Embedding Partial Orderings in Degree Structures

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There exists a computable partial ordering $R = \langle \mathbb{N}, \leq \rangle$ in which every countable partial ordering can be embedded.

Proof.

Let $R = \langle \mathbb{Q}^2, \leq \rangle$, where $\langle a, b \rangle \leq \langle c, d \rangle$ if and only if $a \leq c$ and $b \leq d$.

Conclusion: An embedding of this computable partial ordering gives automatically an embedding of every countable partial ordering.
Independent sequences of sets

Definition (Kleene, Post 1954)
A sequence of sets \( \{A_i\}_{i<\omega} \) is called computably independent if for every \( i \):

\[ A_i \not
\leq_T \bigoplus_{j \neq i} A_j. \]

Theorem (Kleene, Post 1954)
There is a computably independent sequence of sets. This sequence can be constructed uniformly below \( 0' \).

Theorem (Muchnik 1958)
There is a computably independent sequence of c.e. sets.
Putting the two together

**Theorem (Sacks 1963)**

The existence of a computably independent sequence of sets gives an embedding of any computable partial ordering in the Turing degrees.

**Proof.**

Let $\mathcal{R} = \langle \mathbb{N}, \preceq \rangle$ be a computable partial ordering and $\{A_i\}_{i<\omega}$ be a computably independent sequence of sets. The embedding is:

$$\kappa(i) = d_T(\bigoplus_{j \geq i} A_i).$$
Corollary

Every countable partial ordering can be embedded

1. Kleene and Post: in the Turing degrees, even in the $\Delta^0_2$ Turing degrees.
3. Robinson 1971: densely in the c.e. Turing degrees, i.e. in any nonempty interval of c.e. Turing degrees.
The enumeration degrees

- The e-degrees as a proper extension of the Turing degrees, inherit this complexity.
- Case 1971: Any countable partial ordering can be embedded in the e-degrees below the degree of any generic function.
- Copestake 1988: below any 1-generic enumeration degree.
- Cooper and McEvoy 1985: below any nonzero $\Delta^0_2$ e-degree.
- Bianchini 2000: densely in the $\Sigma^0_2$ enumeration degrees.

**Method**: e-independent sequences of sets.
The first observation

**Theorem**

Let \( b < a \) be enumeration degrees such that \( a \) contains a member with a good approximation. Then every countable partial ordering can be embedded in the interval \([b, a]\).

**Idea:** Construct an e-independent sequence of sets above \( b \) and uniformly below \( a \).

**Techniques:** Good approximations combined with a construction inspired by Cooper’s density construction.
The general picture

a ∈ 4-c.e.a
b

a ∈ 3-c.e.a
b

a ∈ TOT
b

c ∈ Π^0_2

d ∈ Π^0_2

a ∈ Σ^0_2
b
Let $S$ be the set of all sequences of sets of natural numbers.

**Definition**

Let $\mathcal{A} = \{A_n\}^{n<\omega}$ be a sequence of sets natural numbers and $V$ be an e-operator. The result of applying the enumeration operator $V$ to the sequence $\mathcal{A}$, denoted by $V(\mathcal{A})$, is the sequence $\{V[n](A_n)\}^{n<\omega}$. We say that $V(\mathcal{A})$ is enumeration reducible ($\leq_e$) to the sequence $\mathcal{A}$.

So $\mathcal{A} \leq_e \mathcal{B}$ is a combination of two notions:

- Enumeration reducibility: for every $n$ we have that $A_n \leq_e B_n$ via, say, $\Gamma_n$.
- Uniformity: the sequence $\{\Gamma_n\}^{n<\omega}$ is uniform.
Basic definitions

With every member $A \in S$ we connect a *jump sequence* $P(A)$.

**Definition**

The *jump sequence* of the sequence $A$, denoted by $P(A)$ is the sequence $\{P_n(A)\}_{n<\omega}$ defined inductively as follows:

- $P_0(A) = A_0$.
- $P_{n+1}(A) = A_{n+1} \oplus P'_n(A)$, where $P'_n(A)$ denotes the enumeration jump of the set $P_n(A)$.

The jump sequence $P(A)$ transforms a sequence $A$ into a monotone sequence of sets of natural numbers with respect to $\leq_e$. Every member of the jump sequence contains full information on previous members.
The $\omega$-enumeration degrees

Let $A, B \in S$.

Definition

- $\omega$-enumeration reducibility: $A \leq_{\omega} B$, if $A \leq_e P(B)$.
- $\omega$-enumeration equivalence: $A \equiv_{\omega} B$ if $A \leq_{\omega} B$ and $B \leq_{\omega} A$.
- $\omega$-enumeration degrees: $d_{\omega}(A) = \{B \mid A \equiv_{\omega} B\}$.
- The structure of the $\omega$-enumeration degrees: $D_{\omega} = \langle\{d_{\omega}(A) \mid A \in S\}, \leq_{\omega}\rangle$, where $d_{\omega}(A) \leq_{\omega} d_{\omega}(B)$ if $A \leq_{\omega} B$.
- The least $\omega$-enumeration degree: $0_{\omega} = d_{\omega}((\emptyset, \emptyset, \emptyset, \ldots))$ or equivalently $d_{\omega}((\emptyset, \emptyset', \emptyset'', \ldots))$. 
The $\omega$-enumeration degrees

Let $A, B \in S$.

Definition

- $\omega$-enumeration reducibility: $A \leq_\omega B$, if $A \leq_e P(B)$.
- $\omega$-enumeration equivalence: $A \equiv_\omega B$ if $A \leq_\omega B$ and $B \leq_\omega A$.
- $\omega$-enumeration degrees: $d_\omega(A) = \{B | A \equiv_\omega B\}$.
- The structure of the $\omega$-enumeration degrees: $D_\omega = \langle\{d_\omega(A) | A \in S\}, \leq_\omega\rangle$, where $d_\omega(A) \leq_\omega d_\omega(B)$ if $A \leq_\omega B$.
- The least $\omega$-enumeration degree: $0_\omega = d_\omega((\emptyset, \emptyset, \emptyset, \ldots))$ or equivalently $d_\omega((\emptyset, \emptyset', \emptyset'', \ldots))$. 
\( D_\omega \) as an upper semilattice with jump operation

- The join and least upper bound: \( A \oplus B = \{ A_n \oplus B_n \}_{n<\omega} \).
  \( d_\omega (A \oplus B) = d_\omega (A) \lor d_\omega (B) \).

- The jump operation: \( d_\omega (A)' = d_\omega (A') \), where \( A' = \{ P_{n+1}(A) \}_{n<\omega} \).
The e-degrees as a substructure

\( \langle D_e, \leq_e, \lor, ' \rangle \) can be embedded in \( \langle D_\omega, \leq_\omega, \lor, ' \rangle \) via the embedding \( \kappa \) defined as follows:

\[
\kappa(d_e(A)) = d_\omega((A, \emptyset, \emptyset, \ldots)) = d_\omega((A, A', A'', \ldots)).
\]

Theorem (Soskov, Ganchev)

- The structure \( D_1 = \kappa(D_e) \) is first order definable in \( D_\omega \).
- The structures \( D_e \) and \( D_\omega \) with jump operation have isomorphic automorphism groups.
The embeddability question

Consider the structure $\mathcal{G}_\omega$ consisting of all degrees reducible to $0'_\omega = d_\omega((\emptyset', \emptyset'', \emptyset''', \ldots ))$ also called the $\Sigma^0_2 \omega$-enumeration degrees.

**Theorem (Soskov)**

*The structure $\mathcal{G}_\omega$ is dense.*

**Theorem**

*Let $b < \omega a \leq \omega 0'_\omega$. Every countable partial ordering can be embedded in the interval $[b, a]$.***
The independent sequence method

Definition
Let \( \{A_i\}_{i<\omega} \) be a sequence of sequences of sets
▶ For every computable set \( C \) set
\[
\bigoplus_{i \in C} A_i = (\bigoplus_{i \in C} A_{0,i}, \bigoplus_{i \in C} A_{1,i}, \bigoplus_{i \in C} A_{2,i}, \ldots ).
\]
▶ The sequence is \( \omega \)-independent if for every \( i \) we have
\[
A_i \not\preceq_\omega \bigoplus_{j \neq i} A_j
\]

Goal: Construct an \( \omega \)-independent sequence of sequences of sets above \( b \) and uniformly below \( a \).
Good approximations to sequences

Definition (Soskov)
Let \( \{ A_n^{\{s\}} \}_{n,s<\omega} \) be a uniformly computable matrix of finite sets. We say that \( \{ A_n^{\{s\}} \}_{s<\omega} \) is a **good approximation** to the sequence \( A = \{ A_n \}_{n<\omega} \) if:

G0: \((\forall s, k)[A_k^{\{s\}} \subseteq A_k \Rightarrow (\forall m \leq k)[A_m^{\{s\}} \subseteq A_m]]\);

G1: \((\forall n, k)(\exists s)(\forall m \leq k)[A_m \upharpoonright n \subseteq A_m^{\{s\}} \subseteq A_m] \) and

G2: \((\forall n, k)(\exists s)(\forall t > s)[A_k^{\{t\}} \subseteq A_k \Rightarrow (\forall m \leq k)[A_m \upharpoonright n \subseteq A_m^{\{t\}}]]\).

Or more intuitively:

- We have a good approximation to every member of the sequence.
- If \( m \leq k \) then every \( k \)-good stage is \( m \)-good.
Proof idea

Theorem (Soskov)

Every $\Sigma_2^0 \omega$-enumeration degree contains a member $A$ such that $A \equiv_e P(A)$ and $A$ has a good approximation.

So fix $A = (A_0, A_1, \ldots)$ in $a$ with the properties listed in the theorem and $B = (B_0, B_1, \ldots)$ in $b$.

Now $B <_\omega A$ can follow by two ways:

- Non-enumeration reducibility: There is an $n$ such that $B_n <_e A_n$.
- Non-uniformity: For every $n$ we have $B_n \equiv_e A_n$ but not uniformly in $n$. 
Let $n$ be such that $B_n <_e A_n$.

- By first theorem there is an independent sequence of sets $\{C_i\}_{i<\omega}$ above $B_n$ and uniformly below $A_n$.
- Define $\{C_i\}_{i<\omega}$ by $C_i = (B_0, B_1, \ldots, B_{n-1}, C_i, B_{n+1}, \ldots)$.
- $\{C_i\}_{i<\omega}$ is an $\omega$-independent.
- For every $i$ we have $B <_\omega C_i <_\omega A$. 

Easy case: From the e-degrees
Easy case: From the e-degrees

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- For every $i$ we have $B <_\omega C_i <_\omega A$. 
Easy case: From the e-degrees

Let \( n \) be such that \( B_n \prec_e A_n \).

- By first theorem there is an independent sequence of sets \( \{C_i\}_{i<\omega} \) above \( B_n \) and uniformly below \( A_n \).
- Define \( \{C_i\}_{i<\omega} \) by \( C_i = (B_0, B_1, \ldots, B_{n-1}, C_i, B_{n+1}, \ldots) \).
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- For every \( i \) we have \( B <_\omega C_i <_\omega A \).
Difficult case

For every $n \ A_n \equiv_e B_n$.

**Idea:** Direct construction building on ideas from first result.

**Difficulties:** Approximate sets of the form $P(V(A))$, where $V$ is the constructed e-operator.

**Techniques:** Good approximations for sequences of sets. Length of agreement function for sequences of sets. Fixed point theorem (Recursion theorem).
Definition (Jockusch, Lerman, Soare and Solovay)
Let \( a \) and \( b \) be c.e. Turing degrees. \( a \sim_\infty b \) iff there exists a natural number \( n \) such that \( a^n = b^n \).

- Induced degree structure \( \mathcal{R}/ \sim_\infty \) with \( [a]_{\sim_\infty} \leq [b]_{\sim_\infty} \) if and only if there exists a natural number \( n \) such that \( a^n \leq_T b^n \).
- Least element \( L = \bigcup_{n<\omega} L_n \).
- Greatest element \( H = \bigcup_{n<\omega} H_n \).
- \( \mathcal{R}/ \sim_\infty \) is a dense structure.
- Lempp: There is a splitting of the highest \( \infty \)-degree and a minimal pair of \( \infty \)-degrees.
Starting with other classes of degree

- $\mathcal{G}_T / \sim_\infty$: the $\Delta^0_2$ Turing degrees modulo iterated jump. Shoenfield, Sacks: The range of the jump operator restricted to the c.e. Turing degrees coincides with the range of the jump operator restricted to the $\Delta^0_2$ Turing degrees. It is namely the set of all Turing degrees c.e. in and above $0'$. Hence:

  $$\mathcal{G}_T / \sim_\infty \cong \mathcal{R} / \sim_\infty.$$  

- $\mathcal{G}_e / \sim_\infty$: the $\Sigma^0_2$ e-degrees modulo iterated jump. McEvoy: The range of the enumeration jump operator restricted to the $\Sigma^0_2$-enumeration degrees coincides with the range of the enumeration jump operator restricted to the $\Pi^0_1$ enumeration degrees. Hence:

  $$\mathcal{R} / \sim_\infty \cong (\Pi^0_1 \text{ e-degrees}) / \sim_\infty \cong \mathcal{G}_e / \sim_\infty.$$
The $\omega$-enumeration degrees modulo iterated jump

Consider $\mathcal{G}_\omega / \sim_\infty$.

- $\mathcal{R} / \sim_\infty$ embeds in $\mathcal{G}_\omega / \sim_\infty$.
  \[ \mathcal{R} \subseteq \mathcal{G}_T \hookrightarrow \iota(\mathcal{G}_T) = \text{Tot} \subseteq \mathcal{G}_e \hookrightarrow \kappa(\mathcal{G}_e) = \mathcal{D}_1 \subseteq \mathcal{G}_\omega \]

- A basic property:

**Lemma**

Let $a$ and $b$ be two $\Sigma^0_2$ $\omega$-enumeration degrees.

1. If $a \leq_\omega b$ then $[a]_{\sim_\infty} \leq [b]_{\sim_\infty}$.
2. If $[a]_{\sim_\infty} \leq [b]_{\sim_\infty}$ then there is a representative $c \in [a]_{\sim_\infty}$ such that $c \leq_\omega b$. 
The almost degrees

Definition
Let $\mathcal{A} = \{A_n\}_{n<\omega}$ be a sequence of sets of natural numbers. We shall say that the sequence $\mathcal{B} = \{B_n\}_{n<\omega}$ is almost-$\mathcal{A}$ if for every $n$ we have that $P_n(\mathcal{A}) \equiv_e P_n(\mathcal{B})$. If $\mathcal{A}$ is almost-$\mathcal{B}$ then we shall say that $d_\omega(\mathcal{A})$ is almost-$d_\omega(\mathcal{B})$.

Lemma
Let $a \leq 0'_{\omega}$ be an $\omega$-enumeration degree.

1. If $b$ is almost-$a$ and $A \in a$ then every $B \in b$ is almost-$A$.
2. The class of almost-$a$ degrees is closed under least upper bound.
3. If $a \leq_\omega c \leq_\omega b$ and $b$ is almost-$a$ then $c$ is almost-$a$.
The almost degrees

Lemma

4. If $a \in D_1$ then $a$ is the least almost-$a$ $\Sigma_2^0$ $\omega$-enumeration degree.

5. If $b$ and $c$ are almost-$a$ $\Sigma_2^0$ $\omega$-enumeration degrees then $[b]_{\sim_\infty} \leq [c]_{\sim_\infty}$ if and only if $b \leq_\omega c$.

6. If $a <_\omega b$ and $a <_\infty b$ then there exists an almost-$a$ degree $z$ such that $a <_\omega z \leq_\omega b$.

Corollary

$G_\omega/ \sim_\infty$ properly extends $R/ \sim_\infty$.
Embedding partial orders in $G_\omega/\sim_\infty$

Corollary
Every countable partial ordering can be embedded densely in $G_\omega/\sim_\infty$.

Proof.
- Let $[a]_{\sim_\infty} < [b]_{\sim_\infty}$.
- We may assume that $a <_\omega b$.
- Let $z$ be an almost-$a$ degree such that $a <_\omega z \leq_\omega b$.
- Then $[a]_{\sim_\infty} < [z]_{\sim_\infty} \leq [b]_{\sim_\infty}$.
- And $[a, z]$ consists entirely of almost-$a$ degrees, hence is isomorphic to $[[a]_{\sim_\infty}, [z]_{\sim_\infty}]$.
- By second result we can embed any countable partial ordering in $[a, z]$. 


Embedding partial orders in $G_\omega/\sim_\infty$

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*Every countable partial ordering can be embedded densely in $G_\omega/\sim_\infty$.*

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Thank you!