Problem 1: Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation. Recall that $T$ has a left inverse if there is an operator $S : W \to V$ such that $S \circ T(v) = v$ for all $v \in V$, and that $T$ has a right inverse if there is an operator $R : W \to V$ such that $T \circ R(w) = w$ for all $w \in W$. Prove that if $T$ has a left inverse $S$ and a right inverse $R$, then $R = S$. (In this case we say that $T$ is invertible.)

Problem 2: Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be a linear transformation.

(a) Suppose that there is linear transformation $S : W \to V$ such that $S \circ T$ is the identity operator on $V$. (That is, suppose that $T$ has a left inverse $S$.) Show that $\ker(T) = (0)$ and hence that $T$ is ‘one-to-one’.

(b) Suppose that there is a linear transformation $R : W \to V$ so that $T \circ R$ is the identity operator on $W$. (That is, suppose that $T$ has a right inverse $R$.) Show that the $\im(T) = W$ and hence that $T$ is ‘onto’.

(c) Suppose that the dimension of $V = W$ and that the dimension of $V$ is finite. Show that if $T$ has a left inverse, then $T$ is invertible.

(d) Suppose that the dimension of $V = W$ and that the dimension of $V$ is finite. Show that if $T$ has a right inverse, then $T$ is invertible.

Problem 3: Let $\mathbb{R}^\infty$ denote the space of all infinite sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$ of real numbers. Then $\mathbb{R}^\infty$ is a vector space if addition and scalar multiplication are defined by

$$\alpha x + \beta y = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \ldots, \alpha x_n + \beta y_n, \ldots).$$

(a) Define $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by the formula

$$T(x_1, x_2, \ldots, x_n, \ldots) = (0, x_1, x_2, \ldots, x_n, \ldots).$$

($T$ is called the “right shift.”) Show that $T$ is a linear transformation, and that $T$ is ‘one-to-one’ but not ‘onto’.

(b) Define $S : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by the formula

$$S(x_1, x_2, \ldots, x_n, \ldots) = (x_2, x_3, \ldots, x_n, \ldots).$$

($S$ is called the “left shift.”) Show that $S$ is a linear transformation, and that $S$ is ‘onto’ but not ‘one-to-one’.

(c) Show that $S \circ T$ is the identity operator on $\mathbb{R}^\infty$. What is $T \circ S$?

Problem 4: Let $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Recall that if $n$ is positive integer, $A^n$ is the $2 \times 2$ matrix obtained by multiplying $A$ by itself $n$ times.

(a) Prove that $A^2 = 2A - I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

(b) Compute $A^{100}$. 
Problem 5: Find all $2 \times 2$ matrices $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Problem 6: Use Gauss-Jordan elimination to find the most general solution of the following system of equations:

\begin{align*}
2x + 3y - z - 5u &= 9 \\
4x - y + z - u &= 5 \\
5x - 3y + 2z + u &= 3.
\end{align*}

Problem 7: Use Gauss-Jordan elimination to find the inverse of the matrix

$$\begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Problem 8: Let $f_1(x), f_2(x), g_1(x), g_2(x)$ be differentiable functions on an interval $(a, b)$.

(a) If $F(x) = \begin{bmatrix} f_1(x) \\ g_1(x) \end{bmatrix}$, show that $F'(x) = \begin{bmatrix} f_1'(x) \\ g_1'(x) \end{bmatrix}$.

(b) Find a generalization of the result in part (a) for the derivative of the determinant

$$F(x) = \det \begin{bmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{bmatrix}.$$

For the last six problems, we will use the following notation. Let $A = \{a_{i,j}\}$ be an $m \times n$ matrix, and let $A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation with this matrix relative to the standard basis in each space. That is

$$A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix}.$$

Let $A^t = \{a_{k,j}\}$ denote the transpose matrix; that is, $A^t$ is the $n \times m$ matrix obtained from $A$ by interchanging rows and columns. Then $A^t$ defines a linear mapping $A^t : \mathbb{R}^m \to \mathbb{R}^n$; i.e.

$$A^t = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{1,1}y_1 + a_{2,1}y_2 + \cdots + a_{m,1}y_m \\ a_{1,2}y_1 + a_{2,2}y_2 + \cdots + a_{m,2}y_m \\ \vdots \\ a_{1,n}y_1 + a_{2,n}y_2 + \cdots + a_{m,n}y_m \end{bmatrix}.$$

Also, let $\langle \cdot, \cdot \rangle_n$ denote the usual inner product in $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle_m$ denote the usual inner product in $\mathbb{R}^m$. If $V \subset \mathbb{R}^n$ is a subspace of $V$, the orthogonal complement is the set

$$V^\perp = \{x \in \mathbb{R}^n : \langle x, v \rangle_n = 0 \text{ for every } v \in V\}.$$

Similarly, if $W \subset \mathbb{R}^m$ is a subspace of $W$, the orthogonal complement is the set

$$W^\perp = \{y \in \mathbb{R}^m : \langle y, w \rangle_m = 0 \text{ for every } w \in W\}.$$

Problem 9: Show that the range $R(A)$ is spanned by the columns of the matrix $A$.

Problem 10: Show that for every $x \in \mathbb{R}^n$ and every $y \in \mathbb{R}^m$ we have

\[ \langle A^t x, y \rangle_m = \langle x, A^t y \rangle_n. \]
Problem 11: Prove that $V^\perp$ is a subspace of $\mathbb{R}^n$ and that $W^\perp$ is a subspace of $\mathbb{R}^m$.

Problem 12: Prove that
\[ \dim(V) + \dim(V^\perp) = n, \]
\[ \dim(W) + \dim(W^\perp) = m. \]

Problem 13: Prove that $R(A)^\perp = N(A^\top)$ and that $R(A^\top)^\perp = N(A)$.

Problem 14: Prove that the dimension of the range $R(A)$ of the mapping $A$ equals the dimension of the range $R(A^\top)$ of the mapping $A^\top$. In other words, prove that the dimension of the subspace of $\mathbb{R}^m$ spanned by the columns of $A$ (called the column rank of $A$) equals the dimension of the subspace of $\mathbb{R}^n$ spanned by the rows of $A$ (called the row rank of $A$).