Math 375  Exam # 1  October 8, 2003

Problem I  Give brief but precise answers to each of the following questions.

(a) (3 points) If $V$ is a vector space and if $S = \{\bar{v}_1, \ldots, \bar{v}_k\}$ is a finite subset of $V$, what is the linear span of $S$? (The linear span of $S$ is often denoted by $L(S)$).

(b) (3 points) If $V$ is a vector space and if $S = \{\bar{v}_1, \ldots, \bar{v}_k\}$ is a finite subset of $V$, what does it mean that the vectors in $S$ are linearly independent?

(c) (3 points) If $V$ is a vector space and if $S = \{\bar{v}_1, \ldots, \bar{v}_k\}$ is a finite subset of $V$, what does it mean that the set $S$ is a basis for $V$?

(d) (3 points) If $V$ and $W$ are vector spaces, what is a linear transformation from $V$ to $W$?

(e) (3 points) If $V$ and $W$ are vector spaces, and if $T : V \rightarrow W$ is a linear transformation, what is the null space of $T$? (The null space is often denoted by $N(T)$).

(f) (3 points) If $V$ and $W$ are vector spaces, and if $T : V \rightarrow W$ is a linear transformation, what is the range of $T$? (The range is often denoted by $R(T)$ or by $T(V)$).

Problem II  Let $S = \{\bar{v}_1, \ldots, \bar{v}_k\}$ be a finite set of linearly independent vectors in a vector space $V$.

(a) (4 points) Prove that if $\bar{x} \in V$, if $\bar{x} = \alpha_1 \bar{v}_1 + \cdots + \alpha_k \bar{v}_k$, and if $\bar{x} = \beta_1 \bar{v}_1 + \cdots + \beta_k \bar{v}_k$, then $\alpha_j = \beta_j$ for $1 \leq j \leq k$.

(b) (10 points) If $\bar{v}_{k+1}$ is a vector in $V$ which is not an element of the linear span of the set $S$, prove that the vectors $\{\bar{v}_1, \ldots, \bar{v}_k, \bar{v}_{k+1}\}$ are linearly independent.

Problem III (12 points) Let $f$ and $g$ be real-valued continuous functions on the interval $[0, 1]$. Without quoting any results established in class, prove directly that

$$\left| \int_0^1 f(x)g(x) \, dx \right|^2 \leq \left( \int_0^1 f(x)^2 \, dx \right) \left( \int_0^1 g(x)^2 \, dx \right).$$

Hint: You should recognize this as an example of an inequality proved in class, and you can use the proof from class to establish this special case.

Problem IV  Let $V$ be a real vector space with inner product. If $\bar{x}, \bar{y} \in V$, the inner product of $\bar{x}$ and $\bar{y}$ is denoted by $\langle \bar{x}, \bar{y} \rangle$.

(a) (6 points) Let $\bar{v}$ be a fixed vector in $V$, and let $S$ be the set of all vectors $\bar{x} \in V$ such that $\langle \bar{x}, \bar{v} \rangle = 0$. Prove that $S$ is a subspace of $V$.

(b) (9 points) Let $V$ be $\mathbb{R}^3$ with the inner product given by the usual dot product. Let $\bar{v} = (1, 2, 3)$, and let $S$ be the set of all vectors $\bar{x} \in V$ such that $\langle \bar{x}, \bar{v} \rangle = 0$. Find a basis for $S$.

Problem V (15 points) Let $V$ and $W$ be vector spaces, and let $T : V \rightarrow W$ be a linear transformation. Prove that if $\{\bar{x}_1, \ldots, \bar{x}_n\}$ are vectors in $V$ and if $\{T(\bar{x}_1), \ldots, T(\bar{x}_n)\}$ are linearly independent vectors in $W$, then $\{\bar{x}_1, \ldots, \bar{x}_n\}$ are linearly independent vectors in $V$.

Problem VI (14 points) Let $V$ denote the space of all real-valued functions on $\mathbb{R}$. Are the three functions $\{1, e^{2x}, e^{3x}\}$ linearly dependent or linearly independent in $V$? Be sure to give a proof that your answer is correct.

Problem VII  Consider $\mathbb{R}^3$ with the usual dot product. Let

$$\bar{e}_1 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \quad \bar{e}_2 = \left( \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right), \quad \bar{e}_3 = \left( \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right).$$

(a) (5 points) Show that the vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ are orthonormal.

(b) (7 points) Find real numbers $\alpha, \beta, \gamma$ so that $(2, -1, 7) = \alpha \bar{e}_1 + \beta \bar{e}_2 + \gamma \bar{e}_3$.

Hint: Use the fact from part (a) that the vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ are orthonormal, and apply a result proved in class.