Problem 1: Give brief but precise answers to the following questions.

(a) If \( V \) is a vector space and if \( S = \{v_1, \ldots, v_k\} \subset V \), what does it mean that the vectors in \( S \) are **linearly independent**?

If \( \alpha_1 v_1 + \cdots + \alpha_k v_k = 0_V \) then \( \alpha_1 = \cdots = \alpha_k = 0 \).

(b) If \( V \) is a vector space and if \( S = \{v_1, \ldots, v_k\} \subset V \), what is the **linear span** of \( S \)?

The linear span of \( S \) is the set of all vectors \( x \) which can be written as \( x = \alpha_1 v_1 + \cdots + \alpha_k v_k \) where \( \alpha_1, \ldots, \alpha_k \) are scalars.

(c) If \( V \) is a vector space and if \( S = \{v_1, \ldots, v_k\} \subset V \), what does it mean that \( S \) is a **basis** for \( V \)?

The set \( S \) is a basis if the vectors \( \{v_1, \ldots, v_k\} \) are linearly independent and if the linear span of \( S \) is the whole vector space \( V \).

(d) If \( V \) and \( W \) are vector spaces and if \( T : V \to W \) is a function, what does it mean that \( T \) is a **linear transformation**?

\( T \) is a linear transformation if and only if \( T(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 T(v_1) + \alpha_2 T(v_2) \) for all vectors \( v_1, v_2 \in V \) and all scalars \( \alpha_1, \alpha_2 \).

(e) If \( V \) and \( W \) are vector spaces, and if \( T : V \to W \) is a linear transformation, what is the **null space** \( N(T) \) and what is the **range** \( R(T) \)?

We have \( N(T) = \{v \in V \mid T(v) = 0\} \), \( R(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\} \).

(f) If \( z = a + ib \) is a complex number, what is the **complex conjugate** \( \overline{z} \) and what is the **absolute value** \( |z| \)?

We have \( \overline{z} = a - ib \) and \( |z| = \sqrt{a^2 + b^2} = \sqrt{z \overline{z}} \).

(g) If \( V \) is a vector space with an inner product and if \( S = \{e_1, \ldots, e_n\} \subset V \), what does it mean that the vectors in \( S \) are **orthonormal**?

The vectors in \( S \) are orthonormal if \( \langle e_j, e_j \rangle = 1 \) and \( \langle e_j, e_k \rangle = 0 \) if \( j \neq k \). That is, each vector has length 1, and any two distinct vectors are perpendicular.

(h) If \( T_1 \) and \( T_2 \) are linear transformations mapping a vector space \( V_1 \) to a vector space \( V_2 \), and if \( \alpha_1 \) and \( \alpha_2 \) are scalars, how does one define the **linear combination** \( \alpha_1 T_1 + \alpha_2 T_2 \)?

The mapping \( \alpha_1 T_1 + \alpha_2 T_2 \) is defined by the equation \( (\alpha_1 T_1 + \alpha_2 T_2)(v) = \alpha_1 T_1(v) + \alpha_2 T_2(v) \) for every \( v \in V_1 \).

Problem 2: (10 points) Let \( V \) be a vector space with an inner product, let \( \{v_1, \ldots, v_k\} \) be a finite set of vectors in \( V \), and suppose that \( \langle v_j, v_k \rangle = \begin{cases} 0 & \text{if } j \neq k, \\ 1 & \text{if } j = k. \end{cases} \) Prove that the vectors \( \{v_1, \ldots, v_n\} \) are linearly independent.

Suppose that there is an equation \( \alpha_1 v_1 + \cdots + \alpha_k v_k = 0_V \). Take the inner product of both sides with one of the vectors \( v_j \). Using the properties of inner product we get
\[
0 = \langle 0_V, v_j \rangle = \langle \alpha_1 v_1 + \cdots + \alpha_k v_k, v_j \rangle = \alpha_1 \langle v_1, v_j \rangle + \cdots + \alpha_k \langle v_k, v_j \rangle = \alpha_j
\]
since all the terms \( \langle v_i, v_j \rangle = 0 \) unless \( i = j \). This says that each \( \alpha_j = 0 \), and so the vectors \( \{v_1, \ldots, v_k\} \) are linearly independent.

Problem 3: Let \( V \) be a real vector space having an inner product. Prove the Cauchy-Schwarz inequality; that is, for any vectors \( x, y \in V \), prove that \( |\langle x, y \rangle| \leq ||x|| \cdot ||y|| \) where \( \langle x, y \rangle \) is the inner product of the vectors \( x \) and \( y \), and \( ||x|| = \sqrt{\langle x, x \rangle} \) and \( ||y|| = \sqrt{\langle y, y \rangle} \).
Proof: If \( y = 0 \), then both sides of the inequality are zero, and there is nothing to prove. Thus we may assume that \( y \neq 0 \). Now for any real number \( t \), the length of the vector \( x + ty \) is greater than or equal to zero. Thus we have

\[
0 \leq ||x + ty||^2 = (x + ty, x + ty) = (x, x) + t(y, x) + t^2(y, y) = ||x||^2 + 2t(x, y) + t^2||y||^2.
\]

(We have used the fact that for a real vector space, \((y, x) = (x, y), (x, ty) = t(x, y), \text{ and } (x, y) = (y, x)\).) The quadratic polynomial on the right takes on its minimum value when \( t = -(x, y)/||y||^2 \). If we plug this value of \( t \) into the inequality \( ||x||^2 + 2t(x, y) + t^2||y||^2 \geq 0 \), we get \( ||x||^2 - \frac{||(x, y)||^2}{||y||^2} \geq 0 \), which then implies \( ||x, y|| \leq ||x|| ||y|| \). This completes the proof.

Problem 4: (10 points) Let \( V \) be a real vector space having an inner product. Prove that for any \( x, y \in V \),

\[
||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2).
\]

We have

\[
||x + y||^2 = (x + y, x + y) = (x, x) + 2(x, y) + (y, y) = ||x||^2 + 2(x, y) + ||y||^2, \]

\[
||x - y||^2 = (x - y, x - y) = (x, x) - 2(x, y) + (y, y) = ||x||^2 - 2(x, y) + ||y||^2.
\]

Adding these two equations we get the desired equality.

Problem 5: Define a linear transformation \( T : \mathbb{R}^4 \rightarrow \mathbb{R}^3 \) by the equation

\[
T(x, y, z, w) = (x - y, y - z, z - x).
\]

(a) Find a basis for the null space \( N(T) \).

The null space is the set of \((x, y, z, w)\) such that \(x - y = 0, y - z = 0\), and \(z - x = 0\). That is, the null space is the set of \((x, y, z, w)\) such that \(x = y = z\), and so is the set of vectors in \(\mathbb{R}^4\) of the form \((x, x, x, w)\). Every such vector can be written uniquely as \(x(1, 1, 1, 0) + w(0, 0, 0, 1)\), and so an example of a basis for \( N(T) \) is the set of two vectors \(((1, 1, 1, 0), (0, 0, 0, 1)) \).

(b) What is the dimension of the range \( R(T) \)? (Explain how you arrived at your answer.)

Using the theorem relating the dimensions of the range and null space, we see that

\[
\text{dim}(N(T)) + \text{dim}(R(T)) = \text{dim}(\mathbb{R}^4) = 4,
\]

and since by part (a) the dimension of \( N(T) \) is 2, it follows that the dimension of \( R(T) \) is also 2.

(c) Show that the range \( R(T) \) is contained in the subspace \( W = \{(a, b, c) \in \mathbb{R}^3 | a + b + c = 0\} \).

If \((a, b, c) \in R(T)\) then \((a, b, c) = (x - y, y - z, z - x)\) for some \((x, y, z, w) \in \mathbb{R}^4\). It follows that \(a = x - y, b = y - z,\) and \(c = z - x\). It follows that \(a + b + c = (x - y) + (y - z) + (z - x) = 0\). Thus the range \( R(T) \) is contained in \( W \).

(d) Is \( R(T) = W \)? Why or why not?

Yes, \( R(T) = W \) because \( W \) is a plane in \( \mathbb{R}^3 \) and hence has dimension 2. But from part (b) we know that \( R(T) \) has dimension 2, and we know \( R(T) \subset W \). Since they have the same dimension, they must be equal.

(e) Show that the set of vectors \( B = \{e_1 = (1, 1, 0, 0), e_2 = (1, 0, 1, 0), e_3 = (0, 1, 1, 0), e_4 = (0, 0, 0, 1)\} \) is a basis for \( \mathbb{R}^4 \).

Suppose we have \( \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 = 0 \) where \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are scalars. This gives the equations \( \alpha_1 + \alpha_2 = 0, \alpha_1 + \alpha_3 = 0, \alpha_2 + \alpha_3 = 0, \text{ and } \alpha_4 = 0 \). It follows from the first equation that \( \alpha_2 = -\alpha_1 \) and from the second equation that \( \alpha_3 = -\alpha_2 \). Plugging into the third equation, we see that \(-2\alpha_1 = 0\) and so \(\alpha_1 = 0\). It now follows easily that all the alphas are zero. This shows that the vectors are linearly independent. Since the dimension of \( \mathbb{R}^4 \) is 4, it also follows that these vectors are a basis.

(f) Find the matrix representation of the linear transformation \( T \) if we use the basis \( B \) in \( \mathbb{R}^4 \) and the standard basis \( \{f_1 = (1, 0, 0, 0), f_2 = (0, 1, 0, 0), f_3 = (0, 0, 1, 0)\} \) in \( \mathbb{R}^3 \).

We need to compute \( T(e_j) \) and write the result as a linear combination of \( \{f_1, f_2, f_3\} \). But

\[
T(e_1) = T(1, 1, 0, 0) = (0, 1, -1) = 0 f_1 + 1 f_2 - 1 f_3,
\]

\[
T(e_2) = T(1, 0, 1, 0) = (1, -1, 0) = 1 f_1 - 1 f_2 + 0 f_3,
\]

\[
T(e_3) = T(0, 1, 1, 0) = (-1, 0, 1) = -1 f_1 + 0 f_2 + 1 f_3,
\]

\[
T(e_4) = T(0, 0, 0, 1) = (0, 0, 0) = 0 f_1 + 0 f_2 + 0 f_3.
\]

Thus the matrix of \( T \) relative to these bases is

\[
\begin{bmatrix}
0 & 1 & -1 & 0 \\
1 & 0 & 1 & 0 \\
-1 & 0 & 1 & 0
\end{bmatrix}.
\]