Assignment # 2

Math 375, Lectures 1 and 2

Due Tuesday, September 25, 2012

Read sections 8, 9, 11, 12, 14, 15, and 16 in Chapter 1 of the text, Calculus, Volume II by Tom M. Apostol, and do the problems given below. Homework will be collected in your recitation section.

Problem 1: Let $V$ be a vector space. If $S \subset V$ is a (possibly infinite) subset of $V$, then the span of $S$, denoted by $L(S)$, is the set of linear combinations of any finite collection of elements in $S$. That is, a vector $y \in L(S)$ if and only if there are elements $x_1, \ldots, x_k \in S$ and scalars $\alpha_1, \ldots, \alpha_k$ so that $y = \alpha_1 x_1 + \cdots + \alpha_k x_k$.

(a) If $S \subset V$ and show that $L(S)$ is a subspace of $V$.
(b) If $S \subset W \subset V$, and if $W$ is a subspace of $V$, show that $L(S) \subset W$.
(c) If $S \subset V$ and $T \subset V$ are two subsets, show that $L(S \cap T) \subset L(S) \cap L(T)$.
(d) Give an example where $L(S \cap T) \neq L(S) \cap L(T)$.
(e) If $S \subset V$ and $T \subset V$ are two subsets, show that $L(S \cup T) = L(L(S) \cup L(T))$.

Problem 2: Consider the system of three equations in four unknowns

\[
\begin{align*}
3x + 2y - 4z + 2w &= a, \\
2x - 7y + 3z - 9w &= b, \\
x - 3y + 2z + 7w &= c,
\end{align*}
\]

where $(x, y, z, w) \in \mathbb{R}^4$ and $(a, b, c) \in \mathbb{R}^3$.

(a) Show that the set of vectors $(a, b, c) \in \mathbb{R}^3$ for which there exists a solution to the system (1) is the same as the linear span in the vector space $\mathbb{R}^3$ of the set of vectors $S = \{(3, 2, 1), (2, -7, -3), (-4, 3, 2), (2, -9, 7)\}$.

(b) Under what conditions on the vector $(a, b, c)$ is the set of solutions to the system (1) a subspace of $\mathbb{R}^3$?

Problem 3: Consider the system of $m$ equations in $n$ unknowns

\[
\begin{align*}
c_{1,1} x_1 + c_{1,2} x_2 + \cdots + c_{1,n-1} x_{n-1} + c_{1,n} x_n &= \alpha_1, \\
c_{2,1} x_1 + c_{2,2} x_2 + \cdots + c_{2,n-1} x_{n-1} + c_{2,n} x_n &= \alpha_2, \\
& \quad \vdots \\
c_{m,1} x_1 + c_{m,2} x_2 + \cdots + c_{m,n-1} x_{n-1} + c_{m,n} x_n &= \alpha_m.
\end{align*}
\]

Here the coefficients $\{c_{i,j}\}$ are thought of as given, the variables $(x_1, \ldots, x_n) \in \mathbb{R}^n$ are the unknowns, and the quantities $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ are also given.

(a) Find a finite set $S$ of vectors in $\mathbb{R}^m$ such that the set of vectors $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ for which there exists a solution to the system (2) is the same as the linear span $L(S)$ of $S$.

(b) Under what conditions on the vector $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ is the set of solutions to the system (2) a subspace of $\mathbb{R}^n$?

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1This is a slight generalization of the definition of linear span given in class, where we assumed that the set $S$ was finite.
2Recall that $X \cap Y$ is the set of elements which belong to both $X$ and $Y$.
3Recall that $X \cup Y$ is the set of elements which belong either to $X$ or to $Y$ or to both.
Problem 4: Let $V$ be the vector space consisting of all real valued functions defined on the real line $\mathbb{R}$. Determine whether each of the following subsets $S \subseteq V$ is linearly dependent or linearly independent, and compute the dimension of the subspace spanned by each set.

(a) $S = \{1 + x, x^2 - 1, 1 - x + x^2\}$;
(b) $S = \{1, \cos(2x), \sin^2(x)\}$;
(c) $S = \{e^{3x}, e^{-2x}, xe^x\}$.

Problem 5: Let $V$ be a finite dimensional vector space and let $W \subseteq V$ be a subspace.

(a) Show that $W$ is finite dimensional, and that the dimension $\dim(W)$ of $W$ is less than or equal to the dimension $\dim(V)$ of $V$.
(b) Show that $\dim(W) = \dim(V)$ if and only if $W = V$.
(c) Show that if $\{w_1, \ldots, w_k\}$ is a basis for $W$, then there is a basis for $V$ containing the vectors $w_1, \ldots, w_k$.
(d) Give an example to show that a basis for $V$ need not contain a subset which is a basis for $W$.

Problem 6: If $z = a + ib$ is a complex number, then the complex conjugate of $z$ is the complex number
$$\overline{z} = a - ib,$$
and the absolute value of $z$ is the non-negative real number
$$|z| = \sqrt{a^2 + b^2}.$$

(a) If $z = a + ib$ is a complex number, show that $|z|^2 = z\overline{z}$.
(b) If $z = a + ib$ and $w = c + id$ are two complex numbers, show that $(zw)\overline{w} = z\overline{w}$.
(c) If $z = a + ib$ and $w = c + id$ are two complex numbers, show that $|zw| = |z||w|$.
(d) If $z = a + ib$ and $w = c + id$ are two complex numbers, show that $|z + w| \leq |z| + |w|$.

Problem 7: If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two vectors in $\mathbb{R}^n$, we define the dot product of these two vectors to be the real number
$$x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in \mathbb{R}.$$
If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are two vectors in $\mathbb{C}^n$, we define the dot product of these two vectors to be the complex number
$$z \cdot w = z_1\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n} \in \mathbb{C}.$$

(a) Show that if $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ then
$$|\langle x_1, x_2 \rangle \cdot \langle y_1, y_2 \rangle| \leq \sqrt{x_1^2 + x_2^2} \sqrt{y_1^2 + y_2^2}$$
(b) Show that if $(z_1, z_2), (w_1, w_2) \in \mathbb{C}^2$ then
$$|\langle z_1, z_2 \rangle \cdot \langle w_1, w_2 \rangle| \leq \sqrt{|z_1|^2 + |z_2|^2} \sqrt{|w_1|^2 + |w_2|^2}$$
(c) Can you conjecture a generalization of the inequalities in equations (3) and (4)?