The first problem deals with the change from Cartesian coordinates \((x, y)\) to polar coordinates \((r, \theta)\) where \(x = r \cos(\theta)\) and \(y = r \sin(\theta)\). We use the following terminology. If \(f(x, y)\) is a function of two variables in Cartesian coordinates, the Laplacian of \(f\) is the function
\[
\Delta f(x, y) = \frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y).
\]
The function \(f\) is said to be harmonic if \(\Delta f(x, y) = 0\).

**Problem 1.** Let \(f(x, y)\) be a function of two variables such that all partial derivatives up to order 2 exist and are continuous. Let \(g(r, \theta) = f(r \cos(\theta), r \sin(\theta))\). [Thus the function \(g\) is the function \(f\) written in polar coordinates.]

(a) Show that
\[
\begin{align*}
\frac{\partial f}{\partial x}(r \cos(\theta), r \sin(\theta)) &= \cos(\theta) \frac{\partial g}{\partial r}(r, \theta) - \frac{\sin(\theta)}{r} \frac{\partial g}{\partial \theta}(r, \theta), \\
\frac{\partial f}{\partial y}(r \cos(\theta), r \sin(\theta)) &= \sin(\theta) \frac{\partial g}{\partial r}(r, \theta) + \frac{\cos(\theta)}{r} \frac{\partial g}{\partial \theta}(r, \theta).
\end{align*}
\]
[This gives the formula for \(\nabla f\), the gradient of \(f\), in polar coordinates.]

(b) Show that
\[
\Delta f(r \cos(\theta), r \sin(\theta)) = \frac{\partial^2 g}{\partial r^2}(r, \theta) + \frac{1}{r} \frac{\partial g}{\partial r}(r, \theta) + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2}(r, \theta).
\]
[This gives the formula for the Laplace operator applied to \(f\) in polar coordinates.]

(c) Show that the function \(f(x, y) = \log(\sqrt{x^2 + y^2})\) is harmonic.
[This can be done directly, but is easier if you use part (b).]

(d) Find all harmonic functions \(f(x, y)\) such that \(g(r, \theta) = f(r \cos(\theta), r \sin(\theta))\) does not depend on the variable \(r\).

(e) If \(n\) is any integer, show that the function \(f(x, y) = (x + iy)^n\) is harmonic.

Let \(V\) be a vector space with an inner product \((\mathbf{v}, \mathbf{w})\). In problems 2, 3, 4, and 5 we deal with linear transformations \(T : V \to V\) such that \((T[\mathbf{v}], T[\mathbf{w}]) = (\mathbf{v}, \mathbf{w})\) for all \(\mathbf{v}, \mathbf{w} \in V\). If \(V\) is a real vector space, a transformation with this property is called orthogonal, while if \(V\) is a complex vector space, such a transformation is called unitary. Recall that the norm on the vector space \(V\) is defined by \(||\mathbf{v}||^2 = (\mathbf{v}, \mathbf{v})\).

**Problem 2.** Let \(V\) be a vector space with an inner and let \(T : V \to V\) be a linear transformation such that \((T[\mathbf{v}], T[\mathbf{w}]) = (\mathbf{v}, \mathbf{w})\) for all \(\mathbf{v}, \mathbf{w} \in V\).

(a) If \(\mathbf{v}, \mathbf{w} \in V\) prove that \(||T[\mathbf{v}] - T[\mathbf{w}]|| = ||\mathbf{v} - \mathbf{w}||\).

(b) If \(V\) is finite dimensional prove that \(T\) is invertible and that \(T^{-1}\) also satisfies \((T^{-1}[\mathbf{v}], T^{-1}[\mathbf{w}]) = (\mathbf{v}, \mathbf{w})\) for all \(\mathbf{v}, \mathbf{w} \in V\).

(c) Prove that if \(\lambda \in \mathbb{C}\) is an eigenvalue of \(T\), then \(|\lambda| = 1\).

(d) Prove that if \(\mathbf{v}, \mathbf{w} \in V\) are eigenvectors of \(T\) corresponding to distinct eigenvalues \(\lambda, \mu\), then \((\mathbf{v}, \mathbf{w}) = 0\).

(e) Prove that if \(\mathbf{v}\) is an eigenvector of \(T\), and if \(\mathbf{w} \in V\) is a vector such that \((\mathbf{v}, \mathbf{w}) = 0\), then \((\mathbf{v}, T[\mathbf{w}]) = 0\).
Problem 3. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation whose matrix, relative to the standard basis of $\mathbb{R}^4$ is given by

$$
T = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) & 0 & 0 \\
\sin(\theta) & \cos(\theta) & 0 & 0 \\
0 & 0 & \cos(\varphi) & -\sin(\varphi) \\
0 & 0 & \sin(\varphi) & \cos(\varphi)
\end{bmatrix} \quad \text{so that} \quad T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \cos(\theta) - x_2 \sin(\theta) \\ x_1 \sin(\theta) + x_2 \cos(\theta) \\ x_3 \cos(\varphi) - x_4 \sin(\varphi) \\ x_3 \sin(\varphi) + x_4 \cos(\varphi) \end{bmatrix}
$$

(a) Show that $T$ is an orthogonal linear transformation.
(b) Show that if neither $\theta$ nor $\varphi$ is an integer multiple of $\pi$ then the linear transformation $T$ does not have any eigenvalues.
(c) Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that the point $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ corresponds to the point $(x_1 + ix_2, x_3 + ix_4) \in \mathbb{C}^2$. Then the mapping $T$ corresponds to the mapping $S : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$
S \begin{bmatrix} x_1 + ix_2 \\ x_3 + ix_4 \end{bmatrix} = \begin{bmatrix} (x_1 \cos(\theta) - x_2 \sin(\theta)) + i(x_1 \sin(\theta) + x_2 \cos(\theta)) \\ (x_3 \cos(\varphi) - x_4 \sin(\varphi)) + i(x_3 \sin(\varphi) + x_4 \cos(\varphi)) \end{bmatrix}
$$

Show that $S$ is a complex linear mapping.
(d) Find the eigenvalues of $S$.

Problem 4. Let $V$ be an $n$-dimensional vector space with an inner product, and let $T : V \to V$ be a linear transformation.

(a) Let $\{e_1, \ldots, e_n\}$ be any basis of $V$. Show that $\langle T[v], T[w] \rangle = \langle v, w \rangle$ for all $v, w \in V$ if and only if $\langle T[e_j], T[e_k] \rangle = \langle e_j, e_k \rangle$ for all $1 \leq j, k \leq n$.

Now suppose that $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $V$, and let $T = \{t_{j,k}\}$ be the $n \times n$ matrix of $T$ relative to this basis. [Recall that this means $T[e_k] = \sum_{j=1}^n t_{j,k} e_j$.]

(b) Prove that $\langle T[v], T[w] \rangle = \langle v, w \rangle$ for all $v, w \in V$ if and only if the columns of the matrix $T$ form an orthonormal basis for $V$.
(c) If $V$ is a real vector space, let $T^t$ be the $n \times n$ matrix obtained from the matrix $T$ by interchanging rows and columns. Prove that $T$ is an orthogonal transformation if and only if $T^t T = I$, the identity operator.
(d) If $V$ is a complex vector space, let $T^\dagger$ be the $n \times n$ matrix obtained from the matrix $T$ by interchanging rows and columns, and then taking the complex conjugate of each entry. Prove that $T$ is a unitary transformation if and only if $T^\dagger T = I$, the identity operator.
(e) If $A$ is any $n \times n$ matrix, prove that the rows of $A$ are orthonormal vectors if and only if the columns of $A$ are orthonormal vectors.

Problem 5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an orthogonal linear transformation, and suppose that the product of the eigenvalues of $T$ is positive. Prove that there is at least one non-zero vector $v \in \mathbb{R}^3$ so that $T[v] = v$. 