-degrees
Theorem (S)

There is an element $g \leq 0^{(8)}$ such that $D_e$ is biinterpretable with second order arithmetic using parameter $g$.

The singleton $\{g\}$ is an automorphism base for $D_e$. 
Automorphism analysis in the enumeration degrees

**Theorem (S)**

There is an element \( g \leq 0^{(8)} \) such that \( D_e \) is biinterpretable with second order arithmetic using parameter \( g \).

The singleton \( \{g\} \) is an automorphism base for \( D_e \).

\( \text{Aut}(D_e) \) is countable and every member has an arithmetically definable presentation.
Theorem (S)

There is an element $g \leq 0^{(8)}$ such that $D_e$ is biinterpretable with second order arithmetic using parameter $g$.

The singleton $\{g\}$ is an automorphism base for $D_e$.

$\text{Aut}(D_e)$ is countable and every member has an arithmetically definable presentation.

Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters.
Theorem (S)

There is an element $g \leq 0^{(8)}$ such that $D_e$ is biinterpretable with second order arithmetic using parameter $g$.

The singleton $\{g\}$ is an automorphism base for $D_e$.

$\text{Aut}(D_e)$ is countable and every member has an arithmetically definable presentation.

Every relation induced by a degree invariant definable relation in Second order arithmetic is definable with parameters.

In particular the total enumeration degrees are definable with parameters in $D_e$. 
Defining total enumeration degrees in $\mathcal{D}_e$

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

If $\{A, B\}$ is a nontrivial $K$-pair in $\mathcal{D}_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$. 

Proof flavor:

Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $A \times B \subseteq W$. 

1. The countable component: we use $W$ to construct an effective labeling of the computable linear ordering $Q$. 

2. The uncountable component: $C$ will be a left cut in this ordering. We label elements of $Q$ with the elements of $\mathbb{N} \cup \mathbb{N}$. 

The goal:

$A = \{m | \exists q \in C (q \text{ is labeled by } m)\}$ and $B = \{k | \exists q \in C (q \text{ is labeled by } k)\}$. 

While $(m, k) \notin W$: 

$Q$: $k \in A, m \in B$. 

Mariya I. Soskova
Definability, automorphisms and e-degrees
Defining total enumeration degrees in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof flavor: Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. 
Defining total enumeration degrees in $\mathcal{D}_e$

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

If $\{A, B\}$ is a nontrivial $K$-pair in $\mathcal{D}_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

**Proof flavor:** Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

1. The countable component: we use $W$ to construct an effective labeling of the computable linear ordering $\mathbb{Q}$. 

Mariya I. Soskova

Definability, automorphisms and e-degrees
Defining total enumeration degrees in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

*If* $\{A, B\}$ *is a nontrivial К-pair in* $\mathcal{D}_e$ *then there is a semi-computable set* $C$, *such that* $A \leq_e C$ *and* $B \leq_e \overline{C}$.

*Proof flavor:* Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

1. The countable component: we use $W$ to construct an effective labeling of the computable linear ordering $\mathbb{Q}$.
2. The uncountable component: $C$ will be a left cut in this ordering.

Mariya I. Soskova

Definability, automorphisms and e-degrees
Defining total enumeration degrees in $\mathcal{D}_e$

Theorem (Cai, Ganchev, Lempp, Miller, S)

If $\{A, B\}$ is a nontrivial $\mathcal{K}$-pair in $\mathcal{D}_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

Proof flavor: Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

1. The countable component: we use $W$ to construct an effective labeling of the computable linear ordering $\mathbb{Q}$.
2. The uncountable component: $C$ will be a left cut in this ordering.

We label elements of $\mathbb{Q}$ with the elements of $\mathbb{N} \cup \overline{\mathbb{N}}$. 
Defining total enumeration degrees in $D_e$

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

If $\{A, B\}$ is a nontrivial $K$-pair in $D_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.

**Proof flavor:** Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

1. **The countable component:** we use $W$ to construct an effective labeling of the computable linear ordering $\mathbb{Q}$.
2. **The uncountable component:** $C$ will be a left cut in this ordering.

We label elements of $\mathbb{Q}$ with the elements of $\mathbb{N} \cup \mathbb{N}$.

The goal: $A = \{m \mid \exists q \in C (q \text{ is labeled by } m)\}$ and $B = \{k \mid \exists q \in \overline{C} (q \text{ is labeled by } k)\}$.
Defining total enumeration degrees in $D_e$

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

*If \( \{A, B\} \) is a nontrivial K-pair in $D_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.***

**Proof flavor:** Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

1. **The countable component:** we use $W$ to construct an effective labeling of the computable linear ordering $\mathbb{Q}$.
2. **The uncountable component:** $C$ will be a left cut in this ordering.

We label elements of $\mathbb{Q}$ with the elements of $\mathbb{N} \cup \mathbb{N}$.

The goal: $A = \{m \mid \exists q \in C (q \text{ is labeled by } m)\}$ and $B = \{k \mid \exists q \in \overline{C} (q \text{ is labeled by } k)\}$.

While $(m, k) \notin W$:

```
Q : --------- k ----------- m
```

Mariya I. Soskova
Definability, automorphisms and e-degrees
Defining total enumeration degrees in $\mathcal{D}_e$

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

*If $\{A, B\}$ is a nontrivial $K$-pair in $\mathcal{D}_e$ then there is a semi-computable set $C$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.***

**Proof flavor:** Let $W$ be a c.e. set such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$.

1. **The countable component:** we use $W$ to construct a labeling of the computable linear ordering $Q$.
2. **The uncountable component:** $C$ will be a left cut in this ordering.

We label elements of $Q$ with the elements of $\mathbb{N} \cup \mathbb{N}$.

The goal: $A = \{ m \mid \exists q \in C (q \text{ is labeled by } m) \}$ and $B = \{ k \mid \exists q \in \overline{C} (q \text{ is labeled by } k) \}$.

If $(m, k) \in W$:

If $(m, k) \in W$:
Theorem (Cai, Ganchev, Lempp, Miller, S)

The total enumeration degrees are first order definable in $D_e$. 

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees.

If $D_T$ is rigid then $D_e$ is rigid.

The automorphism analysis for the enumeration degrees follows.

The total degrees below 0 are an automorphism base of $D_e$. 

Mariya I. Soskova
Definability, automorphisms and e-degrees
Success!

**Theorem (Cai, Ganchev, Lempp, Miller, S)**

The total enumeration degrees are first order definable in $D_e$.

**Corollary**

The total enumeration degrees form a definable automorphism base of the enumeration degrees.
Theorem (Cai, Ganchev, Lempp, Miller, S)

The total enumeration degrees are first order definable in $D_e$.

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees.

- If $D_T$ is rigid then $D_e$ is rigid.
Theorem (Cai, Ganchev, Lempp, Miller, S)

The total enumeration degrees are first order definable in $D_e$.

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees.

- If $D_T$ is rigid then $D_e$ is rigid.
- The automorphism analysis for the enumeration degrees follows.
Theorem (Cai, Ganchev, Lempp, Miller, S)

The total enumeration degrees are first order definable in $D_e$.

Corollary

The total enumeration degrees form a definable automorphism base of the enumeration degrees.

- If $D_T$ is rigid then $D_e$ is rigid.
- The automorphism analysis for the enumeration degrees follows.
- The total degrees below $0_e^{(5)}$ are an automorphism base of $D_e$. 
The relation *c.e. in*

**Definition**

A Turing degree \( a \) is *c.e. in* a Turing degree \( x \) if some \( A \in a \) is c.e. in some \( X \in x \).
The relation \textit{c.e. in}

\begin{definition}
A Turing degree $a$ is \textit{c.e. in} a Turing degree $x$ if some $A \in a$ is c.e. in some $X \in x$.
\end{definition}

Recall that $\iota$ is the standard embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$. 
The relation \textit{c.e. in}

\textbf{Definition}

A Turing degree $a$ is \textit{c.e. in} a Turing degree $x$ if some $A \in a$ is c.e. in some $X \in x$.

Recall that $\iota$ is the standard embedding of $\mathcal{D}_T$ into $\mathcal{D}_e$.

\textbf{Theorem (Cai, Ganchev, Lempp, Miller, S)}

The set $\{\langle \iota(a), \iota(x) \rangle \mid a \text{ is c.e. in } x \}$ is first order definable in $\mathcal{D}_e$.

1 Ganchev, S had observed that if $\mathcal{T\mathcal{O}\mathcal{T}}$ is definable by maximal $\mathcal{K}$-pairs then the image of the relation ‘c.e. in’ is definable for non-c.e. degrees.

2 A result by Cai and Shore allowed us to complete this definition.
Local structures of Turing degrees

Definition

$\mathcal{R}$ is the substructure of the computably enumerable degrees.
Local structures of Turing degrees

**Definition**

\( \mathcal{R} \) is the substructure of the computably enumerable degrees.

\( \mathcal{D}_T(\leq 0') \) is the substructure of all degrees that are bounded by \( 0' \), the \( \Delta^0_2 \) Turing degrees.
The local coding theorem

Definition
A set of degrees $Z$ contained in $D_T(\leq 0')$ is uniformly low if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$ representing the degrees in $Z$, and a computable function $f$ such that $\{f(i)\}_{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

Example: If $\bigoplus_{i<\omega} A_i$ is low then $A = \{d_T(A_i) | i<\omega\}$ is uniformly low.

Theorem (Slaman and Woodin)
If $Z$ is a uniformly low subset of $D_T(\leq 0')$ then $Z$ is definable from finitely many parameters in $D_T(\leq 0')$. 
The local coding theorem

Definition

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq 0')$ is uniformly low if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function $f$ such that $\{f(i)\}^{\emptyset'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$. 

Example: If $\bigoplus_{i<\omega} A_i$ is low then $A = \{d_{\mathcal{D}_T}(A_i) | i<\omega\}$ is uniformly low.

Theorem (Slaman and Woodin)

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq 0')$ then $\mathcal{Z}$ is definable from finitely many parameters in $\mathcal{D}_T(\leq 0')$. 

Mariya I. Soskova

Definability, automorphisms and e-degrees
The local coding theorem

**Definition**

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq 0')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function $f$ such that $\{f(i)\}_0'$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

*Example:* If $\bigoplus_{i<\omega} A_i$ is low then $A = \{d_T(A_i) \mid i < \omega\}$ is uniformly low.
The local coding theorem

**Definition**

A set of degrees $\mathcal{Z}$ contained in $\mathcal{D}_T(\leq 0')$ is *uniformly low* if it is bounded by a low degree and there is a sequence $\{Z_i\}_{i<\omega}$, representing the degrees in $\mathcal{Z}$, and a computable function $f$ such that $\{f(i)\}^{0'}$ is the Turing jump of $\bigoplus_{j<i} Z_j$.

*Example:* If $\bigoplus_{i<\omega} A_i$ is low then $\mathcal{A} = \{d_T(A_i) \mid i < \omega\}$ is uniformly low.

**Theorem (Slaman and Woodin)**

If $\mathcal{Z}$ is a uniformly low subset of $\mathcal{D}_T(\leq 0')$ then $\mathcal{Z}$ is definable from finitely many parameters in $\mathcal{D}_T(\leq 0')$. 
Applications of the coding theorem

Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, s^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$. 

The set $\mathbb{N}^\mathcal{M}$ is definable with parameters $\vec{p}$.

The graphs of $s^\mathcal{M}$, $+^\mathcal{M}$, and the relation $\leq^\mathcal{M}$ are definable with parameters $\vec{p}$.

$\mathbb{N}^\mathcal{M} \mid = \varphi$ iff $D_T(\leq^0' \mid = \varphi_T(\vec{p})$. 

Mariya I. Soskova
Definability, automorphisms and e-degrees
Applications of the coding theorem

Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, s^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$.

- The set $\mathbb{N}^\mathcal{M}$ is definable with parameters $\vec{p}$.
Applications of the coding theorem

Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, s^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$.

1. The set $\mathbb{N}^\mathcal{M}$ is definable with parameters $\vec{p}$.
2. The graphs of $s$, $+$, $\times$ and the relation $\leq$ are definable with parameters $\vec{p}$.
Applications of the coding theorem

Using parameters we can code a model of arithmetic $\mathcal{M} = (\mathbb{N}^\mathcal{M}, 0^\mathcal{M}, s^\mathcal{M}, +^\mathcal{M}, \times^\mathcal{M}, \leq^\mathcal{M})$.

1. The set $\mathbb{N}^\mathcal{M}$ is definable with parameters $\vec{p}$.

2. The graphs of $s$, $+$, $\times$ and the relation $\leq$ are definable with parameters $\vec{p}$.

3. $\mathbb{N} \models \varphi$ iff $\mathcal{D}_T(\leq 0') \models \varphi_T(\vec{p})$
Applications of the coding theorem

If $Z \subseteq \mathcal{D}_T(\leq 0')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^\mathcal{M} \to \mathcal{D}_T(\leq 0')$ such that $\varphi(i^\mathcal{M}) = d_T(Z_i)$.
Applications of the coding theorem

If $Z \subseteq \mathcal{D}_T(\leq 0')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^{\mathcal{M}} \to \mathcal{D}_T(\leq 0')$ such that $\varphi(i^{\mathcal{M}}) = d_T(Z_i)$.

We call such a function an indexing of $Z$. 

Applications of the coding theorem

If $Z \subseteq \mathcal{D}_T(\leq 0')$ is uniformly low and represented by the sequence $\{Z_i\}_{i<\omega}$ then there are parameters that code a model of arithmetic $\mathcal{M}$ and a function $\varphi : \mathbb{N}^\mathcal{M} \rightarrow \mathcal{D}_T(\leq 0')$ such that $\varphi(i^\mathcal{M}) = d_T(Z_i)$.

We call such a function an indexing of $Z$.

Theorem (Slaman and Woodin)

*There are finitely many $\Delta^0_2$ parameters which code a model of arithmetic $\mathcal{M}$ and an indexing of the c.e. degrees: a function $\psi : \mathbb{N}^\mathcal{M} \rightarrow \mathcal{D}_T(\leq 0')$ such that $\psi(e^\mathcal{M}) = d_T(W_e)$.***
An indexing of the c.e. degrees
The Goal

Extend this result to an indexing $\varphi$ of the $\Delta^0_2$ Turing degrees.

We will call $e$ an index for a $\Delta^0_2$ set $X$ if $\{e\}^{\emptyset'}$ is the characteristic function of $X$. 
The Goal

Extend this result to an indexing $\varphi$ of the $\Delta^0_2$ Turing degrees.

We will call $e$ an index for a $\Delta^0_2$ set $X$ if $\{e\}^{\emptyset'}$ is the characteristic function of $X$. 
The Goal

Extend this result to an indexing $\varphi$ of the $\Delta^0_2$ Turing degrees.

We will call $e$ an index for a $\Delta^0_2$ set $X$ if $\{e\}^{\emptyset'}$ is the characteristic function of $X$. 
Extend this result to an indexing $\varphi$ of the $\Delta^0_2$ Turing degrees.

We will call $e$ an index for a $\Delta^0_2$ set $X$ if $\{e\}'$ is the characteristic function of $X$.

*Idea:* We can use a further uniformly low set $\mathcal{Z} = \{d_T(Z_i) \mid i < \omega\}$. 
The Goal

Extend this result to an indexing \( \varphi \) of the \( \Delta^0_2 \) Turing degrees.

We will call \( e \) an index for a \( \Delta^0_2 \) set \( X \) if \( \{e\}^{\emptyset'} \) is the characteristic function of \( X \).

Idea: We can use a further uniformly low set \( \mathcal{Z} = \{d_T(Z_i) \mid i < \omega\} \).
Biinterpretability with parameters

Theorem (Slaman, S)

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $M$ and an indexing of the $\Delta^0_2$ degrees.
Biinterpretability with parameters

**Theorem (Slaman, S)**

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta^0_2$ degrees.

**Proof flavor:**

1. A $\Delta^0_2$ degree can be defined from four low degrees using meet and join.

Mariya I. Soskova
Definability, automorphisms and e-degrees
Biinterpretability with parameters

**Theorem (Slaman, S)**

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $M$ and an indexing of the $\Delta^0_2$ degrees.

**Proof flavor:**

1. A $\Delta^0_2$ degree can be defined from four low degrees using meet and join.
2. There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $x$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$. 
Biinterpretability with parameters

Theorem (Slaman, S)

There are finitely many $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the $\Delta^0_2$ degrees.

Proof flavor:

1. A $\Delta^0_2$ degree can be defined from four low degrees using meet and join.
2. There exists a uniformly low set of Turing degrees $\mathcal{Z}$, such that every low Turing degree $x$ is uniquely positioned with respect to the c.e. degrees and the elements of $\mathcal{Z}$.

If $x, y \leq 0'$, $x' = 0'$ and $y \not\leq x$ then there are $g_i \leq 0'$, c.e. degrees $a_i$ and $\Delta^0_2$ degrees $c_i, b_i \in \mathcal{Z}$ for $i = 1, 2$ such that:

1. $g_i$ is the least element below $a_i$ which joins $b_i$ above $c_i$.
2. $x \leq g_1 \lor g_2$.
3. $y \not\leq g_1 \lor g_2$. 
Applications

Theorem (Slaman, S)

$D_T(\leq 0')$ has a finite automorphism base.
Applications

Theorem (Slaman, S)

1. $\mathcal{D}_T(\leq 0')$ has a finite automorphism base.
2. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
Applications

Theorem (Slaman, S)

1. $\mathcal{D}_T(\leq 0')$ has a finite automorphism base.
2. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
3. Every automorphism of $\mathcal{D}_T(\leq 0')$ has an arithmetic presentation.
Applications

Theorem (Slaman, S)

1. $\mathcal{D}_T(\leq 0')$ has a finite automorphism base.
2. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
3. Every automorphism of $\mathcal{D}_T(\leq 0')$ has an arithmetic presentation.
4. Every relation $R \subseteq \mathcal{D}_T(\leq 0')$ induced by an arithmetically definable degree invariant relation is definable with finitely many $\Delta^0_2$ parameters.
## Applications

**Theorem (Slaman, S)**

1. $\mathcal{D}_T(\leq 0')$ has a finite automorphism base.
2. The automorphism group of $\mathcal{D}_T(\leq 0')$ is countable.
3. Every automorphism of $\mathcal{D}_T(\leq 0')$ has an arithmetic presentation.
4. Every relation $R \subseteq \mathcal{D}_T(\leq 0')$ induced by an arithmetically definable degree invariant relation is definable with finitely many $\Delta^0_2$ parameters.
5. $\mathcal{D}_T(\leq 0')$ is rigid if and only if $\mathcal{D}_T(\leq 0')$ is biinterpretable with first order arithmetic.
Towards a better automorphism base of $D_e$

**Theorem (Slaman, Woodin)**

*There are total $\Delta^0_2$ parameters that code a model of arithmetic $\mathcal{M}$ and an indexing of the image of the c.e. Turing degrees.*
Towards a better automorphism base of $D_e$

**Theorem (Slaman, Woodin)**

*There are total $\Delta^0_2$ parameters that code a model of arithmetic $M$ and an indexing of the image of the c.e. Turing degrees.*

**Idea:** In the wider context of $D_e$ we can reach more elements: non-total elements.
Towards a better automorphism base of $D_e$

**Theorem (Slaman, S)**

If $\vec{p}$ defines a model of arithmetic $M$ and an indexing of the image of the c.e. Turing degrees then $\vec{p}$ defines an indexing of the total $\Delta^0_2$ enumeration degrees.

**Proof flavour:**

The image of the c.e. degrees
→ The low 3-c.e. e-degrees
→ The low $\Delta^0_2$ e-degrees
→ The total $\Delta^0_2$ e-degrees
Moving outside the local structure

1. Extend to an indexing of all total degrees that are “c.e. in” and above some total $\Delta^0_2$ enumeration degree.
   - The jump is definable.
   - The image of the relation “c.e. in” is definable.

2. Relativizing the previous theorem extend to an indexing of $\bigcup_{x \leq_T 0'} \iota([x, x'])$. 
Moving outside the local structure

Extend to an indexing of all total degrees below $0_{e''}$. 

3.
And now we iterate
And now we iterate
And now we iterate
And now we iterate
Theorem (Slaman, S)

Let $n$ be a natural number and $\vec{p}$ be parameters that index the image of the c.e. Turing degrees. There is a definable from $\vec{p}$ indexing of the total $\Delta^0_{n+1}$ degrees.
Consequences

Theorem (Slaman, S)

There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees.
Consequences

Theorem (Slaman, S)

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees.
2. The image of the c.e. Turing degrees is an automorphism base for $D_e$. 
Consequences

Theorem (Slaman, S)

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees.
2. The image of the c.e. Turing degrees is an automorphism base for $D_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.
Consequences

Theorem (Slaman, S)

1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees.
2. The image of the c.e. Turing degrees is an automorphism base for $\mathcal{D}_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Question

Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?
Consequences

Theorem (Slaman, S)
1. There is a finite automorphism base for the enumeration degrees consisting of total $\Delta^0_2$ enumeration degrees.
2. The image of the c.e. Turing degrees is an automorphism base for $D_e$.
3. If the structure of the c.e. Turing degrees is rigid then so is the structure of the enumeration degrees.

Question
Can every automorphism of the Turing degrees be extended to an automorphism of the enumeration degrees?
The best puzzles are the ones that will never be completely solved.

-Ivan Soskov