Problem 1: Recall that a subset $U \subset \mathbb{R}^n$ is open if and only if for every $x \in U$ there exists $\delta > 0$ so that the open ball $B(x, \delta) = \{y \in \mathbb{R}^n : ||x - y|| < \delta \}$ with center $x$ and radius $\delta$ is entirely contained in $U$. Prove the following properties of open subsets of $\mathbb{R}^n$:

(a) The entire set $\mathbb{R}^n$ is open.
(b) The empty set $\emptyset$ is open.
(c) The union of any collection of open sets is open.
(d) The intersection of a finite collection of open sets is open.

Problem 2: Recall that a subset $E \subset \mathbb{R}^n$ is closed if and only if its complement $E^c = \{y \in \mathbb{R}^n : y \notin E \}$ is open. Prove the following properties of closed subsets of $\mathbb{R}^n$:

(a) The entire set $\mathbb{R}^n$ is closed.
(b) The empty set $\emptyset$ is closed.
(c) The intersection of any collection of closed sets is closed.
(d) The union of a finite collection of closed sets is closed.

Problem 3:

(a) Give an example of an infinite collection of open subsets of $\mathbb{R}$ whose intersection is not open.
(b) Give an example of an infinite collection of closed subsets of $\mathbb{R}$ whose union is not closed.

An infinite sequence of points $\{x_1, x_2, \ldots, x_n, \ldots\} \subset \mathbb{R}^n$ converges to a point $a$ if and only if for every $\epsilon > 0$ there exists a positive integer $N$ so that if $j \geq N$ then $x_j \in B(a, \epsilon)$. In this case we write $\lim_{n \to \infty} x_n = a$.

Problem 4: Prove that a subset $E \subset \mathbb{R}^n$ is closed if and only if the following statement is true: for every infinite sequence of points $\{x_1, x_2, \ldots, x_n, \ldots\}$ which all belong to the set $E$, if $\lim_{n \to \infty} x_n = a$ then $a \in E$.

Problem 5: We saw in class that the function

$$f(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ (xy)(x^2 + y^2)^{-1} & \text{if } (x, y) \neq (0, 0), \end{cases}$$

is not continuous at the point $(0, 0)$. Nevertheless show that the first order partial derivatives of $f$ exist at every point of $\mathbb{R}^2$. Computes these partial derivatives.
Problem 6: Define a function $g : \mathbb{R}^2 \to \mathbb{R}$ by the formulas
\[ g(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ (x^2y)(x^4 + y^2)^{-1} & \text{if } (x, y) \neq (0, 0). \end{cases} \]

(a) Show that $\lim_{x \to 0} g(x, 0) = 0$.
(b) Show that $\lim_{y \to 0} g(0, y) = 0$.
(c) For any real number $\lambda \neq 0$, show that $\lim_{t \to 0} g(t, \lambda t) = 0$.
(d) Show that $g$ is not continuous at the point $(0, 0)$.

Problem 7: Find the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, and $\frac{\partial^2 f}{\partial y^2}$ for each of the following functions of two variables $(x, y)$:

(a) $f(x, y) = x^2y^3 - x \cos(xy)$
(b) $f(x, y) = \log\left[2 + \sin(x + 3y)\right]$ (c) $f(x, y) = \exp\left[\frac{xy}{x^2 + y^2}\right]$ if $(x, y) \neq (0, 0)$
(d) $f(x, y) = \arctan\left[\frac{x + y}{1 - xy}\right]$ if $xy \neq 1$

Problem 8: Define $F : \mathbb{R}^2 \to \mathbb{R}^3$ by the equation
\[ F(x, y) = (xy, x - y, x^2 + y^2). \]

For every $(a, b) \in \mathbb{R}^2$, find the matrix representation of the linear transformation $DF_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^3$ such that
\[ F(a + h, b + k) = F(a, b) + DF_{(a,b)}[(h, k)] + E((a, b), (h, k)) \]

where
\[ \lim_{(h, k) \to (0, 0)} \frac{||E((a, b), (h, k))||}{||(h, k)||} = 0. \]