MATH 722: COMPLEX ANALYSIS
SPRING SEMESTER 2012

1. References

There are many possible references for an introductory course in complex analysis. The following books, given with call numbers, should be on reserve in the Kleene Math Library:

1) *Complex Analysis* by Lars V. Ahlfors, Third Edition, McGraw-Hill, (QA331 A45 1979)

   This is a standard text, written by a winner of the Fields medal.

2) *Analytic Function Theory* by Einar Hille, Volume I and II, Chelsea Publishing Company, (QA331 H54 1973)

   The second volume especially has discussions of a number of interesting topics.

3) (a) *Elements of the Theory of Functions* by Konrad Knopp, Dover, (QA331 K6814)

   (b) *Theory of Functions*, Parts I and II, by Konrad Knopp, Dover, (QA331 K713)

   (c) *Problem Book in the Theory of Functions*, Volumes I and II, Dover, (QA331 K663)

   These books give a concise (and inexpensive) introduction to complex analysis. The *Problem Book* is a good source of exercises, with answers.

4) *Complex Analysis in One Variables* by Raghavan Narasimhan, Birkhauser, (QA331 N27 1985)

   This book presents the subject from a modern point of view. It includes a differential-geometric proof of Picard's 'big theorem'.

5) *Analytic Functions* by Stanislaw Saks and Antoni Zygmund, Polich Scientific Publishers, (QA331 S315 1971)

   This is an elegant, classical introduction to the subject.

6) *Complex Analysis* by Elias M. Stein and Rami Shakarchi, Princeton University Press, (QA331 S74 2003)

   This is part of a four-volume introduction to analysis for 'Princeton undergraduates'. Much of the focus is on applications to number theory.

7) *The Theory of Functions* by E.C. Titchmarsh, Oxford University Press, (QA331 T5)

   This is another elegant classic.

2. Holomorphic Functions and the Cauchy-Riemann Equations

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If $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$, recall that f is *differentiable* at $x \in (a, b)$ if and only if the limit

$$f'(x) = \lim_{h\to\mathbf{0}} \frac{f(x+h) - f(x)}{h}$$

exists. The objective of this course is to study the analogous concept for functions f where both the domain and the range are allowed to consist of complex numbers.

2.1. The field $\mathbb{C}$ of complex numbers.

Introduce a product on the vector space $\mathbb{R}^2$ by setting

$$(a, b) \cdot (x, y) = (ax - by, ay + bx).$$

Note that this product is commutative:

$$(a, b) \cdot (x, y) = (x, y) \cdot (a, b).$$

We identify a real number x with the pair $(x, 0)$ in $\mathbb{R}^2$ so that (with an abuse of notation), we have $0 = (0, 0)$, and $1 = (1, 0)$. Note that

$$1 \cdot (x, y) = (1, 0) \cdot (x, y) = (x, y).$$

We also introduce the notation $(0, 1) = i$. Then any pair (x, y) can be written

$$(x, y) = x(1, 0) + y(0, 1) = x + yi = x + iy.$$

When using this product, we call the elements of $\mathbb{R}^2$ *complex numbers*, and denote the set of complex numbers by $\mathbb{C}$. We use the letters $z, w, \ldots$ for complex numbers. Note that using the symbol "i", the product of complex numbers is given by

$$(a + bi) \cdot (x + yi) = (ax - by) + (ay + bx)i.$$

Proposition 2.1. *With this definition of multiplication, the set of complex numbers is a field, and the set of real numbers is a subfield. In particular, we have*

(1) $i^2 = i \cdot i = -1$,

(2) *If* $(x + yi) \neq 0$, *then* $(x + yi)^{-1} = \dfrac{x}{x^2 + y^2} + \dfrac{-y}{x^2 + y^2} i$.

Proof. This is a routine exercise. In particular, one needs to check that if $z, w, u \in \mathbb{C}$, then
(a) $(z \cdot w) \cdot u = z \cdot (w \cdot u)$;
(b) $z \cdot w = w \cdot z$;
(c) $z \cdot (w + u) = z \cdot w + z \cdot u$;
(d) $(x + yi) \cdot \left(\dfrac{x}{x^2 + y^2} + \dfrac{-y}{x^2 + y^2} i\right) = 1$.

The statement about the sub-field of real numbers follows since $(a, 0) \cdot (x, 0) = (ax, 0)$. □

From now on we shall write zw instead of $z \cdot w$. If $z = x + iy \in \mathbb{C}$, we write

$\Re\mathrm{e}[z] = x$	the real part of z,
$\Im\mathrm{m}[z] = y$	the imaginary part of z,
$\bar{z} = (x - iy)$	the complex conjugate of z,
$\lvert z\rvert = \sqrt{x^2 + y^2} \geq 0$	the absolute value of z.

Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Also, if $w = a + ib \in \mathbb{C}$, then

$$|z - w| = \sqrt{(x - a)^2 + (y - b)^2},$$

which is just the Euclidean distance between the points z and w thought of as point in $\mathbb{R}^2$. Thus $\mathbb{C}$ is naturally a metric space with distance $d(z,w) = |z-w|$. We will freely use topological notions like open set, closed set, interior, boundary, *etc.* applied to subset of $\mathbb{C}$. Finally we have:

$$\begin{aligned}|zw| &= |(x+iy)(a+ib)| = |(xa-yb)+i(xb+ya)| = \sqrt{(xa-yb)^2+(xb+ya)^2} \\ &= \sqrt{x^2a^2-2xyab+y^2b^2+x^2b^2+2xyab+y^2b^2} \\ &= \sqrt{x^2a^2+y^2b^2+x^2b^2+y^2b^2} = \sqrt{(x^2+y^2)(a^2+b^2)} = \sqrt{(x^2+y^2)}\sqrt{(a^2+b^2)} \\ &= |z||w|.\end{aligned}$$

2.2. **Complex differentiability and the Cauchy-Riemann Equations.**

Let $\Omega \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$. We say that the function f is *complex differentiable* at a point $z = x+iy \in \Omega$ if and only if the limit

$$f'(z) = \lim_{h\to 0,\ h\in\mathbb{C}} \frac{f(z+h)-f(z)}{h} \tag{2.1}$$

exists. We can write $f(x+iy) = u(x+iy)+iv(x+iy)$ where $u, v : \Omega \to \mathbb{R}$. Suppose that f is complex differentiable at z. Then letting $h \to 0$ with $h \in \mathbb{R}$, we have

$$f'(z) = \lim_{h\to 0, h\in\mathbb{R}} \frac{u(x+h,y)-u(x,y)}{h} + i \lim_{h\to 0, h\in\mathbb{R}} \frac{v(x+h,y)-v(x,y)}{h}$$

exists, which means that the ordinary partial derivatives

$$\begin{aligned}\frac{\partial u}{\partial x}(x,y) &= \lim_{h\to 0, h\in\mathbb{R}} \frac{u(x+h,y)-u(x,y)}{h} \\ \frac{\partial v}{\partial x}(x,y) &= \lim_{h\to 0, h\in\mathbb{R}} \frac{v(x+h,y)-v(x,y)}{h}\end{aligned}$$

both exist, and

$$f'(z) = \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z).$$

However, in the definition of complex differentiability, we can also let $h = ik$ be purely imaginary. Then

$$\begin{aligned}f'(z) &= \lim_{k\to 0, k\in\mathbb{R}} \frac{u(x,y+k)-u(x,y)}{ik} + i \lim_{k\to 0, k\in\mathbb{R}} \frac{v(x,y+k)-v(x,y)}{ik} \\ &= \lim_{k\to 0, k\in\mathbb{R}} \frac{v(x,y+k)-v(x,y)}{k} - i \lim_{k\to 0, k\in\mathbb{R}} \frac{u(x,y+k)-u(x,y)}{k}\end{aligned}$$

exists, which means that the ordinary partial derivatives

$$\begin{aligned}\frac{\partial u}{\partial y}(x,y) &= \lim_{k\to 0, k\in\mathbb{R}} \frac{u(x,y+k)-u(x,y)}{k} \\ \frac{\partial v}{\partial y}(x,y) &= \lim_{k\to 0, k\in\mathbb{R}} \frac{v(x,y+k)-v(x,y)}{k}\end{aligned}$$

both exist, and

$$f'(z) = \frac{\partial v}{\partial y}(z) - i\frac{\partial u}{\partial y}(z).$$

Thus we have established:

Proposition 2.2. *Suppose that $f = u + iv$ is complex differentiable at a point z. Then the first partial derivatives of the real-valued functions u and v also exist at z and we have*

$$\begin{aligned}\frac{\partial u}{\partial x}(z) &= +\frac{\partial v}{\partial y}(z)\\ \frac{\partial v}{\partial x}(z) &= -\frac{\partial u}{\partial y}(z).\end{aligned} \tag{2.2}$$

The equations (2.2) are called the *Cauchy-Riemann* equations. We introduce the following notation:

$$\begin{aligned}\frac{\partial}{\partial z} &= \frac{1}{2}\left[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right],\\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right].\end{aligned} \tag{2.3}$$

If $f = u + iv$ is a complex valued function, and if the first partial derivatives of u and v exist, then we have

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\right],\\ \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right].\end{aligned}$$

Proposition 2.3. *With the notation introduced in (2.3), a function $f : \Omega \to \mathbb{C}$ is complex differentiable at a point $z \in \Omega$ if and only if $\frac{\partial f}{\partial \bar{z}}(z) = 0$, in which case $f'(z) = \frac{\partial f}{\partial z}(z)$.*

Definition 2.4. *If $\Omega \subset \mathbb{C}$ is a non-empty open set, a function $f : \Omega \to \mathbb{C}$ is holomorphic on Ω if and only if f is complex differentiable at every point $z \in \Omega$.*

Remarks:

(a) Note that complex differentiability of f at a point z implies that f is continuous at z. Thus holomorphic functions are continuous.

(b) No hypothesis, such as continuity, is made on the regularity of the derivative $f'(z)$. In fact, we will see that such an hypothesis is non necessary. If a function f is holomorphic, then $f'(z)$ is automatically continuous, and in fact also a holomorphic function, so that holomorphic functions are automatically infinitely differentiable.

(c) In fact, one does not even need to assume that f has partial derivatives. If $f : \Omega \to \mathbb{C}$ is continuous (or even just integrable in the sense of Riemann or Lebesgue) and if for every function $\varphi : \Omega \to \mathbb{C}$ which is infinitely differentiable and has compact support in Ω we have

$$\iint_\Omega f(z)\frac{\partial \varphi}{\partial \bar{z}}(z)\,dx\,dy = 0,$$

then f is holomorphic. (If we could integrate by parts, this would say that $\frac{\partial f}{\partial \bar{z}}(z) = 0$, so we are requiring the Cauchy-Riemann equations to hold in a weak sense).

3. Goursat's Theorem, the Cauchy integral formula, and its consequences

3.1. **Integration on curves.**

We recall the definition of integrals of continuous functions over curves. Let $\gamma : [a, b] \to \Omega$ be a continuously differentiable function. (This means that if we write $\gamma(t) = x(t) + iy(t)$, then the real-valued functions x, $y : [a, b] \to \mathbb{R}$ are continuously differentiable. If $f : \Omega \to \mathbb{C}$ is continuous, we define

$$\int_\gamma f(z)\,dz := \int_a^b f\big(x(t) + iy(t)\big)\,\big(x'(t) + iy'(t)\big)\,dt. \tag{3.1}$$

If $\alpha : [c, d] \to [a, b]$ is continuously differentiable with $\alpha(c) = a$, $\alpha(d) = b$, and $\alpha'(s) > 0$ for all $s \in [c, d]$, then we can make the change of variables $t = \alpha(s)$ in the integral on the right hand side of (3.1), and see that

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_c^d f\big(x(\alpha(s)) + iy(\alpha(s))\big)\,\big(x'(\alpha(s)) + iy'(\alpha(s))\big)\,\alpha'(s)\,ds \\ &= \int_c^d f\big(X(s) + iY(s)\big)\,\big(X'(s) + iY'(s)\big)\,ds \\ &= \int_{\tilde\gamma} f(z)\,dz\end{aligned}$$

where $X(s) = x(\alpha(s))$, $Y(s) = y(\alpha(s))$, and $\tilde\gamma(s) = X(s) + iY(s)$. This shows that the value of the integral $\int_\gamma f(z)\,dz$ is (essentially) independent of the parameterization, and really only depends on the image Γ of the mapping γ. 1/25/12

The length of a curve γ is

$$L(\gamma) = \int_\gamma \sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_\gamma |z'(t)|\,dt,$$

and this too is independent of the parameterization, so we write it as $L(\Gamma)$. We then clearly have the following estimate.

Proposition 3.1. *Let $\Omega \subset \mathbb{C}$ be open, let $f : \Omega \to \mathbb{C}$ be continuous, and let $\Gamma \subset \Omega$ be a piecewise smooth curve. Then*

$$\left|\int_\Gamma f(z)\,dz\right| \le \sup_{z\in\Gamma} |f(z)|\; L(\Gamma).$$

Lemma 3.2. *Let $\Omega \subset \mathbb{C}$ be open, let $f : \Omega \to \mathbb{C}$, and suppose there is a holomorphic function $F : \Omega \to \mathbb{C}$ so that $F'(z) = f(z)$ for all $z \in \Omega$. (If this is the case, we say that f has a primitive on Ω.) Let $\gamma : [a, b] \to \Omega$ be a piecewise continuously differentiable curve with $\gamma(a) = z_0 \in \Omega$ and $\gamma(b) = z_1 \in \Omega$. Then*

$$\int_\gamma f(z)\,dz = F(z_1) - F(z_0).$$

In particular, the value of the integral depends only on the endpoints, and not on the curve.

Proof. Suppose that $\gamma(t) = x(t)+iy(t)$ for $a \le t \le b$. Then using the chain rule and the fundamental theorem of calculus, we have

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_a^b f\big(x(t) + iy(t)\big)\,\big(x'(t) + iy'(t)\big)\,dt \\ &= \int_a^b F'\big(x(t) + iy(t)\big)\,\big(x'(t) + iy'(t)\big)\,dt \\ &= \int_a^b \frac{d}{dt}\left(F(x(t) + iy(t)\right)\,dt \\ &= F(z_1) - F(z_0).\end{aligned}$$

$\square$

Corollary 3.3. *If $f : \Omega \to \mathbb{C}$ is continuous and has a primitive, and if $\gamma : [a,b] \to \mathbb{C}$ is a piecewise continuously differentiable curve with $\gamma(a) = \gamma(b)$, then $\int_\gamma f(z)\,dz = 0$.*

Remark 3.4. *For each integer $n \geq 0$ the function $z \to z^n$ has a primitive on $\mathbb{C}$ given by $z \to (n+1)^{-1} z^{n+1}$. Similarly, for each integer $n \leq -2$, the function $z \to z^n$ has a primitive on $\mathbb{C} \setminus \{0\}$ given by $z \to (n+1)^{-1} z^{n+1}$.*

3.2. Goursat's Theorem.

Theorem 3.5 (Goursat)**.** *Let $\Omega \subset \mathbb{C}$ be open, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $R = [a,b] \times [c,d] \subset \Omega$ be a closed rectangle. Then*

$$\int_{\partial R} f(z)\,dz = 0,$$

where ∂R is the boundary of R consisting of four line segments, taken in the counterclockwise sense.

Before giving the proof, we should observe that if we assume that f is a continuously differentiable function, we could apply Green's theorem as follows. Write $f(z) = u(z) + iv(z)$ with u, v real-valued. Then

$$\begin{aligned} \int_{\partial R} f(z)\,dz &= \int_{\partial R} (u + iv)(dx + i\,dy) \\ &= \int_{\partial R} \big(u + iv\big)\,dx + \big(iu - v\big)\,dy \\ &= \iint_R \frac{\partial\big(iu - v\big)}{\partial x} - \frac{\partial\big(u + iv\big)}{\partial y}\,dx\,dy \\ &= \iint_R -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,dx\,dy \\ &= \iint_R 0\,dx\,dy = 0 \end{aligned}$$

where we have used Green's Theore in the third equality and the Cauchy-Riemann equations in the next to last equality. However, if we only assume that f is holomorphic, then *a priori* we know nothing about the regularity of the various partial derivatives, and we cannot (at least easily) apply Green's theorem. Thus the point of Goursat's theorem is that we establish the result with **no** regularity assumption on $f'(z)$.

Proof of Theorem 3.5. If $R \subset \Omega$ is any closed rectangle, we can subdivide R into four equal closed sub-rectangles R^1, R^2, R^3, R^4 so that

$$\int_{\partial R} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial R^j} f(z)\,dz,$$

and hence

$$\left| \int_{\partial R} f(z)\,dz \right| \leq \sum_{j=1}^{4} \left| \int_{\partial R^j} f(z)\,dz \right|.$$

It follows that for at least one of these sub-rectangles R^j we have

$$\frac{1}{4} \left| \int_{\partial R} f(z)\,dz \right| \leq \left| \int_{\partial R^j} f(z)\,dz \right|.$$

Let R_1 be this sub-rectangle. We have

$$\left|\int_{\partial R} f(z)\,dz\right| \leq 4\left|\int_{\partial R_1} f(z)\,dz\right|.$$

Repeating this procedure, we find a sequence of rectangles $R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ so that for every $n \geq 1$,

$$\left|\int_{\partial R} f(z)\,dz\right| \leq 4^n\left|\int_{\partial R_n} f(z)\,dz\right|. \tag{3.2}$$

Clearly

$$L(\partial R_n) = 2^{-n}\,L(\partial R),$$

and if $d(R) = \sup_{z,w\in R}|z-w|$ is the diameter of the original rectangle, then $d(R_n) = 2^{-n}d(R)$. Since $\{R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots\}$ is a nested sequence of compact sets, it follows that the intersection is non-empty. Moreover, since the diameters of these sets goes to zero, the intersection consists of a single point $z_0 \in \Omega$.

Now the function f is complex differentiable at z_0, so

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists. For $z \in \Omega$ set

$$\psi(z) = \begin{cases} f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0, \\ \\ 0 & \text{if } z = z_0, \end{cases}$$

so that

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + (z-z_0)\psi(z).$$

Then complex differentiability of f implies that the function ψ is continuous on Ω.

We have

$$\begin{aligned}\int_{\partial R_n} f(z), dz &= \int_{\partial R_n}\big[f(z_0)+f'(z_0)(z-z_0)\big]\,dz + \int_{\partial R_n}(z-z_0)\psi(z)\,dz \\ &= \int_{\partial R_n}(z-z_0)\psi(z)\,dz\end{aligned}$$

by Corollary 3.3, since the function $z \to f(z_0) + f'(z_0)(z-z_0)$ has a primitive $z \to f(z_0)(z-z_0) + \frac{1}{2}f'(z_0)(z-z_0)^2$. It follows that

$$\begin{aligned}4^n\left|\int_{\partial R_n} f(z)\,dz\right| &= 4^n\left|\int_{\partial R_n}(z-z_0)\psi(z)\,dz\right| \\ &\leq 4^n\,L(\partial R_n)\sup_{z\in\partial R_n}\Big(|z-z_0||\psi(z)|\Big) \\ &\leq 4^n\,L(\partial R_n)\,d(R_n)\sup_{z\in\partial R_n}|\psi(z)| \\ &= L(R)\,d(R)\sup_{z\in\partial R_n}|\psi(z)| \\ &\leq L(R)\,d(R)\sup_{|z-z_0|\leq 2^{-n}d(R)}|\psi(z)|\end{aligned}$$

Since $\psi(z_0) = 0$ and ψ is continuous at z_0, it follows that

$$\lim_{n\to\infty} 4^n\left|\int_{\partial R_n} f(z)\,dz\right| = 0,$$

and hence $\int_{\partial R} f(z)\,dz = 0$ by the inequality in equation (3.2). This completes the proof. □

3.3. **Local existence of primitives.** 1/27/12

Theorem 3.6. *Let $D = D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$ be the disk of radius R centered at z_0. Let $f : D \to \mathbb{C}$ be a holomorphic function. Then there exists a holomorphic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.*

Proof. For any point $w \in D$ let γ_w be the curve joining z_0 to w consisting of two straight line segments: the first in the horizontal direction, and the second in the vertical direction. Then put

$$F(w) = \int_{\gamma_w} f(z)\, dz.$$

We want to show that $F'(w) = f(w)$. Choose $h \in \mathbb{C}$ with $|h|$ sufficiently small that $w + h \in D$. Then an application of Goursat's theorem shows that

$$F(w + h) - F(w) = \int_{\Gamma(w,h)} f(z)\, dz$$

where $\Gamma(w, h)$ is the curve joining w to $w + h$ consisting of two straight line segments: the first in the horizontal direction and the second in the vertical direction. But

$$\begin{aligned} \int_{\Gamma(w,h)} f(z)\, dz &= \int_{\Gamma(w,h)} f(w)\, dz + \int_{\Gamma(w,h)} (f(z) - f(w))\, dz \\ &= f(w)\, h + \int_{\Gamma(w,h)} (f(z) - f(w))\, dz \end{aligned}$$

by Lemma 3.2, since the constant function $z \to f(w)$ has a primitive $f(w)\, z$, and so

$$\int_{\Gamma(w,h)} f(w)\, dz = f(w)(w + h) - f(w)w = f(w)\, h.$$

Thus for $h \neq 0$ but sufficiently small we have

$$\begin{aligned} \left| \frac{F(w + h) - F(w)}{h} - f(w) \right| &= \frac{1}{|h|} \left| \int_{\Gamma(w,h)} (f(z) - f(w))\, dz \right| \\ &\leq \frac{1}{|h|}\, L\big(\Gamma(w, h)\big) \sup_{z \in \Gamma(w,h)} |f(z) - f(w)|. \end{aligned}$$

But $L\big(\Gamma(w, h)\big) \leq \sqrt{2}|h|$, and so

$$\left| \frac{F(w + h) - F(w)}{h} - f(w) \right| \leq \sqrt{2} \sup_{z \in \Gamma(w,h)} |f(z) - f(w)| \longrightarrow 0$$

as $h \to 0$ since the function f is continuous at w. This completes the proof. □

Corollary 3.7. *Let f be holomorphic in an open disk $D \subset \mathbb{C}$, and let $f : D \to \mathbb{C}$ be a holomorphic function. If γ is any piecewise smooth curve in D with the same initial and final point (i.e. a closed curve), then*

$$\int_\gamma f(z)\, dz = 0.$$

3.4. **Cauchy's Theorem in a simply connected region.**

We introduce the following slightly non-standard definition. Let $\Omega \subset \mathbb{C}$ be a connected open set. We say that Ω is simply connected if and only if it has the following property. Let $\gamma; [a, n] \subset \mathbb{R} \to \Omega$ be any piecewise smooth curve which is closed so that $\gamma(a) = \gamma(b) = z_0 \in \Omega$, Then there exists a continuous function $\Gamma : [a, b] \times [0, 1] \to \Omega$ such that:

(a) $\Gamma(0, t) = \gamma(t)$ for $a \leq t \leq b$;
(b) For each $0 \leq s \leq 1$ the mapping $t \to \Gamma(s, t)$ is piecewise smooth;
(c) $\Gamma(1, t) = z_0$ for $a \leq t \leq b$;
(d) $\Gamma(s, a) = \Gamma(s, b) = z_0$ for $0 \leq s \leq 1$;
(e) We can partition the rectagle $R = [a, b] \times [0, 1]$ into a collection $R_{j,k}$ of rectangle with disjoint interiors so that
  (i) for each j, k the image $\Gamma(R_{j,k)})$ is contained in an open disk $D_{j,k}$ contained in Ω;
  (ii) for each j, k, the restriction of Γ to $\partial R_{j,k}$ is piecewise smooth.

Theorem 3.8 (Cauchy). *Let $\Omega \subset \mathbb{C}$ be a connected and simply connected open set. Let $f : \Omega \to \mathbb{C}$ be holomorphic. If γ is any piecewise smooth closed curve in Ω, then*

$$\int_\gamma f(z)\, dz = 0.$$

Sketch of a proof. Because Γ is identically z_0 on three sides of R, it follows that

$$\int_{\partial R} f(z)\, dz = \int_\gamma f(z)\, dz.$$

Next, we have

$$\int_{\partial R} f(z)\, dz = \sum_{j,k} \int_{\partial R_{j,k}} f(z)\, dz,$$

and each of the terms in the last sum equals zero by Corollary 3.7 and property (e) of the mapping Γ. □

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Corollary 3.9. *Let $D = \{z \in \mathbb{C} : |z - z_0| < R\}$, and let $w_0 \in D$. Let R_1 and ϵ be positive numbers such that $|z_0 - w_0| < R_1 < R$ and $\epsilon < R - |z_0 - w_0$. Let C_1 and C_2 denote the two circles in D parameterized (for example) by*

$$C_1 = \{z_0 + R_1\big(\cos(t) + i \sin(t)\big) \,|\, 0 \leq t \leq 2\pi\},$$
$$C_2 = \{w_0 + \epsilon\big(\cos(t) + i \sin(t)\big) \,|\, 0 \leq t \leq 2\pi\}.$$

Let f be any function which is holomorphic in $D \setminus \{w_0\}$. Then

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

The proof is by drawing a 'toy picture' of the keyhole contour, as in the text. □

3.5. **The Cauchy integral formula and applications.**

Theorem 3.10 (Cauchy Integral Formula). *Let $D = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < R\}$ be an open disk, and let $f : D \to \mathbb{C}$ be a holomorphic function. Let $z \in D$ and let C be the circle parameterized by*

$$C = \{\zeta_0 + r(\cos(t) + i \sin(t)) \,:\, 0 \leq t \leq 2\pi\}$$

with $0 < r < R$. Then if $|z - \zeta_0| < r$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof. We apply Corollary 3.9 to the function $\zeta \to \frac{f(\zeta)}{\zeta - z}$, which is holomorphic in the punctured disk $D \setminus \{z\}$. Then if

$$C_\epsilon = \{z + \epsilon\big(\cos(t) + i\sin(t)\big) \,|\, 0 \le t \le 2\pi\},$$

we have

$$\begin{aligned}
\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \int_{C_\epsilon} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\
&= \int_0^{2\pi} \frac{f(z + \epsilon(\cos(t) + i\sin(t))}{\epsilon\big(\cos(t) + i\sin(t)\big)}\,\epsilon\big(-\sin(t) + i\cos(t)\big)\,dt \\
&= i\int_0^{2\pi} f(z + \epsilon(\cos(t) + i\sin(t))\,dt \\
&= i\int_0^{2\pi} f(z)\,dt + i\int_0^{2\pi} [f(z + \epsilon(\cos(t) + i\sin(t)) - f(z)]\,dt \\
&= 2\pi i f(z) + i\int_0^{2\pi} [f(z + \epsilon(\cos(t) + i\sin(t)) - f(z)]\,dt.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z)\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} [f(z + \epsilon(\cos(t) + i\sin(t)) - f(z)]\,dt\right| \\
&\le \sup_{0 \le t \le 2\pi} \big|f(z + \epsilon(\cos(t) + i\sin(t)) - f(z)\big| \\
&\longrightarrow 0
\end{aligned}$$

as $\epsilon \to 0$ since f is continuous at z. □

Corollary 3.11. *Let $D = \{\zeta \in \mathbb{C} \,:\, |\zeta - \zeta_0| < R\}$ be an open disk, and let $f : D \to \mathbb{C}$ be a holomorphic function. Then f is infinitely differentiable. Let $z \in D$ and let C be the circle parameterized by*

$$C = \{\zeta_0 + r(\cos(t) + i\sin(t)) \,:\, 0 \le t \le 2\pi\}$$

with $0 < r < R$. Then if $|z - \zeta_0| < r$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta.$$

Proof. We that for each non-negative integer n, the function f can be differentiated n-times in the complex sense, with the resulting formula for the n^{th}-derivative. We establish this by induction on n. The case $n = 0$ follows from the fact that holomorphic functions are continuous and the Cauchy integral formula (Theorem 3.10). Assume by induction that f can be differentiated n-times that that $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$. Then if $0 \ne h \in \mathbb{C}$ with $|z + h - \zeta_0| < r$, we have

$$\frac{f^{(n)}(z + h) - f^{(n)}(z)}{h} = \frac{n!}{2\pi i}\int_C f(\zeta)\frac{1}{h}\left[\frac{1}{(\zeta - z - h)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}}\right]d\zeta.$$

But using the usual arguments for taking derivatives,

$$\lim_{h \to 0} \frac{1}{h}\left[\frac{1}{(\zeta - z - h)^{n+1}} - \frac{1}{(\zeta - z)^{n+1}}\right] = (n + 1)\frac{1}{(\zeta - z)^{n+2}},$$

with the convergence uniform for $\zeta \in C$. It follows that

$$\lim_{h \to 0} \frac{f^{(n)}(z + h) - f^{(n)}(z)}{h} = \frac{(n + 1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+2}}\,d\zeta,$$

and this completes the induction step and the proof. □

Corollary 3.12 (Cauchy Estimates)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Let $z \in \Omega$, and suppose that closed disk $\overline{D}(z, R) = \{w \in \mathbb{C} : |z - w| \leq R\} \subset \Omega$. Let $C(s, R) = \{z + R(\cos(t) + i\sin(t)) : 0 \leq t \leq 2\pi\}$ be the boundary circle of $D(z, R)$. Then for every non-negative integer n,*

$$|f^{(n)}(z)| \leq n! R^{-n} \sup_{w \in C(z,R)} |f(w)|.$$

Definition 3.13. *An entire function is a function which is defined and holomorphic on the whole complex plane $\mathbb{C}$.*

Corollary 3.14 (Liouville's Theorem)**.** *Suppose that f is a bounded entire function; i.e. $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and there is a constant $M > 0$ so that $f(z)| \leq M$ for all $z \in \mathbb{C}$. Then the function f is constant.*

Proof. If f is entire and bounded, it follows from the Cauchy estimates (Corollary 3.12) that $f'(z) \equiv 0$ for all $z \in \mathbb{C}$. But if γ_z is the straight line joining the origin 0 to the point z, it follows that

$$f(z) - f(0) = \int_{\gamma_z} f'(w)\, dw = 0$$

and hence $f(z) = f(0)$ is a constant. □

Corollary 3.15 (Fundamental Theorem of Algebra)**.** *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients of degree $n \geq 1$ (so that $a_n \neq 0$). Then the equation $P(z) = 0$ has a complex root.*

Proof. We argue by contradiction. If $P(z) \neq 0$ for all $z \in \mathbb{C}$, then the function $f(z) = P(z)^{-1}$ is an entire function. For $|z| \geq 1$ we have

$$\begin{aligned}
|P(z)| &= \Big|\sum_{j=0}^{n} a_j z^j\Big| \\
&\geq \big|a_n z^n\big| - \sum_{j=0}^{n-1} |a_j||z|^j \\
&= |a_n||z^n| \left[1 - \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}|z|^{j-n}\right] \\
&\geq |a_n||z^n| \left[1 - |z|^{-1} \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}\right].
\end{aligned}$$

Thus if we let $R = \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}$ and we take $|z| \geq \max\{1, 2R\}$, we have

$$|P(z)| \geq |a_n||z|^n \big[1 - \frac{1}{2}\big] \geq \frac{1}{2}|a_n| R^n.$$

Thus the function f is bounded by $2|a_n|^{-1}R^{-n}$ for $|z| \geq \max\{1, 2R\}$, and since it is continuous, it is also bounded on the compact set where $|z| \leq \max\{1, 2R\}$. Thus f is a bounded entire function, and hence is constant by Liouville's Theorem (Corollary 3.14). This would imply that the polynomial P is also constant, but this is a contradiction since the degree of P is at least 1. Thus the equation $P(z) = 0$ must have a root. □

3.6. **Power Series Expansions.** 2/1/12

A function $f : \Omega \to \mathbb{C}$ is holomorphic if it is complex differentiable at each point, but no assumption is made about the continuity of the derivative $f' : \Omega \to \mathbb{C}$. We have seen that (surprisingly!) it follows that holomorphic function are infinitely differentiable in the complex sense. Thought of as a function defined on an open subset of $\mathbb{R}^2$, this means that the real and imaginary parts of a holomorphic function are infinitely differentiable in the sense that every partial derivative exists and is continuous.

In fact, holomorphic functions have an even stronger regularity property; they are given locally by convergent power series, and conversely, convergent power series in $(z - a)$ are holomorphic functions. This is the content of the following two results.

Corollary 3.16 (Power series expansion)**.** *Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $a \in \Omega$ and if the open disk $D(a, R) = \{z \in \mathbb{C} : |z - a| < R\} \subset \Omega$, then the function f can be written as a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

which converges on every compact subset of $D(a, R)$. The coefficients of this power series expansion are given by

$$a_n = \frac{1}{n!} f^{(n)}(a).$$

Proof. Let $0 < R_2 < R_1 < R$, let $C = \{\zeta \in \mathbb{C} : |\zeta - a| = R_1\}$ be the circle of radius R_1 centered at a, and let $|z - a| \leq R_2$ so that $z \in \Omega$. By Cauchy's formula (Theorem 3.10) we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_C f(\zeta) \left[(\zeta - a) - (z - a)\right]^{-1} d\zeta \\ &= \frac{1}{2\pi i} \int_C f(\zeta)((\zeta - a) \left[1 - \frac{z-a}{\zeta - a}\right]^{-1} d\zeta. \end{aligned}$$

Now

$$\left| \frac{z-a}{\zeta - a} \right| \leq \frac{R_2}{R_1} < 1,$$

and we can write

$$\left[1 - \frac{z-a}{\zeta - a}\right]^{-1} = \sum_{n=0}^{\infty} \left(\frac{z-a}{\zeta-a}\right)^n$$

where this geometric series converges absolutely and uniformly for $|z - a| \leq R_1$ and $\zeta \in C$. The uniform convergence allows us to interchange the order of summation and integration, and we have

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (z-a)^n \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\, (z-a)^n. \end{aligned}$$

This completes the proof. □

In order to establish the converse, we need the following results:

Proposition 3.17. *Let C be the circle centered at a point $a \in \mathbb{C}$ of radius $R > 0$, and let $\varphi : C \to \mathbb{C}$ be a continuous function. For $z \in D(a, R) = \{w \in \mathbb{C} : |z - a| < R\}$ (i.e. for z in the open disk centered at a of radius R), set*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta.$$

Then f is holomorphic in $D(a, R)$, and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{\varphi(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Proof. This is really just a repetition of the argument in the proof of Corollary 3.11. □

Remark: In general, the function f given in Proposition 3.17 will not extend continuously to the closure of the disk $D(a, R)$ and equal φ on the boundary. For example, let $\varphi(\zeta) = \bar{\zeta}$. Then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C \frac{\bar{\zeta}}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{\bar{\zeta}}{\zeta} \left[1 - \frac{z}{\zeta}\right]^{-1} d\zeta \\ &= \frac{1}{2\pi i} \int_C \frac{\bar{\zeta}}{\zeta} \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n d\zeta \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} z^n \int_C \bar{\zeta}\, \zeta^{-n-1}\, d\zeta. \end{aligned}$$

But on C, $\bar{\zeta} = \zeta^{-1}$, and so

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \zeta^{-n-2}\, d\zeta = 0$$

since for $n \geq 2$ the holomorphic function ζ^{-n} has a primitive in the punctured plane, and thus each integral equals zero.

Lemma 3.18. *Let $\{f_n\}$ be a sequence of holomorphic functions defined on an open disk $D(a, R)$ and suppose that this sequence converges uniformly on compact subsets of $D(a, R)$ to a function $F : D(a, R) \to \mathbb{C}$. Then F is also a holomorphic function.*

Proof. Choose $0 < r < R$ and let C_r be the circle centered at a of radius r. Then for $|z - a| < r$, Cauchy's integral formula gives

$$f_n(z) = \frac{1}{2\pi i} \int_{C_r} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta.$$

Then

$$\begin{aligned} F(z) &= \lim_{n\to\infty} f_n(z) \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_{C_r} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{C_r} \lim_{n\to\infty} \frac{f_n(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{F(\zeta)}{\zeta - z}\, d\zeta, \end{aligned}$$

where the interchange of limits and integral is justified by the uniform convergence of the functions $\{f_n\}$ to F on the circle C_r. But now it follows from tlhis representation and Proposition 3.17 that

F is holomorphic in the disk centered at a of radius r. Since $r < R$ was arbitrary, this completes the proof. □

We can now establish the converse to Corollary 3.16. Consider a power series

$$\sum_{n=0}^{\infty} a_n(z-a)^n.$$

Then there are three possibilities:

(a) The series converges if and only if $z = a$.

(b) There is a number $R \in (0, \infty)$ so that the series converges for $|z-a| < R$ and diverges for $|z-a| > R$. In this case, the series converges absolutely and uniformly on compact subsets of the open disk $D(a, R)$. Since the partial sums are polynomials and hence holomorphic, it follows that the power series converges to a holomorphic function f on the open disk $D(a, R)$.

(c) The series converges for all $z \in \mathbb{C}$. In this case, the series converges absolutely and uniformly on compact subsets of $\mathbb{C}$. Since the partial sums are polynomials and hence holomorphic, it follows that the power series converges to a holomorphic function on the whole complex plane $\mathbb{C}$.

The number R in (b) above is given by

$$R^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

and this formula also gives $R = 0$ in case (a) and $R = \infty$ in case (c), R is called the *radius of convergence* of the power series.

2/3/2012

4. Local behavior of holomorphic functions

4.1. Isolated zeros.

Lemma 4.1. *Let $\Omega \subset \mathbb{C}$ be a connected open set, let f be holomorphic on Ω, and let $a \in \Omega$. If f is not identically zero on Ω and if $f(a) = 0$, then there exists $\epsilon > 0$ so that $\{z \in \mathbb{C} : |z-a| < \epsilon\} \subset \Omega$, and $f(z) \neq 0$ for all $0 < |z-a| < \epsilon$.*

Proof. The is an open disk $D(a, R) \subset \Omega$, and on this disk we can represent f as a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n.$$

Note that $a_0 = f(a) = 0$.

Suppose first that $a_n \neq 0$ for some $n \geq 1$. Let $N = \min\{n : a_n \neq 0\}$. Then in the disk $D(a, R)$ we can write

$$\begin{aligned} f(z) &= a_N(z-a)^N\Big[1 + \sum_{k=N+1}^{\infty} \frac{a_k}{a_N}(z-a)^{k-N}\Big] \\ &= a_N(z-a)^N\Big[1 + g(z)\Big] \end{aligned}$$

where

$$g(z) = \frac{1}{a_N}\sum_{n=1}^{\infty} a_{N+n}(z-a)^n.$$

It is not hard to check that the power series defining g also converges in the disk $D(a, R)$, and hence defines a holomorphic function there. Moreover, $g(a) = 0$ since the power series has no constant term. But since g is holomorphic, it is continuous, and hence there exists $0 < \epsilon < R$ so that if $|z-a| < \epsilon$ then $|g(z)| = |g(z) - g(a)| \leq \frac{1}{2}$. It follows that if $0 < |z-a| < \epsilon$, then

$$\begin{aligned} |f(z)| &= |a_N||z-a|^N \Big| 1 + g(z) \Big| \\ &\geq |a_N||z-a|^N \Big[1 - |g(z)| \Big] \\ &\geq \frac{1}{2}|a_N||z-a|^N > 0, \end{aligned}$$

so f has no other zeros in this punctured disk. This establishes the result if some coefficient in the Taylor expansion is non-zero.

Now suppose that $a_n = 0$ for all $n \geq 0$. It follows that $f(z) \equiv 0$ in the disk $D(a, R)$. Let U denote the set of points $w \in \Omega$ such that f is identically zero in some neighborhood of w. Then U is open by definition, and is non-empty by our current hypothesis. Note that if $w \in U$, then $f^{(n)}(w) = 0$ for all $n \geq 0$. Let $w_0 \in \Omega$ be a point in the closure of U. Then there exists a sequence $\{w_n\} \subset U$ with $\lim_{n\to\infty} w_n = w_0$. But since each derivative $f^{(n)}$ is a continuous function on Ω, it follows that

$$f^{(n)}(w_0) = \lim_{k\to\infty} f^{(n)}(w_k) = 0.$$

It follows that the Taylor expansion of f about the point $w_0 \in \Omega$ is identically zero, and hence $f(z) \equiv 0$ in a neighborhood of w_0. We have thus shown that U is both open and closed and non-empty. Since Ω is connected, it follows that $U = \Omega$, in which case f is identically zero in Ω. This contradicts the hypothesis of the lemma, and so it cannot happen that $a_n = 0$ for all $n \geq 0$. This completes the proof. □

Corollary 4.2. *Let $\Omega \subset \mathbb{C}$ be a connected open set, and let f and g be holomorphic functions on Ω. Suppose there is a point $w_0 \in \Omega$, and a sequence $\{w_k\} \subset \Omega$ such that $\lim_{k\to\infty} w_k = w_0$, and $f(w_k) = g(w_k)$ for all k. Then $f(z) \equiv g(z)$ for all $z \in \Omega$.*

Proof. Apply Lemma 4.1 to the holomorphic function $f - g$. □

4.2. **Isolated singularities.** 2/6/12

We now study the behavior of a function near a point a which is holomorphic in some punctured disk $D(a, R)^* = \{z \in \mathbb{C} : 0 < |z-a| < R\}$. In this case we say that f has an *isolated singularity* at a.

Definition 4.3. *If f has an isolated singularity at a point a, then a is a removable singularity for f if there is a holomorphic function F in some disk $D(a, R)$ such that $f(z) = F(z)$ for all $z \in D(a, R)^*$.*

Theorem 4.4 (Riemann)**.** *Let f be holomorphic in a punctured disk $D(a, R)^*$, and suppose that there is a constant $M > 0$ so that $|f(z)| \leq M$ for all $z \in D(a, R)^*$. Then f has a removable singularity at a.*

Proof. Let $0 < r < R$, and let $C_{a,r}$ denote the circle centered at a of radius r. Let $C_{a,\epsilon}$ be the circle centered at a of radius $0 < \epsilon << r$. Choose z so that $\epsilon < |z-a| < r$, choose $\delta > 0$ so that the circle $C_{z,\delta}$ centered at z of radius δ is contained inside the circle $C_{a,r}$ but outside the circle $C_{a,\epsilon}$. An application of the 'two keyhole contour' and Cauchy's theorem shows that

$$\frac{1}{2\pi i}\int_{C_{a,r}}\frac{f(\zeta)}{\zeta-z}\,d\zeta = \frac{1}{2\pi i}\int_{C_{a,\epsilon}}\frac{f(\zeta)}{\zeta-z}\,d\zeta + \frac{1}{2\pi i}\int_{C_{z,\epsilon}}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$
$$= \frac{1}{2\pi i}\int_{C_{a,\epsilon}}\frac{f(\zeta)}{\zeta-z}\,d\zeta + f(z)$$

But for $\zeta \in C_{a,\epsilon}$, $|\zeta - z| \geq |z-a| - \epsilon$. It follows that

$$\Big|\frac{1}{2\pi i}\int_{C_{a,\epsilon}}\frac{f(\zeta)}{\zeta-z}\,d\zeta\Big| \leq \frac{1}{2\pi}\,\frac{M}{|z-a|-\epsilon}\,2\pi\epsilon \to 0$$

as $\epsilon \to 0$. (Here we use the fact that the length of the circle $C_{a,\epsilon}$ is $2\pi\epsilon$). It follows that for $0 < |z-a| < ra$ we have

$$f(z) = \frac{1}{2\pi i}\int_{C_{a,r}}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

and this shows that we can extend f to a holomorphic function in the whole disk $D(a,R)$. □

Remark: A slightly more involved argument with keyhole contours gives the following result, which was discussed in class:

Let f be a holomorphic function in the annulus

$$A(a;R_1,R_2) = \{z \in \mathbb{C} :< R_1 < |z-a| < R_2\}.$$

If $z \in A(a;R_1,R_2)$ and if $R_1 < r_1 < |z-a| < r_2 < R_2$, let C_{r_1} and C_{r_2} be the circles centered at a with radii r_1 and r_2 and the usual counterclockwise orientation. Then

$$f9z) = \frac{1}{2\pi i}\int_{C_{r_2}}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{2\pi i}\int_{C_{r_1}}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Definition 4.5. *If f has an isolated singularity at a point a, then f has a pole at a if and only if a is not a removable singularity for f, but f^{-1} has a removable singularity at a.*

Theorem 4.6. *If f has a pole at $a \in \Omega$ then there exists a disk $D(a,\epsilon) \subset \Omega$, a unique positive integer N (called the order of the pole), and a unique holomorphic function h defined on $D(a,\epsilon)$ so that $f(z) = (z-a)^{-N}\,h(z)$ for $z \in D(a,\epsilon)^*$, and $h(z) \neq 0$ for $z \in D(a,\epsilon)$.*

Proof. By hypothesis, f^{-1} has a removable singularity at a. This means that there is a disk $D(a,R)$ so that $f(z) \neq 0$ when $z \in D(a,R)^*$, and the function $g(z) = f(z)^{-1}$ has a holomorphic extension to all of $D(a,R)$. We can expand g in a convergent Taylor expansion in $D(a,R)$, say $g(z) = \sum_{n=0} a_n(z-a)^n$. We make two observations:

(a) $a_0 = 0$, since otherwise $|g(z)|$ is bounded away from zero near a, hence $f(z) = g(z)^{-1}$ is bounded near a, and hence f has a removable singularity at a by Theorem 4.4.

(b) There exists a positive integer n so that $a_n \neq 0$, for otherwise $g(z) \equiv 0$ in $D(a,R)$ in which case $f(z) = g(z)^{-1}$ could not be holomorphic in $D(a,R)^*$.

If we let $N = \min\{n : a_n \neq 0\} \geq 1$, this means we can write

$$g(z) = (z-a)^N \sum_{k=0}^{\infty} a_{N+k}(z-a)^k = (z-a)^N\, g_1(z)$$

where g_1 is holomorphic in $D(a,R)$ and $g_1(a) = a_N \neq 0$. It follows that there exists $0 < \epsilon \leq R$ so that $g_1(z) \neq 0$ for $z \in D(a,\epsilon)$. If we let $h(z) = g_1(z)^{-1}$ in this disk, then for $z \in D(a,\epsilon)^*$ we have $f(z)^{-1} = g(z) = (z-a)^N\, h(z)^{-1}$, and this completes the proof. □

Corollary 4.7. *If f has a pole of order N at a point a, then there exists $\epsilon > 0$ so that for $0 < |z-a| < \epsilon$ we can write*

$$f(z) = \frac{a_{-N}}{(z-a)^N} + \frac{a_{-N+1}}{(z-a)^{N-1}} + \cdots + \frac{a_{-1}}{(z-a)} + g(z) \tag{4.1}$$

where g is holomorphic on the whole disk $D(a,\epsilon)$.

Proof. Expand the function h in Theorem 4.6 in a Taylor series. □

Definition 4.8. *If f has a pole at a point a, the residue of f at the point a is the coefficient a_{-1} in the expansion (4.1) given in Corollary 4.7. We write $a_{-1} = res_a(f)$.*

Lemma 4.9. *Suppose that f is holomorphic in $D(a,R)^*$ and has a pole at a. If C is a circle centered at a of radius r with $0 < r < R$, then*

$$\int_C f(\zeta)\, d\zeta = 2\pi i\, res_a(f).$$

Proof. Suppose that f has a pole of order N at a. It follows from Corolllary 4.7 that

$$\int_C f(z)\, dz = \int_C \frac{a_{-N}}{(z-a)^N}\, dz + \int_C \frac{a_{-N+1}}{(z-a)^{N-1}}\, dz + \cdots + \int_C \frac{a_{-1}}{(z-a)}\, dz + \int_C g(z)\, dz.$$

Now $\int_C g(z)\, dz = 0$ since g has a primitive in $D(a,R)$, and for $k \geq 2$, $\int_C (z-a)^{-k}\, dz = 0$ since $(z-a)^{-k}$ has a primitive $(-k+1)^{-1}(z-a)^{-k+1}$. On the other hand, a direct calculation shows that

$$\int_C \frac{a_{-1}}{(z-a)}\, dz = 2\pi i\, a_{-1}.$$

This completes the proof. □

2/8/12

Definition 4.10. *Suppose that f has an isolated singularity at a point a. Then f has an essential singularity at a if and only if a is neither a removable singularity nor a pole.*

Theorem 4.11 (Casorati-Weierstrass). *Let f be holomorphic in the punctured disk $D(a,R)^*$ and suppose f has an essential singularity at the point a. Then for every $w \in \mathbb{C}$ and every $\epsilon > 0$ there exists a point $z \in D(a,R)^*$ so that $|f(z) - w| < \epsilon$.*

Proof. We argue by contradiction. If the theorem is not true, there exists $w \in \mathbb{C}$ and $\epsilon > 0$ so that $|f(z) - w| \geq \epsilon$ for all $z \in D(a,R)$. Define g on $D(a,R)^*$ by setting $g(z) = (f(z) - w)^{-1}$. Then g also has an isolated singularity at a, and we see that $|g(z)| \leq \epsilon^{-1}$ on the punctured disk. It follows from Riemann's Theorem (Theorem 4.4) that a is a removable singularity for g, and $|g(a)| \leq \epsilon^{-1}$. But for $z \in D(a,R)^*$ we have $f(z) = g(z)^{-1} + w$. Now if $g(a) \neq 0$ it follows that f has a removable singularity at a, while if $g(a) = 0$ it follows that f has a pole at a. This completes the proof. □

5. Evaluation of Integrals

We can use Cauchy's theorem and the Residue theorem (Lemma 4.9) to evaluate certain kinds of definitie integrals.

5.1. **Fourier transforms and Gaussians.**

If the function $f \in L^1(\mathbb{R}^n)$ (or if you are unfamiliar with Lebesgue integration we can assume that f is Riemann integrable and the improper integral $\int_{\mathbb{R}^n} |f(x)|\,dx$ exists), then we define the Fourier transform of f to be the function given for $\xi \in \mathbb{R}^n$ by

$$\mathcal{F}[f](\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{2\pi i \langle x,\xi\rangle} f(x)\,dx. \tag{5.1}$$

Here $\langle x, \xi\rangle = \sum_{j=1}^n x_j \xi_j$. We show that the Gaussian function $\mathcal{G}(x) = \exp[-\pi x^2]$ is its own Fourier transform. Recall from calculus the formula

$$\int_{\mathbb{R}} e^{-\pi t^2}\,dt = 1,$$

which can be derived by using polar coordinates in $\mathbb{R}^2$.

Proposition 5.1. *For all $\xi \in \mathbb{R}$,*

$$\widehat{\mathcal{G}}(\xi) = \int_{-\infty}^{+\infty} e^{2\pi i x\xi} e^{-\pi x^2}\,dx = e^{-\pi\xi^2}.$$

Proof. Using the usual properties of the exponential function, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{2\pi i x\xi} e^{-\pi x^2}\,dx &= \int_{-\infty}^{+\infty} e^{-\pi[x^2 - 2ix\xi]}\,dx \\ &= e^{-\pi\xi^2}\int_{-\infty}^{+\infty} e^{-\pi[x^2-2ix\xi+(i\xi)^2]}\,dx \\ &= e^{-\pi\xi^2}\int_{-\infty}^{+\infty} e^{-\pi[x-i\xi]^2}\,dx.\end{aligned}$$

If in the last integral we could make the change of variables to a new variable of integration t where $x = t + i\xi$, we would then have

$$\int_{-\infty}^{+\infty} e^{2\pi i x\xi} e^{-\pi x^2}\,dx = e^{-\pi\xi^2}\int_{-\infty}^{+\infty} e^{-\pi t^2}\,dt = e^{-\pi\xi^2}.$$

The question is how to justify this change of variables which involves complex numbers.

Assume $\xi > 0$. We apply Cauchy's theorem to the entire holomorphic function $f(z) = e^{-z^2}$. Let Γ_R denote the closed curve which is a rectangle with top on the real axis going from $-R$ to $+R$, then goes vertically downward to $R - i\xi$, then goes horizontally from $R - i\xi$ to $-R - i\xi$, and finally goes vertically upward back to $-R \in \mathbb{R}$. Since f is holomorphic, we have

$$\begin{aligned}0 &= \int_{\Gamma_R} e^{-\pi z^2}\,dz \\ &= \int_{-R}^{+R} e^{-\pi x^2}\,dx + \int_0^{\xi} e^{-\pi(R-is)^2}\,i\,ds + \int_R^{-R} e^{-\pi(x-i\xi)^2}\,dx + \int_{\xi}^{0} e^{-\pi(-R-is)^2}\,ds \\ &= I(R) + II(R) + III(R) + IV(R).\end{aligned}$$

Now

$$III(R) = e^{\pi\xi^2}\int_{+R}^{-R} e^{2\pi i x\xi} e^{-\pi x^2}\,dx = -e^{\pi\xi^2}\int_{-R}^{+R} e^{2\pi i x\xi} e^{-\pi x^2}\,dx$$

and this converges to $e^{\pi\xi^2}\widehat{\mathcal{G}}(\xi)$ as $R \to \infty$. On the other hand

$$\begin{aligned}\big|II(R)\big| &= \Big|\int_0^\xi e^{-\pi(R-is)^2}\, i\, ds\Big| \le \int_0^\xi \big|e^{-\pi(R-is)^2}\big|\, ds\\ &= \int_0^\xi e^{-\pi R^2} e^{\pi s^2}\, ds \le e^{-\pi R^2}\,\xi\, e^{\pi\xi^2} \to 0\end{aligned}$$

as $R \to \infty$, and a similar estimate shows that $|IV(R)| \to 0$ as $R \to \infty$. It follows that

$$\int_{-R}^{+R} e^{2\pi i x\xi} e^{-\pi x^2}\, dx = e^{-\pi\xi^2}\int_{-R}^{+R} e^{-\pi x^2}\, dx + II(R) + IV(R)$$

and letting $R \to \infty$ completes the proof. □

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5.2. **Integrals of rational functions.**

A rational function is the quotient of two polynomials. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, and suppose

(i) the degree of Q is greater than or equal to the degree of P plus 2;
(ii) the polynomial Q has no real roots.

We consider the improper integral

$$I(R) = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\, dx,$$

which makes sense and converges by hypotheses (i) and (ii). Note that Q has degree(Q) complex roots when counted with multiplicity. Choose $R >> 1$ so large that all the roots of Q are contained in the disk $D(0; R)$, and consider the contour Γ_R consisting of the real interval from $-R$ to $+R$, together with the semicircle $S_R = \{R\cos(t) + iR\sin(t) \,:\, 0 \le t \le \pi\}$. Let $w_1, \ldots, w_q$ be the roots of Q which lie in the upper half plane, and let C_j be a circle centered at w_j of radius ϵ, where $\epsilon > 0$ is chosen so small that all these circles lie in the upper half plane and are mutually disjoint. Then it follows from Cauchy's theorem that

$$\int_{-R}^{+R} \frac{P(x)}{Q(x)}\, dx + \int_{\Gamma_R} \frac{P(z)}{Q(z)}\, dz = \sum_{j=1}^{q} \int_{C_j} \frac{P(z)}{Q(z)}\, dz.$$

Now because of hypothesis (i), it follows that there is a constant C so that for $R >> 1$ large enough we have

$$\left|\frac{P(z)}{Q(z)}\right| \le C\,\frac{C}{R^{\deg(Q)-\deg(P)}} \le \frac{1}{R^2}.$$

Since the length of the curve Γ_R is πR, it follows that

$$\Big|\int_{\Gamma_R} \frac{P(z)}{Q(z)}\, dz\Big| \le \frac{C}{R^2}\,(\pi R) = \frac{C\pi}{R} \to 0$$

as $R \to \infty$. Thus

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\, dx = \sum_{j=1}^{q} \int_{C_j} \frac{P(z)}{Q(z)}\, dz$$

Now in principle, we can calculate the right hand side. If w_j is a root of Q, then we can write $Q(z) = (z - w_j)^{d_j} Q_j(z)$ where Q_j is a polynomial of degree $\deg(Q) - d_j$ which is not equal to zero at w_j. Then

$$\int_{C_j} \frac{P(z)}{Q(z)}\, dz = \int_{C_j} \frac{P(z)}{Q_j(z)}\, \frac{dz}{(z - w_j)^{d_j}}$$

The function $\frac{P(z)}{Q_j(z)}$ is holomorphic in a neighborhood of the closed disk $\overline{D(w_j;\epsilon)}$, and so by the corollary to the Cauchy integral theorem,

$$\int_{C_j} \frac{P(z)}{Q_j(z)} \frac{dz}{(z-w_j)^{d_j}} = \frac{2\pi i}{(d_j)!} \frac{d^{d_j-1}}{dz^{d_j-1}} \Big(\frac{P(z)}{Q_j(z)}\Big)_{z=w_j}.$$

Example: Compute $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^2(x^2+4)}\,dx$. Note that the zeros of the denominator occur at $\pm i$, each of order two, and at $\pm 2i$.

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5.3. **Integrals of the form $\int_{\mathbb{R}} R(x)e^{i\alpha x}\,dx$.**

Assume that $R(x)$ is a rational function $\frac{P(x)}{Q(x)}$ where the polynomial Q again has no real roots. Assume this time that the degree of Q is at least equal to the degree of P plus 1. Let $0 \neq \alpha \in \mathbb{R}$. In this case, the improper integral

$$I(\alpha, R) = \int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{i\alpha x}\,dx$$

need not converge absolutely. Nevertheless, we show the existence of and evaluate

$$\lim_{R_1,R_2\to+\infty} \int_{-R_1}^{+R_2} \frac{P(x)}{Q(x)} e^{i\alpha x}\,dx.$$

Suppose without loss of generality that $\alpha > 0$. Consider the contour consisting of the boundary ∂R of the rectangle R with four sides parameterized by

$$\begin{aligned}
\gamma_1 &= \{t \,:\, -R_1 \le t \le R_2\},\\
\gamma_2 &= \{R_2 + is \,:\, 0 \le s \le A\},\\
\gamma_3 &= \{t + iA \,:\, R_2 \ge t \ge -R_1\},\\
\gamma_4 &= \{-R_1 + is \,:\, A \ge s \ge 0\}.
\end{aligned}$$

Choose R_1, R_2, A sufficiently large that all the zeros of the polynomial Q which are in the upper half plane lie inside R. If we call these points $\{w_1, \ldots, w_q\}$, then the residue formula shows that

$$\sum_{j=1}^{r} \int_{\gamma_j} \frac{P(z)}{Q(z)} e^{i\alpha z}\,dz = 2\pi i \sum_{k=1}^{q} \mathrm{Res}_{z=w_k}\Big[\frac{P(z)}{Q(z)} e^{i\alpha z}\Big].$$

Now

$$\int_{\gamma_1} \frac{P(z)}{Q(z)} e^{i\alpha z}\,dz = \int_{-R_1}^{+R_2} \frac{P(x)}{Q(x)} e^{i\alpha x}\,dx,$$

and we want the limit of this expression as R_1, $R_2 \to +\infty$.

Next,

$$\begin{aligned}
\int_{\gamma_3} \frac{P(z)}{Q(z)} e^{i\alpha z}\,dz &= \int_{+R_2}^{-R_1} \frac{P(t+iA)}{Q(t+iA)} e^{i\alpha(t+iA)}\,dt\\
&= -e^{-\alpha A} \int_{-R_1}^{+R_2} \frac{P(t+iA)}{Q(t+iA)} e^{i\alpha t}\,dt
\end{aligned}$$

Now for A large, the fact that the degree of Q is at least one greater than the degree of P implies that

$$\Big|\frac{P(t+iA)}{Q(t+iA)}\Big| \le \frac{C}{A}.$$

Hence

$$\Big|\int_{\gamma_3}\frac{P(z)}{Q(z)}e^{i\alpha z}\,dz\Big|\leq\frac{C}{A}\,e^{-\alpha A}(R_1+R_2).$$

Next we estimate the integrals over the vertical sides of the rectangle. For R_2 sufficiently large we have

$$\left|\frac{P(R_2+is)}{Q(R_2+is)}\right|\leq\frac{C}{R_2},$$

and so we have

$$\begin{aligned}\Big|\int_{\gamma_2}\frac{P(z)}{Q(z)}e^{i\alpha z}\,dz\Big| &= \Big|\int_0^A\frac{P(R_2+is)}{Q(R_2+is)}e^{i\alpha(R_2+is)}\,ds\Big|\leq\int_0^A\Big|\frac{P(R_2+is)}{Q(R_2+is)}\Big|e^{-\alpha s}\,ds\\ &\leq\frac{C}{R_2}\int_0^A e^{-\alpha s}\,ds=\frac{C}{R_2}\frac{1}{-\alpha}\big[e^{-\alpha A}-1\big]\leq\frac{C}{\alpha R_2}.\end{aligned}$$

A similar argument shows that

$$\Big|\int_{\gamma_4}\frac{P(z)}{Q(z)}e^{i\alpha z}\,dz\Big|\leq\frac{C}{\alpha R_1}.$$

It now follows that

$$\left|\int_{-R_1}^{+R_2}\frac{P(x)}{Q(x)}e^{i\alpha x}\,dx-2\pi i\sum_{k=1}^{q}\mathrm{Res}_{z=w_k}\Big[\frac{P(z)}{Q(z)}e^{i\alpha z}\Big]\right|\leq\frac{C}{A}\,e^{-\alpha A}(R_1+R_2)+\frac{C}{\alpha R_1}+\frac{C}{\alpha R_2}.$$

The left hand side is independent of $A>0$, and so if we first let $A\to+\infty$ we get

$$\left|\int_{-R_1}^{+R_2}\frac{P(x)}{Q(x)}e^{i\alpha x}\,dx-2\pi i\sum_{k=1}^{q}\mathrm{Res}_{z=w_k}\Big[\frac{P(z)}{Q(z)}e^{i\alpha z}\Big]\right|\leq\frac{C}{\alpha R_1}+\frac{C}{\alpha R_2}.$$

Now we can let R_1 and R_2 go independently to $+\infty$, and we see that

$$\lim_{R_1,R_2\to+\infty}\int_{-R_1}^{+R_2}\frac{P(x)}{Q(x)}e^{i\alpha x}\,dx=2\pi i\sum_{k=1}^{q}\mathrm{Res}_{z=w_k}\Big[\frac{P(z)}{Q(z)}e^{i\alpha z}\Big].$$

Example: Compute $\displaystyle\int_0^\infty\frac{\cos(x)}{1+x^2}\,dx=\frac{1}{2}\int_{-\infty}^\infty\frac{\cos(x)}{1+x^2}\,dx=\frac{1}{2}\Re\mathrm{e}\left[\int_{-\infty}^\infty\frac{e^{ix}}{1+x^2}\,dx\right].$

5.4. **Integrals involving rational functions with real roots.**

Consider the improper integral

$$I=\int_0^\infty\frac{\sin(x)}{x}\,dx=\frac{1}{2}\int_{\mathbb{R}}\frac{\sin(x)}{x}\,dx.$$

Formally we can write this as

$$I=\frac{1}{2}\Im\mathrm{m}\left[\int_{-\infty}^{+\infty}\frac{e^{ix}}{x}\,dx\right]$$

and the integral inside the brackets has the form considered in the last subsection, except for the fact that the functin $Q(x)=x$ has a real root at $x=0$. We deal with this difficulty by considering a rectangular contour Γ with base the interval $[-R,+R]\subset\mathbb{R}$ and height $A>0$, which we then modify to a contour Γ_ϵ by replacing the interval $[-\epsilon,+\epsilon]\subset\mathbb{R}$ by a semi-circle $C_\epsilon=\{\epsilon\,e^{i\theta}\,:\,\pi\leq\theta\leq2\pi\}$. The function $f(z)=\frac{e^{iz}}{z}$ has one pole at $z=0$ inside this new contour. Thus by the residue theorem,

$$\int_{\Gamma_\epsilon}\frac{e^{iz}}{z}\,dz=2\pi i\mathrm{Res}_{z=0}\left(\frac{e^{iz}}{z}\right)=2\pi i.$$

On the left hand side, the contribution from the semi-circle C_ϵ is

$$\int_\pi^{2\pi} \frac{e^{i(\epsilon e^{i\theta})}}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\, d\theta = i\int_\pi^{2\pi} \exp[i\epsilon\cos(\theta) - \epsilon\sin(\theta)]\, d\theta \longrightarrow \pi i$$

as $\epsilon \to 0$. The contribution over the horizontal side at height A goes to zero as $A \to +\infty$ and the contributions over the vertical sides also go to zero as $R \to +\infty$ as in the earlier section. Thus we are left with

$$\pi i + \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left[\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx + \int_\epsilon^R \frac{e^{ix}}{x}\, dx\right] = 2\pi i$$

Now

$$\int_{-R}^{-\epsilon} \frac{\cos(x)}{x}\, dx + \int_\epsilon^R \frac{\cos(x)}{x}\, dx = 0$$

since $x^{-1}\cos(x)$ is an odd function, while

$$\int_{-R}^{-\epsilon} \frac{\sin(x)}{x}\, dx + \int_\epsilon^R \frac{\sin(x)}{x}\, dx = 2\int_\epsilon^R \frac{\sin(x)}{x}\, dx$$

since $x^{-1}\sin(x)$ is even. It follows that

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_\epsilon^R \frac{\sin(x)}{x}\, dx = \frac{\pi}{2}.$$

5.5. Integrals of the form $\int_0^{2\pi} R(\sin(\theta), \cos(\theta))\, d\theta$.

We can use these methods to evaluate integrals of the form

$$I = \int_0^{2\pi} \frac{P(\sin(\theta), \cos(\theta))}{Q(\sin(\theta), \cos(\theta))}\, d\theta$$

where $P(u, v)$ and $Q(u, v)$ are polynomials in two real variables, and $Q(\sin(\theta), \cos(\theta)) \neq 0$ for $0 \le \theta \le 2\pi$. Consider the function g of a complex variable given by

$$g(z) = \frac{P\left(\frac{1}{2i}\left[z - \frac{1}{z}\right], \frac{1}{2}\left[z + \frac{1}{z}\right]\right)}{Q\left(\frac{1}{2i}\left[z - \frac{1}{z}\right], \frac{1}{2}\left[z + \frac{1}{z}\right]\right)}$$

Since P and Q are polynomials, we can multiply top and bottom by a high enough power of z and observe that $g(z)$ is actually a rational function of z. Let $C = \{e^{i\theta} : 0 \le \theta \le 2\pi\}$ be the boundary of the unit disk. Then for $z = e^{i\theta}$ we have

$$\frac{1}{2i}\left[z - \frac{1}{z}\right] = \sin(\theta); \qquad \frac{1}{2}\left[z + \frac{1}{z}\right] = \cos(\theta).$$

Hence

$$\int_C g(z)\,\frac{dz}{z} = i\int_0^{2\pi} \frac{P(\sin(\theta), \cos(\theta))}{Q(\sin(\theta), \cos(\theta))}\, d\theta$$

and so

$$\int_0^{2\pi} \frac{P(\sin(\theta), \cos(\theta))}{Q(\sin(\theta), \cos(\theta))}\, d\theta = \frac{1}{i}\int_C g(z)\,\frac{dz}{z} = 2\pi \sum_{a\in\mathbb{D}} \mathrm{Res}_{z=a}\left(\frac{g(z)}{z}\right).$$

Example: Compute $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + \cos(\theta)}$ for $a > 1$.

6. Holomorphic logarithms

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6.1. Definition of logarithms.

The exponential function $\exp(z) = e^z$ is an entire function, so if $\Omega \subset \mathbb{C}$ is a connected open set, and if $f : \Omega \to \mathbb{C}$ is holomorphic, the function $g(z) = \exp\big(f(z)\big)$ is also holomorphic on Ω. We now ask: *given a holomorphic function g defined on a connected open set $\Omega \subset \mathbb{C}$, does there exist a holomorphic function f on Ω such that $g(z) = e^{f(z)}$?* We would call such a function a *logarithm* of g.

We derive some necessary conditions for the existence of a logarithm.

(1) If g has a logarithm on the domain Ω, then $g(z) = e^{f(z)} \neq 0$ for all $z \in \Omega$ since the exponential function is never zero.
(2) If $g(z) = e^{f(z)}$, then $g'(z) = e^{f(z)}\, f'(z) = g(z)\, f'(z)$, and so $\frac{g'}{g} = f'$. Thus if g has a holomorphic logarithm on the domain Ω, the function $h(z) = \frac{g'(z)}{g(z)}$ (which is holomorphic on Ω if $g = 0$ has no roots on Ω) has a primitive on Ω.

These two necessary conditions turn out to be sufficient. We have

Lemma 6.1. *Let $\Omega \subset \mathbb{C}$ be a connected open set, let $g : \Omega \to \mathbb{C}$ be a holomorphic function on Ω, suppose that $g(z) \neq 0$ for all $z \in \Omega$, and suppose there is a holomorphic function $f : \Omega \to \mathbb{C}$ which is a primitive for $\frac{g'}{g}$. Then there is a constant $\alpha \in \mathbb{C}$ so that $g(z) = \exp(f(z) + \alpha)$ for all $z \in \Omega$.*

Proof. Consider the function $h(z) = g(z)\, e^{-f(z)}$. Then using the product rule and the formula for the derivative of the exponential function, we have

$$h'(z) = g'(z)\, e^{-f(z)} - g(z)e^{-f(z)}\, f'(z) = 0.$$

It follows that the function $h(z) \equiv c$ where $c \in \mathbb{C}$ is a non-zero constant. Choose $\alpha \in \mathbb{C}$ so that $c = e^{\alpha}$. Then

$$g(z) = h(z)\, e^{f(z)} = e^{\alpha}\, e^{f(z)} = e^{f(z)+\alpha},$$

and so $f(z) + \alpha$ is a logarithm for g. This completes the proof. □

Remark: Note that if f_1 and f_2 are two holomorphic logarithms of g on a connected domain $\Omega \subset \mathbb{C}$, then

$$f_1'(z) = \frac{g'(z)}{g(z)} = f_2'(z)$$

for all $z \in \Omega$, and hence $(f_1 - f_2)'(z) \equiv 0$. Since Ω is connected, it follows that there is a constant C so that $f_1(z) - f_2(z) = C$, and hence

$$e^C = e^{f_1(z)-f_2(z)} = \frac{e^{f_1(z)}}{e^{f_2(z)}} = \frac{g(z)}{g(z)} \equiv 1,$$

and hence $C = 2\pi i n$ for some integer n. Thus if $\log(g)(z)$ is one choice of a holomorphic logarithm of g, the most general holomorphic logarithm has the form $g(z) + 2\pi i n$.

Corollary 6.2. *Suppose that $\Omega \subset \mathbb{C}$ is a connected and simply connected domain. Let g be holomorphic on Ω and suppose that $g(z) \neq 0$ for all $z \in \Omega$. Then g has a holomorphic logarithm on Ω.*

Proof. Fix a point $z_0 \in \Omega$, and for any $z \in \Omega$ let $\gamma(z)$ be a piecewise-smooth curve joining z_0 to z. Define

$$f(z) = \int_{\gamma(z)} \frac{g'(\zeta)}{g(\zeta)}\, d\zeta.$$

This number is independent of the choice of the curve $\gamma(z)$ since the domain is simply connected, and we have seen that in this case, f is indeed a primitive for $\frac{g'}{g}$. We now appeal to Lemma 6.1. This completes the proof. □

Corollary 6.3. *Let $\Omega \subset \mathbb{C}$ be a connected and simply connected open set not containing the point 0. Then there is a holomorphic function $\log(z)$ defined on Ω so that $\exp\big[\log(z)\big] \equiv z$ for all $z \in \Omega$. If $\log_1$ and $\log_2$ are two such holomorphic logarithms of z, there is an integer n so that $\log_1(z) - \log_2(z) \equiv 2\pi i n$ for all $z \in \Omega$.*

When defining $\log(z)$, it is often useful to consider the complex plane $\mathbb{C}$ slit along a line going from the origin to infinity. For example, we can consider the complex plane slit along the negative real axis

$$\mathbb{C}^\dagger = \{z = x + iy \in \mathbb{C} \,:\, y = 0 \Longrightarrow x > 0\}.$$

Then if $z \in \mathbb{C}^\dagger$, there exists unique real numbers $r > 0$ and $\theta \in (-\pi, +\pi)$ so that $z = r\cos(\theta) + ir\sin(\theta) = re^{i\theta}$. The set $\mathbb{C}^\dagger$ is simply connected, and we can define a branch of $\log(z)$ on this set by setting

$$\log(z) = \log(r) + i\theta,$$

where $\log(r)$ is the usual function defined for $r > 0$ by

$$\log(r) = \int_1^r \frac{1}{t}\,dt.$$

With this definition of $\log(z)$, for any $\alpha \in \mathbb{R}$ we can also define a branch of z^α on the set $\mathbb{C}^\dagger$ by setting

$$z = r\,e^{i\theta} \quad\Longrightarrow\quad z^\alpha = e^{\alpha\log(z)} = e^{\alpha\log(r) + i\alpha\theta} = r^\alpha e^{i\alpha\theta}, \quad -\pi < \theta < +\pi.$$

6.2. Example: Evaluate $\displaystyle\int_0^\infty \frac{t^{\frac{1}{5}}}{1+t^2}\,dt$.

Consider the branch of the holomorphic function $z^{\frac{1}{5}}$ defined in the plane slit along the negative imaginary axis; *i.e.* for $z = r\,e^{i\theta}$ with $r > 0$ and $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Let $C = C(R,\epsilon)$ be the closed contour consisting of the interval I_- from $-R$ to $-\epsilon$, then the semicircle $C_\epsilon = \epsilon\,e^{i\theta}$ with θ going from π to 0, then the interval I_+ from ϵ to R, and finally the semicircle $C_R = R\,e^{i\theta}$ where $0 \leq \theta \leq \pi$. The function $f(z) = z^{\frac{1}{5}}(1+z^2)^{-1}$ has one pole inside this contour, at $z = i$. Since

$$f(z) = z^{\frac{1}{5}}(z+i)^{-1}(z-i)^{-1},$$

we have

$$\text{Res}[f(z)]_{z=i} = i^{\frac{1}{5}}(2i)^{-1},$$

and since $\log(i) = \log|i| + i\frac{\pi}{2} = \frac{i\pi}{2}$ we have

$$i^{\frac{1}{5}} = e^{\frac{1}{5}\,\log(i)} = e^{\frac{i\pi}{10}} = \cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right).$$

Thus

$$\int_{C(R,\epsilon)} z^{\frac{1}{5}}(1+z^2)^{-1}\,dz = \pi\left[\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right].$$

Now

$$\lim_{R\to\infty}\int_{C_R} z^{\frac{1}{5}}(1+z^2)^{-1}\,dz = 0,$$
$$\lim_{\epsilon\to 0}\int_{C_\epsilon} z^{\frac{1}{5}}(1+z^2)^{-1}\,dz = 0$$

by the usual arguments. Also on the integral over the interval $-R$ to $-\epsilon$, we have

$$z^{\frac{1}{5}} = |z|^{\frac{1}{5}} e^{\frac{i\pi}{5}}.$$

Thus making the substitution $x = -t$ we have

$$\begin{aligned}\int_{I_-} z^{\frac{1}{5}}(1+z^2)^{-1}\,dz &= \int_{-R}^{-\epsilon} |x|^{\frac{1}{5}} e^{\frac{i\pi}{5}}(1+x^2)^{-1}\,dx \\ &= e^{\frac{i\pi}{5}} \int_{\epsilon}^{R} t^{\frac{1}{5}}(1+t^2)^{-1}\,dt \\ &= e^{\frac{i\pi}{5}} \int_{I_+} z^{\frac{1}{5}}(1+z^2)^{-1}\,dz\end{aligned}$$

It follows that

$$\left(1+e^{\frac{i\pi}{5}}\right)\int_0^\infty x^{\frac{1}{5}}(1+x^2)^{-1}\,dx = \pi\left[\cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)\right] = \pi e^{\frac{i\pi}{10}}$$

or

$$\int_0^\infty x^{\frac{1}{5}}(1+x^2)^{-1}\,dx = \frac{\pi}{e^{i\frac{\pi}{10}} + e^{-i\frac{\pi}{10}}} = \frac{\pi}{2}\sec\left(\frac{\pi}{10}\right).$$

6.3. **The Argument Principle.** 2/17/12

Theorem 6.4. *Suppose that f is meromorphic in an open set Ω containing a closed disk $\overline{D(a,R)}$. Let $C = C(a,R)$ be the boundary of this disk, and suppose that f has no zeros and no poles on C. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \#(Z) - \#(P),$$

where $\#(N)$ is the number of zeros of f in $D(z,R)$ counted with multiplicity, and $\#(P)$ is the number of poles of f in $D(z,R)$ counted with multiplicity.

Proof. We know that the integral on the left hand side is equal to the sum of the residues of $\frac{f'}{f}$ inside the disk $D(a,R)$. If $c \in D(a,R)$ is a zero of order m, then near c we have $f(z) = (z-c)^m g(z)$ where g is holomorphic and non-vanishing near c. Then $f'(z) = m(z-c)^{m-1}g(z) + (z-c)^m g'(z)$, and so near c we have

$$\frac{f'(z)}{f(z)} = \frac{m}{z-c} + \frac{g'(z)}{g(z)}.$$

It follows that the residue of f'/f at c is m.

Similarly, if If $c \in D(a,R)$ is a pole of order n, then near c we have $f(z) = (z-c)^{-n}g(z)$ where g is holomorphic and non-vanishing near c. Then $f'(z) = -n(z-c)^{-n-1}g(z) + (z-c)^{-n}g'(z)$, and so near c we have

$$\frac{f'(z)}{f(z)} = \frac{-n}{z-c} + \frac{g'(z)}{g(z)}.$$

It follows that the residue of f'/f at c is $-n$. This completes the proof. □

We now give two applications of this result.

Theorem 6.5 (Rouché)**.** *Suppose that f and g are holomorphic in an open set Ω containing a closed dism $\overline{D(a,R)}$. Suppose for every $z \in C = C(a,R)$ we have*

$$|f(z)| > |g(z)|.$$

Then f and $f+g$ have the same number of zeros on the open disk $D(a,R)$.

Proof. For $0 \le t \le 1$ set $F_t(z) = f(z) + tg(z)$. Then F_t is holomorphic in Ω, and F_t has no zeros on $C(a, R)$. Let

$$\#(Z_t) = \frac{1}{2\pi i} \int_C \frac{F_t'(z)}{F_t(z)}\, dz.$$

Then $\#(Z_t)$ is the number of zeros of F_t in the disk $D(a, R)$, and hence is an integer. On the other hand, it is not hard to see that the integral is a continuous function of t. But continuous, integer valued functions are constant. This completes the proof. $\square$

Theorem 6.6 (Open Mapping Theorem)**.** *Let f be holomorphic and non-constant on a connected open set $\Omega \subset \mathbb{C}$. Let $z_0 \in \Omega$, and let $w_0 = f(z_0)$. Write*

$$f(z) = w_0 + \sum_{k=N}^{\infty} a_k (z - z_0)^k$$

where $a_N \neq 0$. Then there exist constants $\epsilon > 0$ and $\delta > 0$ so that $\overline{D(z_0, \delta)} \subset \Omega$, and for any $w \in \mathbb{C}$ with $0 < |w - w_0| < \epsilon$, there exist exactly N distinct points $\{z_1, \dots, z_N\}$ such that

(a) $|z_j - z_0| < \delta$ *for* $1 \le j \le N$;

(b) $f(z_j) = w$ *for* $1 \le j \le N$.

In particular, the mapping f is open.

Proof. Our hypothesis is that the function $g(z) = f(z) - w_0$ has a zero of order N at z_0. We already know that this zero is isolated. Thus there exists $\delta > 0$ so that $\overline{D(z_0, \delta)} \subset \Omega$ and so that $f(z) \neq w_0$ if $|z - z_0| \le \delta$. Shrinking δ if necessary, we can also assume that $f'(z) \neq 0$ for $0 < |z - z_0| \le \delta$. The image of the circle $C = \{z \in \mathbb{C} : |z - z_0| = \delta\}$ is a compact set and does not contain w_0. Thus there exists $\epsilon > 0$ so that if $|w - w_0| < \epsilon$, then w is also not an element of C.

For $|w - w_0| < \epsilon$ let $g_w(z) = f(z) - w$. Then g_w is holomorphic in a neighborhood of the closed disk $\overline{D(z_0, \delta)}$, and $g_w(z) \neq 0$ for all $z \in C$ and $|w - w_0| < \epsilon$. Consider

$$n(w) = \frac{1}{2\pi i} \int_C \frac{g_w'(z)}{g_w(z)}\, dz.$$

Then by Theorem 6.4, $n(w)$ is the number of zeros of g_w insider the disk $D(z_0, \delta)$. Thus $n(w)$ is integer valued. On the other hand,

$$n(w) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - w}\, dz$$

which makes it clear that $n(w)$ is a continuous function of w, and hence is constant. Finally, when $w = w_0$ we know that $n(w) = N$. Thus $f(z) - w$ has exactly N zeros insider the disk $D(z_0, \delta)$. Moreover, each of these zeros is simple since we have arranged that $f'(z) \neq 0$ at these points. This completes the proof. $\square$

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Theorem 6.7 (Maximum Modulus Principle)**.** *Let f be a non-constant holomorphic function on a connected open set $\Omega \subset \mathbb{C}$. Then $|f(z)|$ cannot attain a local maximum at any point $z \in \Omega$. In particular, if $K \subset \Omega$ is a compact subset, then*

$$\sup_{z \in K} |f(z)| = \sup_{z \in \partial K} |f(z)|.$$

7. The Riemann sphere

7.1. Definiton and complex structure.

It is often convenient to add a *point at infinity* to the set $\mathbb{C}$ of complex numbers. We introduce a symbol ∞, and define the Riemann sphere $\widehat{\mathbb{C}}$ to be the set $\mathbb{C} \cup \{\infty\}$. We put a topology on $\widehat{\mathbb{C}}$ by declaring that a set $U \subset \widehat{\mathbb{C}}$ is open if and only if either $\infty \notin U$ and U is open in $\mathbb{C}$, or $\infty \in U$, and $U \setminus \{\infty\}$ is open in $\mathbb{C}$ and contains the complement of some compact set in $\mathbb{C}$. Thus a basis for the open neighborhoods of $\{\infty\}$ are the sets $\{\zeta \in \mathbb{C} \,\big|\, |\zeta| > R\}$ for $R > 0$. The Riemann sphere $\widehat{\mathbb{C}}$ is called the *one-point compactification* of the complex plane $\mathbb{C}$.

We can make an explicit identification between the Riemann sphere $\widehat{\mathbb{C}}$ and the unit sphere $S_2 = \{(x, y, z) \in \mathbb{R}^3 \,\big|\, x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. We identify the complex plane with the hyperplane $\{(x, y, z) \in \mathbb{R}^3 \,:\, z = 0\}$ so that $x + iy \in \mathbb{C}$ corresponds to the point $(x, y, 0)$ in the hyperplane. For every point $p = (x, y, z) \in S_2$ with $z \neq 1$ we let $\pi(p)$ be the point of intersection of the line joining $(0, 0, 1)$ and $p = (x, y, z)$ with this hyperplane. We also let $\pi(0, 0, 1) = +\infty$.

Explicitly, we can check that $\pi : S_2 \to \widehat{\mathbb{C}}$ is given algebraically by setting

$$\pi(x, y, z) = \begin{cases} \frac{x}{1-z} + \frac{y}{1-z} i & \text{if } z < 1, \\ \\ \infty & \text{if } z = 1. \end{cases}$$

Then the mapping π is one-to-one and onto, and that the inverse of Φ is given by

$$\begin{aligned} \pi^{-1}(a + bi) &= (\lambda a, \lambda b, 1 - \lambda), \\ \pi^{-1}(\infty) &= (0, 0, 1) \end{aligned}$$

where

$$\lambda = \frac{a^2 + b^2 - 1}{a^2 + b^2 + 1}.$$

We now understand $\widehat{\mathbb{C}}$ as a set and as a topological space, but we also want to understand it as a *compled manifold*; that is, we want to understand what it means for functions define on or with values in $\widehat{\mathbb{C}}$ be be holomorphic. We do this by considering two special open subset of $\widehat{\mathbb{C}}$:

$$\begin{aligned} U_1 &= \{p \in \widehat{\mathbb{C}} \,:\, p \neq +\infty\}, \\ U_2 &= \{p \in \widehat{\mathbb{C}} \,:\, p \neq 0\}. \end{aligned}$$

U_1 is nothing but our original copy of the complex plane $\mathbb{C}$, and

$$U_1 \cap U_2 = \{z \in \mathbb{C} \,:\, z \neq 0\} = \mathbb{C}^*.$$

Moreover there is a homeomorphism $\Phi : U_2 \to \mathbb{C}$ given by

$$\Phi(p) = \begin{cases} z^{-1} & \text{if } U_2 \ni p = z \neq +\infty, \\ \\ 0 & \text{if } U_2 \ni p = +\infty. \end{cases}$$

We think of Φ as giving complex coordinates on the open set U_2. Now if $W \subset \widehat{\mathbb{C}}$ is open an $p \in W$, there are three possibilities:

(a) $p = 0$, in which case there exists $\epsilon > 0$ so that $\{z \in U_1 \,:\, |z| < \epsilon\} \subset W$;

(b) $p = +\infty$, in which case there exists $R > 0$ so that $\{z \in U_2 \,:\, |z| > R\} \subset W$;

(c) $p \neq 0$ and $p \neq +\infty$, in which case there exists $\delta > 0$ so that $\{z \in \mathbb{C} \,:\, |z - p| < \delta\} \subset U_1 \cap U_2$.

Now let $W \subset \widehat{C}$ be an open set, and let $f : W \to \widehat{\mathbb{C}}$ be a continuous function. We want to define what it means for f to be holomorphic. Let $p \in W$ and let $f(p) = q \in \widehat{\mathbb{C}}$. There are four possibilities:

(1) $p \neq +\infty$ and $q \neq +\infty$: In this case there is disk $D(p,\epsilon) \subset W \cap U_1$, and we require that f restricted to that disk be holomorphic in the usual sense.

(2) $p \neq +\infty$ and $q = +\infty$: In this case there is disk $D(p,\epsilon) \subset W \cap U_1$, and we require that f restricted to the punctured disk $D(p,\epsilon)^* = \{z \in \mathbb{C} : 0 < |z-p| < \epsilon\}$ have a pole at p.

(3) $p = +\infty$ and $q \neq +\infty$: In this case the function $g(z) = f(z^{-1})$ has an isolated singularity at 0, and we require that 0 is a removable singularity, with $g(0) = q$.

(4) $p = +\infty$ and $q = +\infty$: In this case the function $g(z) = f(z^{-1})$ has an isolated singularity at 0, and we require that 0 is a pole; *i.e.* the function $h(z) = f(z^{-1})^{-1}$ has a removable singularity at $z = 0$ with $h(0) = 0$.

7.2. **The Complex Projective Line.**

We can also identify $\widehat{\mathbb{C}}$ with the complex projective line. We define an equivalence relation on $\mathbb{C}^2$ by requiring that $(z_1, z_2) \sim (w_1, w_2)$ if and only if there is a non-zero complex number λ such that $\lambda(z_1, z_2) = (w_1, w_2)$. Let $\mathbb{CP}^1$ denote the set of equivalence classes of $\mathbb{C}^2 \setminus \{(0,0)\}$. If $(z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, then we write $[z_1, z_2]$ for the corresponding point in $\mathbb{CP}^1$. (These are called the homogeneous coordinates.) Note that

$$\begin{aligned} (z_1, z_2) &\sim (z_1/z_2, 1) && \text{if } z_2 \neq 0, \\ (z_1, z_2) &\sim (1, 0) && \text{if } z_2 = 0 \end{aligned} .$$

We can then identify $z \in \mathbb{C}$ with the equivalence class of $(z, 1)$ in $\mathbb{CP}^1$, and identify $\{\infty\}$ with the equivalence class of $(1, 0)$.

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7.3. **Fractional linear transformations.**

Let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible 2×2 complex matrix. Then T defines an invertible linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^2$ given by

$$T \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{bmatrix}.$$

If $\sim$ is the equivalence relation defining the complex projective line, it is clear that

$$(z_1, z_2) \sim (w_1, w_2) \quad \Longrightarrow \quad T \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Thus T induces a mapping $\widehat{T} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. It is easy to check that the mapping is given explicitly by

$$\widehat{T}(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C} \text{ and } cz + d \neq 0, \\ \infty & \text{if } z \in \mathbb{C} \text{ and } cz + d = 0, \\ \frac{a}{c} & \text{if } z = \infty \text{ and } c \neq 0, \\ \infty & \text{if } z = \infty \text{ and } c = 0. \end{cases}$$

We usually abbreviate this by simply writing

$$\widehat{T}(z) = \frac{az+b}{cz+d} = \frac{a+b/z}{c+d/z}$$

with the understanding that dividing by 0 gives ∞, and dividing by ∞ gives 0. It is clear that $\widehat{T}$ is a holomorphic mapping from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. Note that if $\lambda \in \mathbb{C}$ with $\lambda \neq 0$, then the fractional linear transformation corresponding to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the same as that corresponding to the matrix $\begin{bmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{bmatrix}$. Thus if $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$, the corresponding mapping is also given by a matrix with determinant 1. Thus we usually restrict our attention to the mapping

$$SL(2,\mathbb{C}) \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \widehat{T}(z) = \frac{az+b}{cz+d}.$$

Both $SL(2\mathbb{C})$ and the set $\mathcal{H}(\widehat{C})$ of holomorphic mappings of $\widehat{\mathbb{C}}$ to itself are groups, with group operation given either by matrix multiplication or composition.

Proposition 7.1. *The mapping* $\widehat{\ } : SL(2,\mathbb{C}) \to \mathcal{H}(\widehat{C})$ *is a group homomorphism, and the kernel consists of* $\pm I$.

Proof. The proof that $\widehat{\ }$ is a group homomorphism is an easy algebraic computation. If $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2\mathbb{C})$ and if $\widehat{T}$ is the identity mapping, then for every $z \in \mathbb{C}$ we have

$$z = \frac{az+b}{cz+d}.$$

Setting $z = 0$ shows that $b = 0$ and setting $z = +\infty$ shows that $c = 0$. Then $\frac{a}{d} = 1$. But since $ad = 1$ it follows that either $a = d = 1$ or $a = d = -1$. □

Corollary 7.2. *Every fractional linear transformation* $z \to \frac{az+b}{cz+d}$ *is a one-to-one and onto mapping of* $\widehat{\mathbb{C}}$ *to itself, and the inverse is again a fractional linear transformation.*

Corollary 7.3. *If a fractional linear transformation* $z \to \frac{az+b}{cz+d}$ *fixes the point* 0, 1 *and* $+\infty$, *then the mapping is the identity.*

Lemma 7.4. *If* a, b, c *are three distinct points in* $\widehat{\mathbb{C}}$, *then there is a unique fractional linear transformation taking* a *to* 0, b *to* 1, *and* c *to* ∞.

Proof. If none of a, b, c are the point $+\infty$, the fractional linear transformation

$$\widehat{T}(z) = \frac{z-a}{z-c}\,\frac{b-c}{b-a}$$

has the required properties. If $a = +\infty$, the fractional linear transformation

$$\widehat{T}(z) = \frac{b-c}{z-c}$$

has the required properties. If $b = +\infty$, the fractional linear transformation

$$\widehat{T}(z) = \frac{z-a}{z-c}$$

has the required properties. Finally, if $c = +\infty$, the fractional linear transformation

$$\widehat{T}(z) = \frac{z-a}{b-a}$$

has the required properties. Uniqueness of the mapping follows from Corollary 7.3. This completes the proof. □

Theorem 7.5. *Let $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic functions which is one-to-one. Then F is given by a fractional linear transformation $z \to \frac{az+b}{cz+d}$ with $ad - bc = 1$, and the mapping is also onto, with a holomorphic inverse.*

Proof. Let $F(0) = a$, $F(1) = b$ and $F(+\infty) = c$. Since the mapping f is one-to-one, it follows that $\{a, b, c\}$ are three distinct points in $\widehat{\mathbb{C}}$. Let S be the (uniques) fractional linear transformation which carries a to 0, b to 1 and c to $+\infty$. Then the composition $G(z) = S\big(F(z)\big)$ is also a one-to-one holomorphic mapping of $\widehat{\mathbb{C}}$ to itself which fixes 0, 1, and ∞. It suffices to show that G is the identity mapping, since then $F(z) = S^{-1}(z)$.

The function $H(z) = G(z^{-1})$ restricted to $z \in \mathbb{C}^*$ has an isolated singularity at $z = 0$. Since this mapping is one-to-one, the singularity cannot be essential. Since $|H(z)| \to \infty$ as $z \to 0$, the singularity cannot be removable. Thus it must be a pole, and since H is one-to-one, the order of the pole must be exactly 1. It follows that

$$G(z^{-1}) = \frac{1}{z} g(z)$$

where $g(z) = zG(z^{-1})$ is holomorphic on $\widehat{\mathbb{C}}$. But

$$\lim_{z\to\infty} g(z) = \lim_{w\to 0} \frac{G(w)}{w}$$

exists since $G(0) = 0$. Thus g is a bounded entire function, and hence is constant. The constant must be 1 since $G(1) = 1$. It follows that $G(z) = z$. This completes the proof. □

Definition 7.6. *Let $\{z_1, z_2, z_3, z_4\}$ be four distinct points on the Riemann sphere $\widehat{\mathbb{C}}$. The* cross ratio *of the points is the complex number*

$$C(z_1, z_2, z_3, z_4) = \begin{cases} \frac{z_1-z_3}{z_1-z_4}\frac{z_2-z_4}{z_2-z_3} & \text{if } \{z_1, z_2, z_3, z_4\} \subset \mathbb{C}, \\ \frac{z_2-z_4}{z_2-z_3} & \text{if } z_1 = \infty, \\ \frac{z_1-z_3}{z_1-z_4} & \text{if } z_2 = \infty, \\ \frac{z_2-z_4}{z_1-z_4} & \text{if } z_3 = \infty, \\ \frac{z_1-z_3}{z_2-z_3} & \text{if } z_4 = \infty. \end{cases}$$

Note that if F is the unique fractional linear transformation taking z_2 to 1, z_3 to 0, and z_4 to ∞, then $F(z_1) = C(z_1, z_2, z_3, z_4)$.

Lemma 7.7. *Let $G(z) = (az + b)(cz + d)^{-1}$ be an arbitrary fractional linear transformation, and let $\{z_1, z_2, z_3, z_4\}$ be four distinct points on the Riemann sphere. Then*

$$C(z_1, z_2, z_3, z_4) = C(G(z_1), G(z_2), G(z_3), G(z_4)).$$

In other words, the cross-ratio is invariant under fractional linear transformations.

Proof. Consider the two fractional linear transformations

$$z \to C(z, z_2, z_3, z_4),$$
$$z \to C(G(z), G(z_2), G(z_3), G(z_4)).$$

Since they both take z_2 to 1, z_3 to 0, and z_4 to ∞, they are equal, and the lemma is that statement with $z = z_1$. □

Lemma 7.8. *Let $G(z) = (az+b)(cz+d)^{-1}$ be an arbitrary fractional linear transformation. Then the image under G of the real axis is either a straight line or a circle.* 2/24/12

Proof. Let $t \in \mathbb{R}$ and let $w_t = G(t) = (at+b)(ct+d)^{-1}$. Then we can solve for t in terms of w_t and get

$$\frac{-dw_t + b}{cw_t - a} = t \in \mathbb{R}.$$

Thus w_t satisfies the equation

$$\Im\mathrm{m}\left[\frac{-dw_t + b}{cw_t - a}\right] = 0.$$

If we write this out in terms of $w_t = x_t + iy_t$ we see that we either get the equation of a circle or of a straight line. □

Lemma 7.9. *The image of a straight line or circle under an arbitrary fractional linear transformation is again either a straight line or circle.*

7.4. **Conformal mappings.**

Let f be a function which is holomorphic in a neighborhood of a point $p \in \mathbb{C}$, and suppose $f(p) = q \in \mathbb{C}$. The f induces a mapping from the set of vectors with tails at p (the *tangent space* at p) to the set of vectors with tails at q (the *tangent space* at q) as follows. Let $\gamma : (-\epsilon, +\epsilon) \to \mathbb{C}$ be a differentiable curve with $\gamma(0) = p$. Then $\gamma'(0)$ is a tangent vector at p. The new curve $\Gamma(t) = f\big(\gamma(t)\big)$ then has the property that $\Gamma(0) = q$, and $\Gamma'(0)$ is a tangent vector at q. But it follows from the Cauchy-Riemann equations that

$$\Gamma'(0) = \frac{d}{dt}\Big(f\big(\gamma(t)\big)\Big)\Big|_{t=0} = f'(\gamma(0))\,\gamma'(0) = f'(p)\gamma'(0).$$

Thus the vector $\gamma'(0)$ is transformed to the vector $f'(p)\gamma'(0)$, which is just multiplication by a complex number. Since multiplication involves a stretching and a rotation, we see that holomorphic mapping preserve the angle between curves. We say that holomorphic functions are *conformal*.

7.5. **Automorphisms of the unit disk.** 2/27/12

We have seen that any holomorphic mapping $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which is one-to-one is given by a fractional linear transformation, and hence is a biholomorphic mapping. Thus we have characterized the group $Aut(\widehat{\mathbb{C}})$ of biholomorphic mappings of the Riemann sphere. We now want to ask similar questions about two other domains: the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the upper half-plane $\mathbb{U} = \{z \in \mathbb{C} : \Im\mathrm{m}[z] > 0\}$. In this section, we deal with the case of the unit disk.

We begin by giving some examples of $Aut(\mathbb{D})$. Let $\alpha \in \mathbb{D}$, and consider the fractional linear transformation

$$\psi_\alpha(z) = \frac{z - \alpha}{\bar{\alpha}z - 1} = \frac{\alpha - z}{1 - \bar{\alpha}z}.$$

Note that

$$\det\begin{bmatrix} 1 & -\alpha \\ \bar{\alpha} & -1 \end{bmatrix} = -1 + |\alpha|^2 < 0$$

so this fractional linear transformation is non-degenerate. (If we divide by $|\alpha|^2 - 1$, the corresponding matrix belongs to $SL(2,\mathbb{C})$). Thus ψ_α is a biholomorphic automorphism of $\widehat{\mathbb{C}}$. It follows from a homework problem that ψ_α maps $\mathbb{D}$ to itself. We can also check this by observing that

$$|\psi_\alpha(1)| = \left|\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{1-\alpha}{\overline{1-\alpha}}\right| = 1,$$
$$|\psi_\alpha(-1)| = \left|\frac{-1-\alpha}{-\bar{\alpha}-1}\right| = \left|\frac{1+\alpha}{\overline{1+\alpha}}\right| = 1,$$
$$|\psi_\alpha(i)| = \left|\frac{i-\alpha}{i\bar{\alpha}-1}\right| = \left|\frac{i-\alpha}{\overline{i-\alpha}}\right| = 1.$$

Since ψ_α takes three points on the unit circle to points on the unit circle, and since fractional linear transformations carry circles to circles, it follows that ψ_α carries the unit circle to itself. Note also that

$$\psi_\alpha(0) = \alpha \qquad\qquad \psi_\alpha(\alpha) = 0.$$

In fact

$$\psi_\alpha \circ \psi_\alpha(z) = \frac{\psi_\alpha(z)-\alpha}{\bar{\alpha}\psi_\alpha(z)-1} = \frac{(z-\alpha)-\alpha(\bar{\alpha}z-1)}{\bar{\alpha}(z-\alpha)-(\bar{\alpha}z-1)} = \frac{z-|\alpha|^2z}{1-|\alpha|^2} = z.$$

Thus ψ_α has period 2.

Next, let $\theta \in \mathbb{R}$, and denote by R_θ the fractional linear transformation

$$R_\theta(z) = e^{i\theta}z.$$

This clearly carries the unit disk to itself, and is just a rotation by the angle θ. It now follows that if $\alpha \in \mathbb{D}$and $\theta \in \mathbb{R}$, then the composition

$$R_\theta \circ \psi_\alpha(z) = e^{i\theta}\frac{\alpha - z}{1-\bar{\alpha}z}$$

is also an element of $Aut(\mathbb{D})$.

Lemma 7.10. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, one-to-one, and onto, with holomorphic inverse f^{-1}. Then there exist $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta}\frac{\alpha - z}{1-\bar{\alpha}z}.$$

Proof. Since $f : \mathbb{D} \to \mathbb{D}$ is one-to-one and onto, there is a unique number $\alpha \in \mathbb{D}$ with $f(\alpha) = 0$. Consider the mapping

$$g(z) = f \circ \psi_\alpha(z).$$

Then $g : \mathbb{D} \to \mathbb{D}$, and $g(0) = 0$. It follows from the Schwarz lemma (see the homework) that for every $z \in \mathbb{D}$ we have

$$|g(z)| \le |z|.$$

Similarly, the inverse $g^{-1} = f^{-1} \circ \psi_\alpha^{-1} \circ f^{-1} = \psi_\alpha \circ f^{-1}$ maps the unit disk to itself, and $g^{-1}(0) = 0$ Again applying the Schwarz lemma, we have

$$|g^{-1}(w)| \le |w|.$$

Putting these two estimates together, it follows that $|g(z)| = |z|$ for all $z \in \mathbb{D}$, and using the Schwarz lemma one more time, it follows that $g(z) = e^{i\theta}z$. Since $\psi_\alpha \circ \psi_\alpha$ is the identity, this gives the desired result. ∎

7.6. Automorphisms of the upper half-plane.

Lemma 7.11. *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R})$ *be a real* 2×2 *matrix with determinant* 1. *Then the fractional linear transformation*

$$T_M(z) = \frac{az+b}{cz+d}$$

is a biholomorphic mapping of the upper half plane to itself. Conversely, every automorphism of $\mathbb{U}$ *is of this form. The mapping* $SL(2,\mathbb{R}) \to Aut(\mathbb{U})$ *is a group homomorphism which is surjective, and whose kernel is* $\pm I$.

8. The inhomogeneous Cauchy-Riemann equations

We have seen that holomorphic functions on an open set $\Omega \subset \mathbb{C}$ are precisely the solutions on Ω of the homogeneous equation

$$\frac{\partial u}{\partial \bar z}(z) = 0.$$

As we will see, it is of interest and importance to consider the corresponding inhomogeneous equation

$$\frac{\partial u}{\partial \bar z}(z) = g(z),$$

where g is a 'given' function or measure or distribution on Ω, and u is the unknown.

8.1. The generalized Cauchy formula. 3/2/2012

Recall that if $z = x+iy$ so that $\bar z = x - iy$, we write $dz = dx + idy$ and $d\bar z = dx - idy$. Also recall that the wedge product of differential forms is skew-symmetric. Thus we have

$$dz \wedge d\bar z = (dx+idy)\wedge(dx-idy) = dx\wedge dx - idx\wedge dy + idy \wedge dx + dy\wedge dy = -2idx\wedge dy.$$

Theorem 8.1 (Complex form of Green's Theorem)**.** *Let* $\Omega \subset \mathbb{C}$ *be a bounded open set whose boundary consists of a finite number of piecewise-smooth curves. Let* φ *be a function of class* $\mathcal{C}^1$ *on the closure of* Ω. *Then*

$$\int_{\partial\Omega} \varphi(\zeta)\, d\zeta = -\iint_\Omega \frac{\partial \varphi}{\partial \bar\zeta}(\zeta)\, d\zeta \wedge d\bar\zeta$$

where $d\zeta \wedge d\bar\zeta = -2i\, dx\, dy$.

Proof. Writing $\zeta = x+iy$ and using the real version of Green's theorem in the plane we have

$$\begin{aligned}
\int_{\partial\Omega} \varphi(\zeta)\, d\zeta &= \int_{\partial\Omega} \varphi(x,y)\, dx + i\varphi(x,y)\, dy \\
&= \iint_\Omega \Big[i\frac{\partial\varphi}{\partial x}(x,y) - \frac{\partial \varphi}{\partial y}(x,y)\Big]\, dx\, dy \\
&= \iint_\Omega \frac{1}{2}\Big[\frac{\partial\varphi}{\partial x}(x,y) + i\frac{\partial\varphi}{\partial y}(x,y)\Big] 2i\, dx\, dy \\
&= -\iint_\Omega \frac{\partial\varphi}{\partial\bar\zeta}(\zeta)\, d\zeta\wedge d\bar\zeta.
\end{aligned}$$

□

Theorem 8.2. *Let $\Omega \subset \mathbb{C}$ be a bounded open set whose boundary consists of a finite number of piecewise-smooth curves. Let φ be a function of class $\mathcal{C}^1$ on the closure of Ω. Then for $z \in \Omega$*

$$\varphi(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\iint_{\Omega} \frac{\partial \varphi}{\partial \bar z}(\zeta)\, \frac{d\zeta \wedge d\bar\zeta}{\zeta - z}.$$

where $d\zeta \wedge d\bar\zeta = -2i\, dx\, dy$.

Proof. Fix $z \in \Omega$, and choose $\epsilon > 0$ so that the disk $\overline{D(z, 2\epsilon)} = \{\zeta \in \mathbb{C} : |z - \zeta| < \epsilon\}$ is contained in Ω. Now apply Theorem 8.1 to the open set $\Omega_\epsilon = \Omega \setminus \overline{D(z, \epsilon)}$, and the function $\zeta \to \varphi(\zeta)(\zeta - z)^{-1}$. Since $(\zeta - z)^{-1}$ is a holomorphic function of ζ in Ω_ϵ, we get

$$\int_{\partial\Omega_\epsilon} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = -\iint_{\Omega_\epsilon} \frac{\partial \varphi}{\partial \bar z}(\zeta)\, \frac{d\zeta \wedge d\bar\zeta}{\zeta - z}.$$

Now

$$\int_{\partial\Omega_\epsilon} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta = \int_{\partial\Omega} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta - \int_{\partial D(z,\epsilon)} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta.$$

But as $\epsilon \to 0$ we have

$$\iint_{\Omega_\epsilon} \frac{\partial \varphi}{\partial \bar z}(\zeta)\, \frac{d\zeta \wedge d\bar\zeta}{\zeta - z} \to \iint_{\Omega} \frac{\partial \varphi}{\partial \bar z}(\zeta)\, \frac{d\zeta \wedge d\bar\zeta}{\zeta - z},$$
$$\int_{\partial D(z,\epsilon)} \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta \to 2\pi i \varphi(z).$$

This completes the proof. □

Remark: Note that in terms of real coordinates

$$\frac{1}{2\pi i}\iint_{\Omega} f(\zeta)\, \frac{d\zeta \wedge d\bar\zeta}{\zeta - z} = \frac{1}{\pi}\iint_{\Omega} f(\zeta)\, \frac{1}{z - \zeta}\, dx\, dy$$

3/5/12

Theorem 8.3. *Let μ be a measure with compact support in $\mathbb{C}$. Set*

$$u(z) = \iint_{\mathbb{C}} (\zeta - z)^{-1}\, d\mu(\zeta).$$

Then

(1) *u is a holomorphic function in the complement of the support of μ.*

(2) *If $\omega \subset \Omega$ is an open set and if $\mu = \frac{1}{2\pi i}\varphi\, dz \wedge d\bar z$ on ω where $\varphi \in \mathcal{C}^k(\omega)$ for some $k \geq 1$, then $u \in \mathcal{C}^k(\omega)$ and*

$$\frac{\partial u}{\partial \bar z}(z) = \varphi(z)$$

for all $z \in \omega$.

Proof. The proof of statement (1) is clear. To prove (2), suppose first that $\omega = \mathbb{C}$. Then φ is a compactly supported $\mathcal{C}^k$ function on $\mathbb{C}$, and we can write

$$\begin{aligned}
u(z) &= \iint_{\mathbb{C}} (\zeta - z)^{-1}\, d\mu(\zeta) \\
&= \frac{1}{2\pi i} \iint_{\mathbb{C}} \varphi(\zeta)\, (\zeta - z)^{-1}\, d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{\pi} \iint_{\mathbb{C}} \varphi(\zeta)\, (z - \zeta)^{-1}\, dx\, dy \\
&= \frac{1}{\pi} \iint_{\mathbb{C}} \varphi(z - (x + iy))\, (x + iy)^{-1}\, dx\, dy.
\end{aligned}$$

Since $(x+iy)^{-1}$ is integrable, (check this using polar coordinates) we can differentiate under the integral sign k-times, and see that $u \in \mathcal{C}^k(\mathbb{C})$. Moreover, we get

$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}}(z) &= \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{z}}(z - (x + iy))\, (x + iy)^{-1}\, dx\, dy \\
&= \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{z}}(\zeta)\, (z - \zeta)^{-1}\, dx\, dy \\
&= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial \varphi}{\partial \bar{z}}(\zeta)\, (z - \zeta)^{-1}\, d\zeta \wedge d\bar{\zeta} \\
&= \varphi(z),
\end{aligned}$$

where the last equality follows from Theorem 8.2 with Ω a large disk containing the support of φ.

Now for an arbitrary ω containing z, choose $\chi \in \mathcal{C}_0^k(\omega)$ with $\chi(\zeta) \equiv 1$ for ζ near z. Then $\mu = \chi\mu + (1-\chi)\mu$. We can apply our first argument to $\chi\mu = \frac{1}{2\pi i}\chi\varphi$, and z is outside the support of $(1-\chi)\mu$, and so this contributes a holomorphic error. □

8.2. **The Runge Approximation Theorem.** 3/7/12

Theorem 8.4 (Runge)**.** *Let $\Omega \subset \mathbb{C}$ be an open set, and let $K \subset \Omega$ be a compact subset. The following conditions are equivalent:*

1. *Every function which is holomorphic in some open neighborhood of K can be uniformly approximated on K by a function which is holomorphic on Ω.*
2. *The open set $\Omega \setminus K = \Omega \cap K^c$ has no component which is relatively compact in Ω.*
3. *For every $z_0 \in \Omega \setminus K$, there exists a function f holomorphic on Ω such that*

$$|f(z_0)| > \sup_{z \in K} |f(z)|.$$

Proof that (2) $\Longrightarrow$ (1)*.* Suppose that (2) is satisfied. We want to show that the closure of the set of restrictions to K of functions holomorphic on Ω is dense in the set of functions holomorphic in some open neighborhood of K. If this were not the case, we could find a linear functional on the Banach space of all continuous functions on K which is zero on the restrictions of functions holomorphic on Ω, but is not zero on some function F which is holomorphic in some (smaller) open neighborhood W of K. The set of bounded linear functionals on $C(K)$ is the set of finite measures on K.

Thus we must show that if $d\mu$ is a measure on K and if $\int_K f(z)\, d\mu(z) = 0$ for every f which is holomorphic on Ω, then it follows that $\int_K F(z)\, d\mu(z) = 0$ for every function F holomorphic in some (smaller) open neighborhood W of K.

For $z \in \mathcal{C} \setminus K$ set

$$\varphi(z) = \int_K (\zeta - z)^{-1}\, d\mu(\zeta).$$

For $z \notin \Omega$, we have

$$\varphi^{(k)}(z) = k! \int_K (\zeta - z)^{-k-1}\, d\mu(\zeta) = 0$$

since the function $\zeta \to (\zeta - z)^{-k-1}$ is holomorphic on Ω. It follows that $\varphi(z) \equiv 0$ for $z \notin \Omega$. Since φ is holomorphic in the complement of K, it follows from the identity theorem that $\varphi(z) \equiv 0$ on every component of the complement of K which intersects $\mathcal{C} \setminus K$. But according to (1), every component of the complement of K intersects the complement of Ω. It follows that $\varphi(z) \equiv 0$ for $z \notin K$.

Now suppose that F is holomorphic in an open set W containing K. Choose $\chi \in \mathcal{C}_0^\infty(W)$ such that $\chi(z) \equiv 1$ for z in some neighborhood of K. Let $z \in K$. Then from the generalized Cauchy integral formula, we have

$$\begin{aligned} F(z) &= F(z)\chi(z) \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\partial (F\chi)}{\partial \bar\zeta}(\zeta) \frac{d\zeta \wedge d\bar\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} F(\zeta)\, \frac{\partial \chi}{\partial \bar\zeta}(\zeta) \frac{d\zeta \wedge d\bar\zeta}{\zeta - z}, \end{aligned}$$

since F is holomorphic on the support of χ. We now integrate F against $d\mu$, and use Fubini's theorem to get

$$\begin{aligned} \int_K F(z)\, d\mu(z) &= \int_K \Big[\frac{1}{2\pi i} \iint_{\mathbb{C}} F(\zeta)\, \frac{\partial \chi}{\partial \bar\zeta}(\zeta) \frac{d\zeta \wedge d\bar\zeta}{\zeta - z}\Big]\, d\mu(z) \\ &= \frac{1}{2\pi i} \iint_{\mathbb{C}} F(\zeta)\, \frac{\partial \chi}{\partial \bar\zeta}(\zeta) \Big[\int_K \frac{1}{\zeta - z}\, d\mu(z)\Big]\, d\zeta \wedge d\bar\zeta \\ &= -\frac{1}{2\pi i} \iint_{\mathbb{C}} F(\zeta)\, \frac{\partial \chi}{\partial \bar\zeta}(\zeta)\varphi(\zeta)\, d\zeta \wedge d\bar\zeta. \end{aligned}$$

Now $\varphi(\zeta) \equiv 0$ for $\zeta \notin K$, and $\frac{\partial \chi}{\partial \bar\zeta}(\zeta) \equiv 0$ in some neighborhood of K since χ is identically 1 there. It follows that the integral is zero, which completes the proof of (1). □

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Proof that (1) $\Longrightarrow$ (2). We show the contrapositive: if (2) is false, then (1) is false. Thus suppose that there is a connected component U of $\Omega \setminus K$ which is relatively compact in Ω. Choose $z_0 \in U$. Then the function $f_0(z) = (z - z_0)^{-1}$ is holomorphic in a neighborhood of K. If statement (1) were true, we could find a sequence of functions $\{f_n\}$ holomorphic on Ω which converges uniformly on K to f_0. By the maximum principle applied to the function $f_m - f_n$ it follows that the sequence $\{f_n\}$ also converges uniformly on U to a function F_0. We have $F = f_0$ on the boundary of U, and so $(z - z_0)F(z) \equiv 1$ for $z \in \partial U$. It follows that $(z - z_0)F(z) \equiv 1$ for all $z \in U$, and this leads to a contradiction when $z = z_0$. Thus (1) cannot be true. □

Note that so far, we have established that (1) $\Longleftrightarrow$ (2).

Proof that (3) $\Longrightarrow$ (2). Again we show the contrapositive: if (2) is false, then (3) is false. Suppose that there is a connected component U of $\Omega \setminus K$ which is relatively compact in Ω. Then the closure $\overline{U} \subset \Omega$, and in particular, the boundary of U is contained in K. Let $z_0 \in U$ so $z \notin K$. By the maximum principle, for every f which is holomorphic on Ω we have

$$|f(z_0)| \le \sup_{z \in U} |f(z)| = \sup_{z \in \partial U} |f(z)| \le \sup_{z \in K} |f(z)|,$$

which contradicts (3). □

Proof that (1) *and* (2) $\Longrightarrow$ (3). Let $z_0 \in \Omega \setminus K$, and choose a closed disk $\overline{D(z_0,\epsilon)}$ contained in the connected component of $\Omega \setminus K$ containing z_0. In (2), replace K by $K \cup \overline{D(z_0,\epsilon)}$. This is a compact subset of Ω, and the components of $\Omega \setminus \big(K \cup \overline{D(z_0,\epsilon)}\big)$ are the same as those of $\Omega \setminus K$ except that $\overline{D(z_0,\epsilon)}$ has been removed from one of them. Thus we can apply (1). The function which equals 0 near K and equals 1 near $\overline{D(z_0,\epsilon)}$ is holomorphic. By (1) we can find a function f holomorphic on Ω with $|f(z)| < \frac{1}{2}$ on K and $|f(z) - 1| < \frac{1}{2}$ on $\overline{D(z_0,\epsilon)}$. This gives (3). □

3/12/12

8.3. The Mittag-Leffler Theorem.

Theorem 8.5 (Mittag-Leffler). *Let $\Omega \subset \mathbb{C}$ be open, and let $\{z_1, z_2, \ldots\}$ be a countable discrete subset of Ω. For each j let*

$$P_j(\zeta) = \sum_{k=1}^{N_j} a_{j,k}\zeta^k$$

be a polynomial of degree N_j. Then there exists a meromorphic function $f : \Omega \to \widehat{\mathbb{C}}$ with poles only at the points $\{z_j\}$ such that for each j the function

$$f(z) - P_j\Big(\frac{1}{z - z_j}\Big) = f(z) - \sum_{k=1}^{N_j} \frac{a_{j,k}}{(z - z_j)^k}$$

has a removable singularity at the point z_j.

Proof. Suppose first that $\Omega \neq \mathbb{C}$. For $w \in \Omega$ let

$$d_\Omega(w) = \inf_{\zeta \notin \Omega} |w - \zeta|$$

be the distance from w to the complement of Ω. Then d_Ω is a continuous function on Ω. In fact, if $w_1, w_2 \in \Omega$ then for any $\zeta \notin \Omega$ we have

$$d_\Omega(w_2) \le |\zeta - w_2| \le |\zeta - w_1| + |w_2 - w_1|$$

and hence

$$d_\Omega(w_2) - |w_2 - w_1| \le \inf_{\zeta \notin \Omega} |\zeta - w_1| = d_\Omega(w_1).$$

It follows that

$$d_\Omega(w_2) - d_\Omega(w_1) \le |w_2 - w_1|$$

and reversing the roles of w_1 and w_2, we see that

$$\big|d_\Omega(w_2) - d_\Omega(w_1)\big| \le |w_2 - w_1|,$$

which shows that d_Ω is Lipschitz continuous. Given $w \in \Omega$, let $\zeta_j \in \mathbb{C} \setminus \Omega$ be a sequence of points such that $d_\Omega(w) = \lim_{j\to\infty} |w - \zeta_j|$. The set of points $\{\zeta_j\}$ is bounded, and so we can find a convergent subsequence. It follows that if $w \in \Omega$, there exists at least one point $\zeta_w \notin \Omega$ so that

$$d_\Omega(w) = |w - \zeta_w|.$$

We begin the proof of Theorem 8.5 by constructing an exhaustion of the open set Ω by compact subsets. Thus for each positive integer N let

$$K_N = \begin{cases} \{w \in \Omega \,:\, |w| \le N \text{ and } d_\Omega(w) \ge N^{-1}\} & \text{if } \Omega \neq \mathbb{C}, \\ \\ \{w \in \Omega \,:\, |w| \le N\} & \text{if } \Omega = \mathbb{C}. \end{cases}$$

Clearly K_N is a bounded subset of Ω. In fact we show that it is closed, and hence compact. For suppose $\{w_l\}$ is a sequence of points in K_N converging to a point w_0. First of all we have

$|w_0| \leq \limsup_{l\to\infty} |w_l| \leq N$. Next, suppose that $d_\Omega(w_0) < N^{-1}$. Then there exists $\zeta \notin \Omega$ and $0 < \epsilon < N^{-1}$ so that $|w_0 - \zeta| < N^{-1} - \epsilon$. Since $\lim_{l\to\infty} w_l = w_0$, there exists L so that for $l \geq L$ we have $|w_l - w_0| < \epsilon$. But then

$$|w_l - \zeta| = |(w_l - w_0) + (w_0 - \zeta)| \leq |w_l - w_0| + |w_0 - \zeta| < \epsilon + N^{-1} - \epsilon = N^{-1},$$

and so $d_\Omega(w_l) < N^{-1}$, which is a contradiction. It follows that $w_0 \in \Omega$ and $d_\Omega(w_0) \geq N^{-1}$. Thus each K_N is a compact subset of Ω.

Now let U be a component of $\Omega \setminus K_N$. Suppose that U is relatively compact in Ω. Then $\overline{U} \subset \Omega$ is compact. The function d_Ω is continuous on $\overline{U}$, and attains its minimum at some point $w_0 \in \Omega$. This minimum cannot be zero, since $d_\Omega(w) > 0$ for all $w \in \Omega$. Also, the minimum cannot occur at an interior point of U, since we could always move in a direction to decrease the distance to the complement of Ω. Thus the minimum must occur on the boundary of Ω, which is contained in K_N. It follows that $d_\Omega(w) \equiv N^{-1}$ for all $w \in \overline{U}$, and so in fact $U \subset K_N$, which is a contradiction. Thus no component of the complement of K_N in Ω is relatively compact in Ω, which means we can apply Runge's theorem (Theorem 8.4) to the pair $K_N \subset \Omega$. Note that the sequence of compact sets $\{K_n\}$ is an *exhaustion* of Ω in the sense that

$$K_N \subset \operatorname{int}(K_{N+1}) \subset K_{N+1}, \quad \text{and} \quad \Omega = \bigcup_{N=1}^{\infty} K_N.$$

Since the points $\{z_j\}$ are a discrete subset of Ω, for each N, only finitely many of the points belong to each K_N. Let

$$\begin{aligned} E_1 &= \{j \in \mathbb{N} : z_j \in K_1\}, \quad \text{and} \\ E_N &= \{j \in \mathbb{N} : z_j \in K_{N+1} \setminus K_N\} \quad \text{if } N > 1. \end{aligned}$$

Then each E_N is a finite set, and every $j \geq 1$ belongs to exactly one E_N. For $N \geq 2$ the function

$$g_N(z) = \sum_{j \in E_N} P_j\Big(\frac{1}{z - z_j}\Big)$$

is a finite sum of meromorphic functions, and is holomorphic in a neighborhood of the compact set K_N. Using the Runge approximation theorem, there is a function h_N holomorphic on Ω such that

$$\sup_{z \in K_N} |g_N(z) - h_N(z)| \leq 2^{-N}.$$

Note that $g_N - h_N$ is meromorphic on all of Ω, and is holomorphic on K_N. It follows that the function

$$\begin{aligned} f(z) &= \sum_{j \in E_1} P_j\Big(\frac{1}{z - z_j}\Big) + \sum_{N=2}^{\infty} \big(g_N(z) - h_N(z)\big) \\ &= \sum_{j \in E_1} P_j\Big(\frac{1}{z - z_j}\Big) + \sum_{N=2}^{\infty} \Big[\sum_{j \in E_N} P_j\Big(\frac{1}{z - z_j}\Big) - h_N(z)\Big] \end{aligned}$$

converges uniformly on compact subsets of Ω to a meromorphic function with the desired properties. This completes the proof. □

8.4. **Solving the inhomogeneous Cauchy-Riemann equation in $\mathcal{C}^\infty(\Omega)$.**

It follows from Theorem 8.3 that if $\varphi \in \mathcal{C}_0^\infty(\mathbb{C})$, there is a function $u \in \mathcal{C}^\infty(\mathbb{C})$ such that $\frac{\partial u}{\partial \bar z}(z) = \varphi(z)$. In fact, we can take

$$u(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}} \varphi(\zeta)\,\frac{d\zeta \wedge d\bar\zeta}{\zeta - z}.$$

We can extend this result as follows:

Theorem 8.6. *Let $\Omega \subset \mathbb{C}$ be open, and let $\varphi \in \mathcal{C}^\infty(\Omega)$. Then there exists $u \in \mathcal{C}^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar z}(z) = \varphi(z)$ for all $z \in \Omega$.*

Proof. Choose a sequence of compact subssets $\{K_N\}$ of Ω so that $K_N \subset \mathrm{int}(K_{N+1}) \subset K_{N+1}$, so that $\Omega = \bigcup_{N=1}^\infty K_N$, and so that we can apply the Runge approximation theorem to each pair (K_N, Ω). For each $N \geq 1$, choose a function $\psi_N \in \mathcal{C}_0^\infty(\mathbb{C})$ such that $\psi_N(z) \equiv 1$ in a neighborhood of K_N, and the support of ψ_N is a compact subset of K_{N+1}.

The function $\varphi_N(z) = \psi_N(z)\varphi(z)$ belongs to $\mathcal{C}_0^\infty(\mathbb{C})$, and we can choose a function $u_N \in \mathcal{C}^\infty(\mathbb{C})$ so that

$$\frac{\partial u_N}{\partial \bar z}(z) = \varphi_N(z) = \psi_N(z)\varphi(z).$$

Note that

$$\frac{\partial}{\partial \bar z}[u_{N+1} - u_N](z) = \left(\psi_{N+1}(z) - \psi_N(z)\right)\varphi(z)$$

and this is zero in a neighborhood of K_N. Thus $u_{N+1} - u_N$ is holomorphic in a neighborhood of K_N. By the Runge approximation theorem, choose a function h_N which is holomorphic on Ω, and such that

$$\sup_{z\in K_N} \left|u_{N+1}(z) - u_N(z) - h_N(z)\right| \leq 2^{-N}. \tag{8.1}$$

Consider the sum

$$u(z) = u_1(z) + \sum_{N=1}^\infty \left[u_{N+1}(z) - u_N(z) - h_N(z)\right].$$

By the estimate in equation (8.1), this series converges uniformly on compact subsets of Ω to a continuous limit. Moreover, on the compact set K_N, the terms in the sum

$$\sum_{M=N}^\infty [u_{M+1}(z) - u_M(z) - h_M(z)]$$

are all holomorphic. Thus uniform convergence implies that this 'tail' is holomorphic, and hence of class $\mathcal{C}^\infty$ on the set K_N. Since the sum $u_1 + \sum_{M=1}^{N-1}[u_{M+1} - u_M - h_M]$ is a finite sum of functions of class $\mathcal{C}^\infty$, it is also infinitely differentiable. It follows that the function u is infinitely differentiable, and it is now clear that $\frac{\partial u}{\partial \bar z}(z) = \varphi(z)$ for all $z \in \Omega$. □

8.5. **The cohomology version.**

Theorem 8.7. *Let $\Omega \subset \mathbb{C}$ be open, and suppose that $\Omega = \bigcup_{j=1}^\infty \Omega_j$ where each Ω_j is open. Let $g_{i,j}$ be a holomorphic function on $\Omega_i \cap \Omega_j$, and suppose the collection of these functions satisfy:*

(a) $g_{i,j}(z) + g_{j,i}(z) = 0$ *for all* $z \in \Omega_i \cap \Omega_j$*;*

(b) $g_{i,j}(z) + g_{j,k}(z) + g_{k,i}(z) = 0$ *for all* $z \in \Omega_i \cap \Omega_j \cap \Omega_k$*.*

Then for each j there exists a function g_j holomorphic on the open set Ω_j so that

$$g_{i,j}(z) = g_i(z) - g_j(z)$$

for all $z \in \Omega_i \cap \Omega_j$.

Proof. Let $\{\varphi_i\}$, $i = 1, \dots,$ be a partition of unit subordinate to the cover $\{\Omega_i\}$; this means that $\varphi_i \in \mathcal{C}_0^\infty(\Omega_{m_i})$, all but finitely many of these functions vanish on any compact subset of Ω, and $\sum_i \varphi(z) \equiv 1$ on Ω. Now for $z \in \Omega_k$ define

$$h_i(z) = -\sum_k \varphi_k(z)\, g_{m_k,i}(z).$$

Then h_i extends (as zero) to a function in $\mathcal{C}^\infty(\Omega_i)$.

First observe that if $z \in \Omega_i \cap \Omega_j$ we have

$$\begin{aligned} h_i(z) - h_j(z) &= \sum_k \varphi_k(z)(-g_{m_k,i}(z) + g_{m_k,j}(z)) \\ &= \sum_k \varphi_k(z)(i, g_{m_k}(z) + g_{m_k,j}(z)) \\ &= -\sum_k \varphi_k(z) g_{j,i}(z) \\ &= g_{i,j}(z). \end{aligned}$$

Thus $\{h_i\}$ is a almost a solution, except that it is not holomorphic. However

$$\frac{\partial h_i}{\partial \bar{z}} - \frac{\partial h_j}{\partial \bar{z}} = \frac{\partial g_{i,j}}{\partial \bar{z}} = 0,$$

so

$$\varphi(z) = \frac{\partial h_i}{\partial \bar{z}}(z) \quad \text{if } z \in \Omega_i$$

is a well-defined smooth function on Ω. Choose $u \in \mathcal{C}^\infty$ so that $\frac{\partial u}{\partial \bar{z}} = \varphi$. Then $h_i - u$ is holomorphic, and satisfies the requirements of the theorem. □

9. The Gamma and Beta Functions

9.1. **Preliminary definition of the Gamma function.**

For $\Re\mathrm{e}[z] > 0$ define

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt = \lim_{\substack{\epsilon \to 0 \\ N \to +\infty}} \int_\epsilon^N e^{-t} t^z \, \frac{dt}{t},$$

where the improper integral converges at $+\infty$ for any complex value of z, and the improper integral converges at 0 since $\Re\mathrm{e}[z] > 0$. The integral converges uniformly for z in compact subsets of the right half plane, so this integral formula defines a holomorphic function, called the *Gamma function* in the right half plane. Note that the measure $\frac{dt}{t}$ on the positive real axis has the property that if $\alpha > 0$ then

$$\int_0^\infty f(\alpha t)\, \frac{dt}{t} = \int_0^\infty f(t)\, \frac{dt}{t}.$$

9.2. **Elementary properties.** 3/19/12

We have $\Gamma(1) = \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \int_\epsilon^N e^{-t}\,dt = \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} -e^{-t}\Big|_\epsilon^N = \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \left[e^{-\epsilon} - e^{-N}\right] = 1$. Next, using integration by parts, if $\Re\mathrm{e}[z] > 0$ we have

$$\begin{aligned}
\Gamma(z+1) &= \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \int_\epsilon^N e^{-t}t^z\,dt \\
&= \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \int_\epsilon^N -\frac{d}{dt}\big(e^{-t}\big)t^z\,dt \\
&= \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \int_\epsilon^N -\frac{d}{dt}\big(e^{-t}t^z\big)\,dt + \lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \int_\epsilon^N e^{-t}\frac{d}{dt}\big(t^z\big)\,dt \\
&= -\lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \left[e^{-N}N^z - e^{-\epsilon}\epsilon^z\right] + z\lim_{\substack{\epsilon\to 0\\ N\to+\infty}} \int_\epsilon^N e^{-t}t^{z-1}\,dt \\
&= z\,\Gamma(z).
\end{aligned}$$

It now follows easily that we have the following facts:

(A) If $\Re\mathrm{e}[z] > 0$ then

$$\Gamma(z+1) = z\Gamma(z).$$

(B) For any positive integer $n \geq 1$, we have

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t}t^n\,dt.$$

(C) For any complex number z with $\Re\mathrm{e}[z] > 0$ and any positive integer $m \geq 1$ we have

$$\Gamma(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}.$$

(D) For any real numbers $s, p > 0$ we have

$$\frac{1}{s^p} = \frac{1}{\Gamma(p)}\int_0^\infty e^{-st}t^{p-1}\,dt$$

9.3. **Meromorphic continuation.**

Lemma 9.1. *There is a meromorphic function defined on the complex plane which agrees with the Gamma function $\Gamma(z)$ for $\Re\mathrm{e}[z] > 0$. This function (also denoted by Γ) has poles precisely at 0 and the negative integers. The poles are all of order 1, and if m is a non-negative integer, the residue of Γ at $-m$ is $(-1)^m[m!]^{-1}$.*

First Proof. For $\Re\mathrm{e}[z] > -m$ define the function

$$g_m(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}.$$

Since $\Re\text{e}[z+m] > 0$, this function is clearly meromorphic, and has simple poles precisely at the points $\{0, -1, -2, \ldots, -m+1\}$. Moreover, the residue at the point $-m+1$ is

$$\begin{aligned}\operatorname{Res}(g_m)\big|_{z=-m+1} &= \frac{\Gamma(-m+1+m)}{(-m+1)(-m+1+1)\cdots(-m+1+m-2)} \\ &= \frac{\Gamma(1)}{(-m+1)(-m+2)\cdots(-2)(-1)} \\ &= \frac{(-1)^{m-1}}{(m-1)!}.\end{aligned}$$

But now if $\Re\text{e}[z] > 0$, it follows that $g_m(z) = \Gamma(z)$ by remark (C) above. Thus the Gamma function extends as a meromorphic function to $\Re\text{e}[z] > -m$, with the required poles. Since m is arbitrary, this completes the proof. □

Second Proof. Put

$$H(z) = \int_1^\infty e^{-t} t^{z-1}\, dt.$$

This improper integral converges for all complex numbers z, and the convergence is uniform on any compact set in $\mathbb{C}$, so H is in fact an entire function. On the other hand, for $0 \le t \le 1$, we can expand the function e^{-t} as a Taylor series

$$e^{-t} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} t^m$$

which converges absolutely and uniformly. Thus for $\Re\text{e}[z] > 0$ we can write

$$\begin{aligned}\Gamma(z) - H(z) &= \int_0^1 e^{-t} t^{z-1}\, dt \\ &= \int_0^1 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} t^{m+z-1}\, dt \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_0^1 t^{m+z-1}\, dt \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{z+m}.\end{aligned}$$

But this series converges for all complex numbers $z \notin \{0, -1, -2, \ldots\}$. In fact, if $|z| \le M$, then for $m \ge 2M$ we have $|z+m| \ge m - |z| \ge 2M - M = M$ and so

$$\sum_{m=2M}^{\infty} \left| \frac{(-1)^m}{m!} \frac{1}{z+m} \right| \le \frac{1}{M} \sum_{m=2M}^{\infty} \frac{1}{m!} \le \frac{e}{M}.$$

Thus the infinite series $\sum_{m=2M}^{\infty} \frac{(-1)^m}{m!} \frac{1}{z+m}$ converges to a holomorphic function on the disk $\{z \in \mathbb{C} : |z| \le M$, while the finite sum $\sum_{m=0}^{2M-1} \frac{(-1)^m}{m!} \frac{1}{z+m}$ is a rational function with simple poles at the points $\{0, -1, \ldots, -2M+1\}$, and with the specified residues. This completes the proof. □

9.4. **The Beta function.**

For $\Re[\alpha] > 0$ and $\Re[\beta] > 0$, define the *Beta function*

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1}\, dt.$$

Making the change of variables $t = 1 - s$ we see that

$$B(\alpha,\beta) = B(\beta,\alpha). \tag{9.1}$$

We can also make the change of variables

$$\begin{aligned} t &= \frac{s}{1+s} = 1 - \frac{1}{1+s}, \\ dt &= \frac{1}{(1+s)^2}\, ds \\ 1-t &= \frac{1}{1+s}. \end{aligned}$$

and we have

$$\begin{aligned} B(\alpha,\beta) &= \int_0^\infty (1+s)^{1-\alpha} s^{\beta-1} (1+s)^{1-\beta} \frac{ds}{(1+s)^2} \\ &= \int_0^\infty \frac{s^{\beta-1}}{(1+s)^{\alpha+\beta}}\, ds. \end{aligned} \tag{9.2}$$

This function of two complex variables is closely connected to the Gamma function. We prove:

Lemma 9.2. *For* $\Re[\alpha] > 0$ *and* $\Re[\beta] > 0$

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} = B(\alpha,\beta).$$

Proof. Making the change of variables $v = us$ we have

$$\begin{aligned} \Gamma(\alpha)\,\Gamma(\beta) &= \int_0^\infty e^{-u} u^\alpha \frac{du}{u} \int_0^\infty e^{-v} v^\beta \frac{dv}{v} \\ &= \int_0^\infty \Big[\int_0^\infty e^{-u-v} u^\alpha v^\beta \frac{dv}{v} \Big] \frac{du}{u} \\ &= \int_0^\infty \Big[\int_0^\infty e^{-u-us} u^{\alpha+\beta} s^\beta \frac{ds}{s} \Big] \frac{du}{u} \\ &= \int_0^\infty s^\beta \Big[\int_0^\infty e^{-u(1+s)} u^{\alpha+\beta} \frac{du}{u} \Big] \frac{ds}{s} \\ &= \int_0^\infty \frac{s^\beta}{(1+s)^{\alpha+\beta}} \Big[\int_0^\infty e^{-u} u^{\alpha+\beta} \frac{du}{u} \Big] \frac{ds}{s} \\ &= \Gamma(\alpha+\beta) \int_0^\infty \frac{s^{\beta-1}}{(1+s)^{\alpha+\beta}}\, ds \\ &= \Gamma(\alpha+\beta)\, B(\alpha,\beta), \end{aligned}$$

which gives the required identity. □

9.5. **The Euler reflection formula.**

Theorem 9.3. *For all $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{9.3}$$

Proof. First suppose that $u \in (0,1)$ so that $1-u \in (0,1)$. Then

$$\begin{aligned}\Gamma(u)\,\Gamma(1-u) &= \Gamma(u+1-u)\,B(u,1-u)\\ &= \Gamma(1)\,B(u,1-u)\\ &= B(u,1-u)\\ &= \int_0^\infty \frac{s^{u-1}}{1+s}\,ds.\end{aligned}$$

We evaluate this integral using a contour $\Sigma = \Sigma(R,\epsilon)$ consisting of the interval (ϵ, R), then a circle centered at the origin of radius R taken counterclockwise, then the interval (R,ϵ), then a circle centered at the origin of radius ϵ taken in the clockwise sense. We also take a branch of the function z^{u-1} with value $|z|^{u-1}$ on the first interval (ϵ, R), but has the value $e^{2\pi i(u-1)}|z|^{u-1} = e^{2\pi i u}|z|^{u-1}$ when we return to the real axis. The only pole of the function $z^{u-1}(1+z)^{-1}$ inside Σ is at the point $z=-1$. Thus

$$\int_\Sigma \frac{z^{u-1}}{1+z}\,dz = 2\pi i \operatorname{Res}\Big(\frac{z^{u-1}}{1+z}\Big)\Big|_{z=-1} = 2\pi i\,(-1)^{u-1} = 2\pi i\, e^{\pi i(u-1)} = -2\pi i e^{\pi i u}.$$

The integral over the circle of radius R tends to zero as R tends to infinity. In fact the length of the curve is on the order of R, but the size of the integrand is on the order of R^{u-2}, and so we can estimate the integral over the large circle by a constant times R^{u-1} which tends to zero since $u-1<0$. Similarly, the integral over the circle of radius ϵ tends to zero as ϵ tends to zero since the length of the curve is ϵ, while the size of the integrand is ont he order of ϵ^{u-1}, and we estimate the integral by $\epsilon^u \to 0$ since $u>0$. Thus we get

$$\int_0^\infty \frac{s^{u-1}}{1+s}\,ds\Big[1-e^{2\pi i u}\Big] = -2\pi i\, e^{\pi i u}$$

and hence

$$\begin{aligned}\int_0^\infty \frac{s^{u-1}}{1+s}\,ds &= \frac{2\pi i e^{\pi i u}}{e^{2\pi i u}-1}\\ &= \frac{2\pi i}{e^{\pi i u}-e^{-\pi i u}}\\ &= \frac{\pi}{\sin(\pi u)}.\end{aligned}$$

Thus we have established that

$$\Gamma(u)\,\Gamma(1-u) = \frac{\pi}{\sin(\pi u)}$$

for all $u \in (0,1)$. However both sides are meromorphic functions, and it follows that we have equality everywhere. This completes the proof. □

Corollary 9.4. $\displaystyle\int_{\mathbb{R}} e^{-x^2}\,dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Corollary 9.5. *The Gamma function $\Gamma(z)$ is never equal to zero. The function $\Gamma(z)^{-1}$ is an entire function with simple zeros at the points $\{0,-1,-2,\dots\}$, and it vanishes nowhere else.*

9.6. **Sterling's Formula.** 3/23/12

We want to study the behavior of $\Gamma(x+1)$ for x real and large. We have

Theorem 9.6 (Sterling's Formula)**.** *For x large and positive*

$$\Gamma(x+1) \sim \left(\frac{x}{e}\right)^x \sqrt{2\pi x}.$$

Precisely, this means that

$$\lim_{x\to+\infty} \Gamma(x+1)\left[\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\right]^{-1} = 1.$$

Proof.

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x\, dt = \int_0^\infty e^{-t+x\log(t)}\, dt = \int_0^\infty e^{-\phi(t)}\, dt$$

where $\phi(t) = t - x\log(t)$. The function ϕ is convex, and in general, the main contribution to integrals of this sort come from the region near where $\phi(t)$ takes on its minimum. But $\phi'(t) = 1 - \frac{x}{t}$ and $\phi''(t) = \frac{x}{t^2}$ so the minimum occurs when $t = x$. If we make the change of variables $t = s + x$, then the maximum occurs when $s = 0$. We have

$$\Gamma(x+1) = \int_{-x}^\infty e^{-s-x+x\log(s+x)}\, ds = e^{-x+x\log(x)} \int_{-x}^\infty e^{-s+x\log\left(1+\frac{s}{x}\right)}\, ds,$$

so

$$\begin{aligned}
\Gamma(x+1)\left[\left(\frac{x}{e}\right)^x \sqrt{2\pi x}\right]^{-1} &= \frac{1}{\sqrt{2\pi x}} \int_{-x}^\infty e^{-s+x\log\left(1+\frac{s}{x}\right)}\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^\infty e^{-t\sqrt{x}+x\log\left(1+\frac{t}{\sqrt{x}}\right)}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^\infty \exp\left[-x\left(\frac{t}{\sqrt{x}} - \log\left(1+\frac{t}{\sqrt{x}}\right)\right)\right] dt.
\end{aligned}$$

Thus to prove Sterling's formula, it suffices to prove that

$$\lim_{x\to+\infty} \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\left(\frac{t}{\sqrt{x}}\right)\right] dt = 1.$$

where for $-1 < s < +\infty$,

$$\psi(s) = s - \log(1+s).$$

Let us begin by trying to see why this is plausible. (A rigorous argument will come later.) For $|s| < 1$ we have

$$\psi(s) = s - \log(1+s) = s - \sum_{m=1}^\infty \frac{(-1)^{m-1}}{m} s^m = \sum_{m=2}^\infty \frac{(-1)^m}{m} s^m.$$

Thus for $|t| < \sqrt{x}$,

$$\begin{aligned}
x\psi\left(\frac{t}{\sqrt{x}}\right) &= x \sum_{m=2}^\infty \frac{(-1)^m}{m} \frac{t^m}{x^{\frac{m}{2}}} = \sum_{m=2}^\infty \frac{(-1)^m}{m} \frac{t^m}{x^{\frac{m}{2}-1}} \\
&= \frac{1}{2}t^2 - \frac{1}{3}\frac{t^3}{\sqrt{x}} + \frac{1}{4}\frac{t^4}{x} - \frac{1}{5}\frac{t^5}{x\sqrt{x}} + \cdots.
\end{aligned}$$

For t fixed, it is clear that

$$\lim_{x\to+\infty} x\psi\Big(\frac{t}{\sqrt{x}}\Big) = \frac{1}{2}t^2.$$

Thus Sterling's formula would seem to follow from the fact that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}t^2\right]\,dt = 1.$$

To make this argument rigorous, we need the following result.

Proposition 9.7. *for $-1 < s < +\infty$, let $\psi(s) = s - \log(s+1)$. Then*

(a) *There is a constant $c > 0$ so that for $\frac{1}{2} \leq s \leq 1$ we have $\psi(s) \geq cs^2$.*
(b) *There is a constant $c > 0$ so that for $1 \leq s < +\infty$ we have $\psi(s) > cs$.*
(c) *There are constants $\{A_j\}$ so that for any positive integer M, if $|s| \leq \frac{1}{2}\sqrt{x}$ we have*

$$\exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right] - \exp\left[-\frac{1}{2}t^2\right] = \sum_{k=1}^{M} A_k$$

Now pick a (large) positive number N, and take $\sqrt{x} >> N$. Using (a) we have

$$\begin{aligned}\int_{-\sqrt{x}}^{-N} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt &\leq \int_{-\sqrt{x}}^{N} e^{-cs^2}\,ds \\ &\leq \int_{-\infty}^{-N} e^{-cs^2} \\ &\leq C\,\exp\big[-\frac{c}{2}N^2\big],\end{aligned}$$

and similarly,

$$\begin{aligned}\int_{N}^{\sqrt{x}} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt &\leq \int_{N}^{\sqrt{x}} e^{-cs^2}\,ds \\ &\leq \int_{N}^{\infty} e^{-cs^2}\,ds \\ &\leq C\,\exp\big[-\frac{c}{2}N^2\big]\end{aligned}$$

where C is an absolute constant. Using (b) we have

$$\begin{aligned}\int_{\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt &\leq \int_{\sqrt{x}}^{\infty} \exp\left[-c\sqrt{x}t\right]\,dt \\ &\leq C\,\exp\big[-\frac{c}{2}x\big].\end{aligned}$$

Thus

$$\left|\int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-N}^{+N} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt\right| \leq 2C\exp\big[-\frac{c}{2}N^2\big] + C\exp\big[-\frac{c}{2}x\big].$$

We now fix N large, and let $x \to +\infty$. We get

$$\limsup_{x\to+\infty}\left|\int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-N}^{+N} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt\right| \le 2C\exp\big[-\frac{c}{2}N^2\big].$$

However,

$$\lim_{x\to\infty} x\psi\left(\frac{t}{\sqrt{x}}\right) = \frac{1}{2}t^2$$

and the convergence is uniform for $|t| \le N$. Therefore

$$\begin{aligned}
&\Big|\int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-\infty}^{+\infty} \exp\Big(-\frac{1}{2}t^2\Big)\,dt\Big| \\
&\quad\le \Big|\int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-N}^{+N} \exp\Big(-\frac{1}{2}t^2\Big)\,dt\Big| + 2\int_N^\infty e^{-\frac{1}{2}t^2}\,dt \\
&\quad\le \Big|\int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-N}^{+N} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt\Big| \\
&\qquad + \Big|\int_{-N}^{+N} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-N}^{+N} \exp\Big(-\frac{1}{2}t^2\Big)\,dt\Big| + 2\int_N^\infty e^{-\frac{1}{2}t^2}\,dt \\
&\quad\le 2C\exp\big[-\frac{c}{2}N^2\big] + \Big|\int_{-N}^{+N} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-N}^{+N} \exp\Big(-\frac{1}{2}t^2\Big)\,dt\Big| + 2\int_N^\infty e^{-\frac{1}{2}t^2}\,dt \\
&\quad\longrightarrow 2C\exp\big[-\frac{c}{2}N^2\big] + 2\int_N^\infty e^{-\frac{1}{2}t^2}\,dt
\end{aligned}$$

as $x \to \infty$. Thus

$$\limsup_{x\to+\infty}\Big|\int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt - \int_{-\infty}^{+\infty} \exp\Big(-\frac{1}{2}t^2\Big)\,dt\Big| \le 2C\exp\big[-\frac{c}{2}N^2\big] + 2\int_N^\infty e^{-\frac{1}{2}t^2}\,dt$$

Now letting $N \to \infty$ we see that

$$\lim_{x\to+\infty} \int_{-\sqrt{x}}^{\infty} \exp\left[-x\psi\Big(\frac{t}{\sqrt{x}}\Big)\right]\,dt = \sqrt{2\pi},$$

and this completes the proof. □

10. Infinite Products

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10.1. Definitions.

In analogy with the definition of infinite sums as the limit of partial sums, it is tempting to say that a (formal) infinite product $\prod_{n=1}^{\infty} z_n$ converges and equals P if and only if $P = \lim_{N\to\infty} P_N$ exists where $P_N = \prod_{n=1}^{N} z_n$ is the partial product. Note however that if any one of the terms $z_n = 0$, then the infinite produce would converge, while one should expect that the convergence or divergence of an infinite product should not depend on the behavior of any finite number of terms, but should only depend on their "eventual behavior". We can avoid this difficulty with the following slightly involved definition.

Definition 10.1. *If $\{z_n\}$ is an infinite sequence of complex numbers, the infinite product $\prod_{n=1}^{\infty} z_n$ converges to a complex number P if and only if*

(a) *At most finitely many of the complex numbers $\{z_n\}$ are equal to zero.*
(b) *If $z_n \neq 0$ for $n \geq M$, and if $P_{N,M} = \prod_{n=M}^{N} z_n$, then $P_M = \lim_{N\to\infty} P_{N,M}$ exists.*
(c) $P = \left[\prod_{n=1}^{M-1} z_n\right] P_M$.

Proposition 10.2. *If the infinite product $\prod_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} a_n = 1$.*

Proof. Suppose that $z_n \neq 0$ for $n \geq M$ and that $P_{N,M} = \prod_{n=M}^{N} z_n$. Then

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{P_{n,M}}{P_{n-1,M}} = \frac{\lim_{n\to\infty} P_{n,M}}{\lim_{n\to\infty} P_{n-1,M}} = \frac{P_M}{P_M} = 1.$$

□

In view of this proposition, it is natural to write $z_n = 1 + a_n$ and consider the infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ where now at most finitely many of the $a_n = -1$, and a necessary condition for convergence is that $\lim_{n\to\infty} a_n = 0$. Also, it is tempting to try to take the logarithm of a product to turn it into a sum. In order to do this we need to choose a branch of the logarithm function. We will use the following "principal branch": For any $z \in \mathbb{C}$ with $z \neq 0$, we cab write $z = re^{i\theta}$ uniquely with $r > 0$ and $-\pi < \theta \leq \pi$. Then

$$\mathrm{Log}(z) = \log(r) + i\theta = \log|z| + i\mathrm{Arg}(z).$$

Proposition 10.3. *Suppose that $1 + a_n \neq 0$ for all n. Then the infinite produce $\prod_{n=1}^{\infty}(1 + a_n)$ converges (to a complex number P) if and only if the infinite sum $\sum_{n=1}^{\infty} Log(1 + a_n)$ converges (to a complex number S), in which case $e^S = P$.*

Proof. Let

$$P_N = \prod_{n=1}^{N}(1 + a_n),$$
$$S_N = \sum_{n=1}^{N}\mathrm{Log}(1 + a_n).$$

Then since $e^{a+b} = e^a e^b$ for any complex numbers a and b we have

$$e^{S_N} = \prod_{n=1}^{N} e^{\mathrm{Log}(1+a_n)} = \prod_{n=1}^{N} e^{\log(r_n)+i\theta_n} = \prod_{n=1}^{N} r_n e^{i\theta_n} = \prod_{n=1}^{N}(1 + a_n) = P_N$$

where we have written each $1+a_n = r_n e^{i\theta_n}$ with $r_n > 0$ and $-\pi < \theta_n \leq +\pi$. Now if $\lim_{n\to\infty} S_N = S$, then since the exponential function is continuous, it follows that

$$\lim_{N\to\infty} P_N = \lim_{N\to\infty} e^{S_N} = e^S.$$

Conversely, suppose that $\lim_{N\to\infty} P_N = P$ exists. Then since $\lim_{N\to\infty} \frac{P_N}{P} = 1$, it follows that

$$\lim_{N\to\infty} \text{Log}\left(\frac{P_N}{P}\right) = 0.$$

Since we are working with a particular branch of the logarithm, it is not always true that $\text{Log}(ab) = \text{Log}(a) + \text{Log}(b)$. However, it is true that

$$\text{Log}(ab) - \text{Log}(a) - \text{Log}(b) = 2\pi i h$$

for some integer h. Thus for each N there is an integer h_N so that

$$\text{Log}\left(\frac{P_N}{P}\right) - S_N + \text{Log}(P) = \text{Log}\left(\frac{P_N}{P}\right) - \sum_{n=1}^{N} \text{Log}(1 + a_{n+1}) + \text{Log}(P) = 2\pi i h_N.$$

It follows that

$$2\pi i(h_{N+1} - h_N) = \text{Log}\left(\frac{P_{N+1}}{P}\right) - \text{Log}(1 + a_{N+1}) - \text{Log}\left(\frac{P_N}{P}\right),$$

and hence

$$2\pi(h_{N+1} - h_N) = \text{Arg}\left(\frac{P_{N+1}}{P}\right) - \text{Arg}\left(\frac{P_N}{P}\right) - \text{Arg}(1 + a_{N+1}).$$

Now

$$\lim_{N\to\infty} \text{Arg}\left(\frac{P_{N+1}}{P}\right) = \lim_{N\to\infty} \text{Arg}\left(\frac{P_N}{P}\right) = 0$$

since $P_{n+1}/P \to 1$ and $P_n/P \to 1$, and $|\text{Arg}(1 + a_{N+1})| \leq \pi$. It follows that eventually we must have $h_{N+1} = h_N = H \in \mathbb{Z}$. It follows that

$$2\pi i H = \lim_{N\to\infty} \left[\text{Log}\left(\frac{P_N}{P}\right) - S_N + \text{Log}(P)\right] = \text{Log}(P) - \lim_{N\to\infty} S_N,$$

and so $\lim_{N\to\infty} S_N$ exists. □

Corollary 10.4. *If the infinite produce $\prod_{n=1}^{\infty}(1+a_n)$ converges to a complex number P, then $P = 0$ if and only if at least one (and at most finitely many) of the terms $1 + a_n = 0$.*

Proof. We have seen that if $1 + a_n \neq 0$ and if the product converges, then $P = e^S \neq 0$ where $S = \sum_{n=1}^{\infty} \text{Log}(1 + a_n)$. □

The product $\prod_{n=1}^{\infty}(1+a_n)$ is said to *converge absolutely* if and only the sum $\sum_{n=1}^{\infty} |\text{Log}(1+a_n)| < +\infty$. Since the absolute convergence of the sum implies that the sum can be arbitrarily rearranged, it follows that if an infinite product converges absolutely, then the terms can also be rearranged arbitrarily. Note that if $|a_n| \leq \frac{1}{2}$, there is a constant $C > 1$ so

$$C^{-1}|a_n| \leq |\text{Log}(1 + a_n)| \leq C|a_n|.$$

It follows that the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

10.2. **Infinite products of holomorphic functions.**

Theorem 10.5. *Let $\{F_n\}$ be an infinite sequence of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$. Suppose that for each compact subset $K \subset \Omega$ there are constants $C_n(K) > 0$ so that*

(a) *For every $z \in K$ and every $n \geq 1$ we have $|F_n(z) - 1| \leq C_n(K)$.*
(b) $\sum_{n=1}^{\infty} C_n(K) < +\infty$.

Then the sequence $P_n(z) = \prod_{n=1}^{N} F_n(z)$ converges uniformly on compact subsets of Ω to a holomorphic function $P(z)$. Moreover, if $z \in \Omega$ and if $F_n(z) \neq 0$ for all n, then $P(z) \neq 0$ and

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Proof. We have $F_n(z) = 1+(F_n(z)-1)$, and the convergence of the product follows from the remarks above. The infinite series $\sum_{n=1}^{\infty}(F_n(z)-1)$ converges absolutely and uniformly on compact subsets of Ω, and hence the same is true of $\sum_{n=1}^{\infty} \mathrm{Log}(F_n(z))$. Since $P_N = \exp[\sum_{n=1}^{N} \mathrm{Log}(F_n(z))$ and since the exponential function is continuous, it follows that P_N converges uniformly on compact subsets. We know that the limit must be holomorphic.

Finally, since P_N converges uniformly to P, it follows that P_N' converges uniformly to P', and on any compact set where $P(z) \neq 0$, it follows that P_N'/P_N converges uniformly to P'/P. Since

$$\frac{P_N'(z)}{P_N(z)} = \sum_{n=1}^{N} \frac{F_N'(z)}{F_N(z)},$$

this completes the proof. □

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10.3. **Example:** $z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = \frac{1}{\pi} \sin(\pi z)$**.**

We begin with the following:

Lemma 10.6. *For $z \in \mathbb{C} \setminus \mathbb{Z}$ we have*

$$\pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)} = \lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Proof. Observe that the series

$$G(z) = \lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

converges to a meromorphic function defined on $\mathbb{C}$ with poles only at the integers, each pole being of order one, with residue 1. Also note that $G(z+1) = G(z)$. Next, let $F(z) = \pi \cot(\pi z)$. Then the function F has the same properties: it is meromorphic with poles only at the integers, each pole of order one with residue 1, and $F(z+1) = F(z)$.

Now it follows that the function $H(z) = F(z) = G(z)$ is an entire function, since the poles are cancelled, and $H(z+1) = H(z)$. If we can prove that H is bounded, it follows that H is constant. Since H is periodic with period 1, it suffices to prove that $|H(z)|$ is bounded for $|\Re\mathrm{e}[z]| \leq 1$. Now being continuous, $|H(z)|$ is certainly bounded on the compact set where $|RE[z]| \leq 1$ and $|\Im\mathrm{m}[z]| \leq 1$. Thus we consider $|H(x+iy)|$ with $|x| \leq 1$ and $|y| \geq 1$. We prove that both $|F(x+iy)|$ and $|G(x+iy)|$ are uniformly bounded when $|y| \geq 1$.

First consider $F(x+iy)$. We have

$$F(x+iy) = \pi i \, \frac{e^{i\pi x}e^{-\pi y} + e^{-i\pi x}e^{+\pi y}}{e^{i\pi x}e^{-\pi y} - e^{-i\pi x}e^{+\pi y}} \quad \text{which we write}$$

$$= \begin{cases} \pi i \big[e^{2\pi i x}e^{-2\pi y} + 1\big]\big[e^{2\pi i x}e^{-2\pi y} - 1\big]^{-1} & \text{when } y \text{ is large positive,} \\ \\ \pi i \,\big[1 + e^{-2\pi i x}e^{+2\pi y}\big]\big[1 - e^{-2\pi i x}e^{+2\pi y}\big]^{-1} & \text{when } y \text{ is large negative.} \end{cases}$$

This is clearly bounded for all $|y| \geq 1$. Next, for $|x| \leq 1$ and $|y| \geq 1$

$$\begin{aligned} |G(x+iy)| &= \left|\frac{1}{x+iy}\right| + \left|\sum_{n=1}^{\infty} \frac{2(x+iy)}{x^2 - y^2 - n^2 + 2ixy}\right| \\ &\leq 1 + 2\sum_{n=1}^{\infty} \frac{1+|y|}{y^2+n^2} \\ &\leq 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{n\leq |y|}^{\infty} \frac{|y|}{y^2+n^2} + 2\sum_{n\geq |y|}^{\infty} \frac{|y|}{y^2+n^2} \\ &\leq 1 + 2\sum_{n=1}^{\infty} \frac{1}{n^2} + 2\sum_{n\leq |y|}^{\infty} \frac{|y|}{y^2} + 2\sum_{n\geq |y|}^{\infty} \frac{|y|}{n^2} \\ &\leq C. \end{aligned}$$

It now follows that $H(z)$ is constant. However, both F and G are odd functions, and so the same is true of H, in which case it follows that $H(z) \equiv 0$. This completes the proof. □

Theorem 10.7. *For all* $z \in \mathbb{C}$ *we have* $z \displaystyle\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) = \frac{1}{\pi}\, \sin(\pi z).$

Proof. First observe that since $\sum_{n=1}^{\infty}\left|\frac{z^2}{n^2}\right| < C|z|^2 < +\infty$, the infinite product converges absolutely and uniformly on compact subsets of $\mathbb{C}$, and defines an entire function $g(z)$. Also, it follows from Theorem 10.5 that we have

$$\frac{g'(z)}{g(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \pi\, \cot(\pi z).$$

Now consider $f(z) = \sin(\pi z)\, g(z)^{-1}$. This is an entire function, since the zeros of g in the denominator are cancelled by the zeros of $\sin(\pi z)$ is the numerator. We have

$$\begin{aligned} f'(z) &= \pi \cos(\pi z) g(z)^{-1} - \sin(\pi z) g(z)^{-2} g'(z) \\ &= \sin(\pi z)\, g(z)^{-1}\left[\pi\, \frac{\cos(\pi z)}{\sin(\pi z)} - \frac{g'(z)}{g(z)}\right] = 0, \end{aligned}$$

so f is constant. However

$$f(z) = \frac{\sin(\pi z)}{\pi z}\, \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)}$$

and it follows that $\lim_{z\to 0} f(z) = 1$. This completes the proof. □

Corollary 10.8 (Wallis Formula)**.**

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{(2n)^2}\right) = \frac{1\cdot 3}{2\cdot 2}\,\frac{3\cdot 5}{4\cdot 4}\,\frac{5\cdot 7}{6\cdot 6}\cdots$$

10.4. **Weierstrass' theorem.** 3/30/12

If f is an entire holomorphic function, then its zeros, counted with multiplicity, form a finite or countable sequence, and in the later case, this sequence has no finite accumulation point. Given any finite set of complex numbers $\{a_1, \dots, a_N\}$, possibly with repetitions, there is a polynomial vanishing precisely on this set:

$$P(z) = \prod_{n=1}^{N}\left(1 - \frac{z}{a_n}\right).$$

For an arbitrary countable sequence, the corresponding infinite product may not converge. However, by introducing appropriate weight functions, we establish the following theorem due to Weierstrass.

Theorem 10.9 (Weierstrass)**.** *Let $\{a_1, a_2, \dots, a_n, \dots\}$ be an infinite sequence of complex numbers (possibly with repetitions) with no finite accumulation point. Then there exists an entire holomorphic function F whose zeros are precisely the element of this sequence. The most general entire holomorphic function with this set of zeros is then of the form $f(z) = F(z)\exp\big[g(z)\big]$ where g is any entire function.*

For the proof, we will need to use *canonical factors* that were defined on the last homework. We want to construct functions $E(z)$ which are equal to zero for $z = 1$ such that $|E(z) - 1|$ is small when $|z|$ is small. For any integer $k \geq 0$ we set

$$E_k(z) = \begin{cases} (1-z) & \text{if } k = 0, \text{ and} \\ \\ (1-z)\exp\big[P_k(z)\big] & \text{if } k \geq 1, \end{cases}$$

where

$$P_k(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^k}{k}.$$

Note that for $|z| < 1$

$$-\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^k}{k} + \cdots,$$

so for $|z| < 1$

$$\begin{aligned} 1 &= (1-z)\frac{1}{(1-z)} \\ &= (1-z)\ \exp\big[-\log(1-z)\big] \\ &= (1-z)\ \exp\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^k}{k} + \cdots\right] \\ &= (1-z)\exp\big[P_k(z)\big]\ \exp\Big[\sum_{k+1}^{\infty}\frac{z^n}{n}\Big] \\ &= E_k(z)\ \exp\Big[\sum_{k+1}^{\infty}\frac{z^n}{n}\Big]. \end{aligned}$$

Thus the canonical factors are truncations of this equation. The following result is Problem 1(a) on the homework:

Lemma 10.10. *There is a constant C independent of k so that if $|z| \leq \frac{1}{2}$, then*

$$\big|1 - E_k(z)\big| \leq C\,|z|^{k+1}.$$

Proof. We have

$$1 - E_k(z) = 1 - \exp\big[-\sum_{n=k+1}^{\infty}\frac{z^n}{n}\big].$$

Now if $|z| \leq \frac{1}{2}$,

$$\Big| - \sum_{n=k+1}^{\infty}\frac{z^n}{n}\Big| \leq \frac{|z|^{k+1}}{k+1}\Big[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\Big] \leq \frac{2}{k+1}|z|^{k+1}.$$

The result follows since $|1 - e^w| \leq C_R|w|$ for $|w| \leq R$. □

Proof of the theorem of Weierstrass. Let $\{a_n\}$ be an infinite sequence of complex numbers so that $\lim_{n\to\infty}|a_n| = \infty$. Consider the infinite product

$$\prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right).$$

If we can show that this product converges absolutely for z in any compact susbet of $\mathbb{C}$, it follows that the product defines an entire function which equals zero if and only if one of the factors equals zero. But $E_n\left(\frac{z}{a_n}\right) = 0$ if and only if $z = a_n$.

Fix $R > 0$ and fix $|z| \leq R$. There exists $N(R)$ so that $n \geq N(R)$ implies $|a_n| \geq 2R$. Thus if $|z| \leq R$ and $n \geq N(R)$ then $|z|/|a_n| \leq \frac{1}{2}$, and so we have

$$\big|1 - E_n\left(\frac{z}{a_n}\right)\big| \leq C\left(\frac{|z|}{|a_n|}\right)^{n+1} \leq C\,\frac{1}{2^{n+1}}.$$

□

10.5. **Jensen's formula.** 4/9/12

Theorem 10.11 (Jensen's Formula)**.** *Let f be holomorphic in an open neighborhood of the closure $\overline{\mathbb{D}}$ of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Suppose that $f(0) \neq 0$ and that, counting multiplicities,*

$$\{z \in \mathbb{D} : f(z) = 0\} = \{a_1, \ldots, a_N\}.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\,d\theta = \log|f(0)| - \sum_{j=1}^{N}\log|a_j|.$$

Proof. First suppose that $\{z \in \mathbb{D} : f(z) = 0\} = \emptyset$. Then there is a function g holomorphic in an open neighborhood of $\overline{\mathbb{D}}$ such that $f(z) = \exp[g(z)]$. It follows that $\log|f(z)| = \Re\mathrm{e}[g(z)]$, which is an harmonic function, and hence satisfies the mean value property. Thus in the case we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\Re\mathrm{e}[g(e^{i\theta})]\,d\theta = \Re\mathrm{e}[g(0)] = \log|f(0)|,$$

which is the desired formula.

Now suppose $\{z \in \mathbb{D} : f(z) = 0\} = \{a_1, \ldots, a_N\} \neq \emptyset$. Consider the product of Blaschke factors

$$h(z) = \prod_{j=1}^{N}\frac{z - a_j}{1 - \bar{a}_j z}.$$

Note that

$$|h(0)| = \prod_{j=1}^{N} |a_j|.$$

Then $\{z \in \mathbb{D} : h(z) = 0\} = \{a_1, \ldots, a_N\}$, h is holomorphic in an open neighborhood of $\overline{\mathbb{D}}$, and $|h(z)| = 1$ when $|z| = 1$. It follows that the function $F(z) = f(z)\,h(z)^{-1}$ is also holomorphic in a neighborhood of $\overline{\mathbb{D}}$, has no zeros in $\mathbb{D}$, and $|F(z)| = |f(z)|$ when $|z| = 1$. Using our first result, we have

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\,d\theta \\
&= \log|F(0)| \\
&= \log|f(0)h(0)^{-1}| \\
&= \log|f(0)| - \log|h(0)| \\
&= \log|f(0)| - \sum_{j=1}^{N} \log|a_j|.
\end{aligned}$$

This completes the proof. ☐

Now let F be an entire function with $F(0) \neq 0$ whose set of zeros, with multiplicity, is given by the sequence $\{a_1, a_2, \ldots, a_n, \ldots\}$. We can assume that $0 < |a_1| \leq |a_2| \leq \cdots \leq |a_n| \leq \cdots$, and $\lim_{n\to\infty} |a_n| = +\infty$. Choose $R > 0$ and an integer N so that $|a_N| < R < |a_{N+1}|$. The function $f_R(z) = F(Rz)$ is certainly holomorphic in a neighborhood of $\overline{\mathbb{D}}$, does not equal zero at $z = 0$ or when $|z| = 1$, and $\{z \in \mathbb{D} : f_R(z) = 0\} = \{a_1R^{-1}, \ldots, a_NR^{-1}\}$. If we apply Jensen's formula, we get

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \log|f_R(e^{i\theta})|\,d\theta \\
&= \log|f_R(0)| - \sum_{j=1}^{N} \log|a_jR^{-1}| \\
&= \log|F(0)| - \sum_{j=1}^{N} \log\left|\frac{a_j}{R}\right|
\end{aligned}$$

We also introduce a function which counts the number of zeros of an entire function. Set

$$\mathfrak{n}_F(r) = \text{ the number of zeros } \{a_j\} \text{ of } F \text{, counted with multiplicity, with } |a_j| < r.$$

Note that if $F(0) \neq 0$ then $\mathfrak{n}_F(r) = 0$ on some interval $r \in [0, \epsilon)$.

Proposition 10.12. *Let F be an entire function with $F(0) \neq 0$, and suppose that the zeros of F, counted with multiplicity and in order of increasing absolute value are $\{a_1, a_2, \ldots, a_N, \ldots\}$. Then for any $R > 0$, if $|a_N| < R < |a_{N+1}|$, then*

$$\int_0^R \mathfrak{n}(r)\,\frac{dr}{r} = \sum_{j=1}^{N} \log\left|\frac{R}{a_j}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(Re^{i\theta})|\,d\theta - \log|F(0)|.$$

Proof. Let

$$\chi_j(r) = \begin{cases} 1 & \text{if } r > |a_j|, \text{ and} \\ 0 & \text{if } r \leq |a_j|. \end{cases}$$

Then

$$\begin{aligned}\sum_{j=1}^{N} \log\left|\frac{R}{a_j}\right| &= \sum_{j=1}^{N} \int_{|a_j|}^{R} \frac{dr}{r} \\ &= \sum_{j=1}^{N} \int_0^R \chi_j(r)\,\frac{dr}{r} \\ &= \int_0^R \Big[\sum_{j=1}^{N} \chi_j(r)\Big]\,\frac{dr}{r} \\ &= \int_0^R \mathfrak{n}(r)\,\frac{dr}{r}.\end{aligned}$$

The rest of the identity follows from Jensen's formula. □

10.6. **Functions of finite order.**

We say that an entire function F has an *order of growth* $\rho > 0$ if F satisfies an inequality

$$|F(z)| \leq Ce^{\sigma|z|^\rho}.$$

The precise order of growth for F is then

$$\rho_F = \inf\big\{\rho > 0 \,:\, \big(\exists C, \sigma\big)\big(|F(z)| \leq Ce^{\sigma|z|^\rho}\big)\big\}.$$

Note that if $|F(z)| \leq Ce^{\sigma|z|^\rho}$, then for $|z|$ sufficiently large

$$\log|F(z)| \leq \log C + \sigma|z|^\rho \leq \sigma'|z|^\rho$$

and so

$$\log\log|F(z)| \leq \log\sigma' + \rho\,\log|z|.$$

Thus

$$\frac{\log\log|F(z)|}{\log|z|} \leq \frac{\log\sigma'}{\log|z|} + \rho.$$

It follows that the precise order of growth of F is

$$\rho_F = \limsup_{|z|\to+\infty} \frac{\log\log|F(z)|}{\log|z|}.$$

Theorem 10.13. *Let F be an entire function with precise order of growth less than or equal to ρ, and suppose that $F(0) \neq 0$. Then*

(a) *There is a constant $C = C(F) > 0$ and an $R = R(F) > 0$ so that for all $r \geq R$ we have $\mathfrak{n}_F(r) \leq Cr^\rho$.*

(b) *If the zeros of F are given by $\{a_1, a_2, \ldots, a_N, \ldots\}$ with $0 < |a_1| \leq |a_2| \leq \cdots \leq |a_N| \leq \cdots$, then for any $s > \rho$,*

$$\sum_{j=1}^{\infty} |a_j|^{-s} < +\infty.$$

Proof. Since $\mathfrak{n}_F(t)$ is an increasing function of t, we have

$$\begin{aligned}
\mathfrak{n}_F(r)\,\log(2) &= \mathfrak{n}_F(r)\int_r^{2r}\frac{dt}{t}\\
&\le \int_r^{2r}\mathfrak{n}_F(t)\,\frac{dt}{t}\\
&\le \int_0^{r}\mathfrak{n}_F(t)\,\frac{dt}{t}\\
&= \frac{1}{2\pi}\int_0^{2\pi}\log|F(2re^{i\theta})|\,d\theta - \log|F(0)|\\
&\le \frac{1}{2\pi}\int_0^{2\pi}\log|Ce^{\sigma(2r)^\rho}|\,d\theta - \log|F(0)|\\
&= \log C - \log|F(0)| + \sigma 2^\rho r^\rho
\end{aligned}$$

For large r this is bounded by $C\,r^\rho$, which establishes (a).

To establish (b), let $s > \rho$. Then

$$\begin{aligned}
\sum_{|a_j|\ge 1}|a_j|^{-s} &= \sum_{k=0}^{\infty}\left(\sum_{2^k\le|a_j|<2^{k+1}}|a_k|^{-s}\right)\\
&\le \sum_{k=0}^{\infty}2^{-ks}\mathfrak{n}_F(2^{k+1})\\
&\le \sum_{k=0}^{\infty}2^{-ks}\,2^{(k+1)\rho}\\
&= 2^\rho\sum_{k=1}^{\infty}2^{-k(s-\rho)} < +\infty
\end{aligned}$$

since $s-\rho>0$. This completes the proof. □

10.7. **Hadamard's Factorization Theorem.**

Theorem 10.14 (Hadamard)**.** *Suppose that F is an entire function with precise order of growth ρ_F. Let k be the unique non-negative integer such that $k \le \rho_F < k+1$. Suppose that F has a zero of order $m \ge 0$ at the origin $z=0$, and that the non-zero solutions of $F(z)=0$ are given by $\{a_1,a_2,\dots,a_N,\dots\}$. The*

$$F(z) = z^m\prod_{n=1}^{\infty}E_k\left(\frac{z}{a_n}\right)\exp\left[P(z)\right]$$

where P is a polynomial of degree at most k.

Example 1: Let $F(z)=\sin(\pi z)$. It is not hard to check that the precise order of growth of F is $\rho_F=1$. Thus in Hadamard's theorem, we take $k=1$. $F(z)=0$ is satisfies at $\{0,\pm1,\pm2,\dots,\}$, and it follows that there is a polynomial $P(z)=A+Bz$ of degree 1 so that

$$\sin(\pi z) = z\prod_{n\ne 0}E_1\left(\frac{z}{n}\right)e^{A+Bz} = z\prod_{n\ne0}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\,e^{A+Bz}.$$

We can rewrite the product

$$\sin(\pi z) = z \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \left(1 - \frac{z}{-n}\right) e^{-\frac{z}{n}} \right] e^{A+Bz}$$
$$= z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) e^{A+Bz}.$$

It follows easily that e^{A+Bz} is an even function of z, which forces $B = 0$. Then letting $z = 0$ we see that $A = \log(\pi)$.

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Example 2: Let $F(z) = \Gamma(z)^{-1}$. We already know that

$$\frac{1}{\Gamma(z)} = \Gamma(1-z)\, \frac{\sin(\pi z)}{\pi}$$

and the poles of $\Gamma(1-z)$ which occur at $1, 2, \ldots, N, \ldots$ are cancelled by the zeros of $\sin(\pi z)$. It follows that $\Gamma(z)^{-1}$ is an entire function, with simple zeros at $0, -1, -2, \ldots, -N, \ldots$.. We will prove later that Γ^{-1} is of order 1. It then follows from Hadamard's theorem that

$$\frac{1}{\Gamma(z)} = z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\, e^{A+Bz}.$$

Now

$$\lim_{z\to 0} \frac{1}{z\Gamma(z)} = 1.$$

It follows that $A = 0$. Then putting $z = 1$, we have

$$e^{-B} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}}$$
$$= \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 + \frac{1}{n}\right) e^{-\frac{1}{n}}$$
$$= \lim_{N\to\infty} \exp\left[\sum_{n=1}^{N} \log\left(1 + \frac{1}{n}\right) - \frac{1}{n} \right]$$
$$= \lim_{N\to\infty} \exp\left[\log(N+1) - \sum_{n=1}^{N} \frac{1}{n} \right]$$
$$= e^{-\gamma}$$

where

$$\gamma = \lim_{N\to\infty} \left[\sum_{n=1}^{N} \frac{1}{n} - \log(N+1) \right]$$

is called Euler' constant. Note that this limit exists since

$$\sum_{n=1}^{N} \frac{1}{n} - \log(N+1) = \sum_{n=1}^{N} \frac{1}{n} - \int_1^{N+1} \frac{dt}{t}$$
$$= \sum_{n=1}^{N} \int_n^{n+1} \left[\frac{1}{n} - \frac{1}{t} \right] dt.$$

But

$$\left| \int_n^{n+1} \left[\frac{1}{n} - \frac{1}{t} \right] dt \right| \leq C\, n^{-2}$$

and so the series converges. Thus once we prove that $\Gamma(z)^{-1}$ is an entire function of order 1, we have

Theorem 10.15. *The entire function* $\Gamma(z)^{-1}$ *has the product representation*
$$\frac{1}{\Gamma(z)} = e^{-\gamma} z \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

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Lemma 10.16. *There are constants* $c_1, c_2 > 0$ *so that for all* $z \in \mathbb{C}$ *we have*
$$\left|\frac{1}{\Gamma(z)}\right| \le c_1\, e^{c_2|z|\log(|z|)}.$$

Proof. Recall that for $\Re\mathrm{e}[z] > 0$ we have
$$\begin{aligned}\Gamma(z) &= \int_0^1 e^{-t}t^{z-1}\,dt + \int_1^\infty e^{-t}t^{z-1}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^1 t^{n+z-1}\,dt + \int_1^\infty e^{-t}t^{z-1}\,dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{n+z} + \int_1^\infty e^{-t}t^{z-1}\,dt,\end{aligned}$$
and we have seen that the second integral defines an entire function of z, while the first sum converges absolutely and uniformly on compact subsets of $\mathbb{C}$ that do not contain any of the points in the set $\{0, -1, -2, \ldots\}$. Now by Euler's reflection formula, we have
$$\frac{\pi}{\Gamma(z)} = \Gamma(1-z)\,\sin(\pi z)$$
Thus for all $z \notin \{1, 2, 3, \ldots\}$ we have
$$\frac{\pi}{\Gamma(z)} = \sin(\pi z)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{n+1-z} + \sin(\pi z)\int_1^\infty e^{-t}t^{-z}\,dt$$
We estimate the two terms separately.

We begin with the integral. Write $\Re\mathrm{e}[z] = x$, and choose n so that $n \le |x| < n+1$. Then
$$\begin{aligned}\left|\int_1^\infty e^{-t}t^{-z}\,dt\right| &\le \int_1^\infty e^{-t}\left|t^{-z}\right|dt\\ &= \int_1^\infty e^{-t}t^{-\Re\mathrm{e}[z]}\,dt\\ &\le \int_1^\infty e^{-t}t^{|x|}\,dt\\ &\le \int_0^\infty e^{-t}t^{n+1}\,dt\\ &= n!\\ &\le n^n = e^{n\log(n)}\\ &\le e^{(|x|+1)\log(|x|+1)}\\ &\le C_1 e^{|z|\log(|z|)}.\end{aligned}$$
Since $|\sin(\pi z)| \le Ce^{\pi|z|}$, it follows that the entire function $\sin(\pi z)\int_1^\infty e^{-t}t^{-z}\,dt$ has order 1.

We now consider the term $\sin(\pi z)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{n+1-z}$. (Note that this too is an entire function since the zeros of $\sin(\pi z)$ cancel the poles of the sum.) If $|\Im\text{m}[z]|\geq 1$ it is easy to check that

$$\begin{aligned}\left|\sin(\pi z)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{n+1-z}\right| &\leq \big|\sin(\pi z)\big|\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\left|\frac{1}{n+1-z}\right| \\ &\leq e\,\big|\sin(\pi z)\big| \\ &\leq C\,e^{\pi|z|}.\end{aligned}$$

Finally, suppose that $|\Im\text{m}[z]|\leq 1$, and choose the integer k so that $k-\frac{1}{2}\leq x=\Re\text{e}[z]<K+\frac{1}{2}$. If $k\leq 0$, then again we can dominate the term $\left|\frac{1}{n+1-z}\right|$ by a constant, and we can estimate everything by a constant. If $k\geq 1$ we have

$$\sin(\pi z)\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{n+1-z}=(-1)^{k-1}\frac{\sin(\pi z)}{(k-1)!(k-z)}+\sin(\pi z)\sum_{n\neq k}\frac{(-1)^n}{n!}\frac{1}{n+1-z}.$$

The first term is bounded since the zero of $\sin(\pi z)$ at $z=k$ cancels the term $(k-z)$, and the second sum is bounded as before. This completes the proof. □

10.8. **Proof of Hadamard's Theorem.**

To prove Hadamard's theorem, we follow Stein and first establish several lemmas. Recall that the canonical product $E_k(z)$ is given by

$$E_k(z)=(1-z)\exp\left[z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}\right].$$

Lemma 10.17. *Let $\epsilon>0$. There are constants c and c' independent of k so that*

$$|E_k(z)|\geq\exp\big[-c|z|^{k+1}\big]\qquad\text{if } |z|\leq\frac{1}{2},$$

and

$$|E_k(z)|\geq|1-z|\,\exp\big[-c'|z|^k\big]\qquad\text{if } |z|\geq\frac{1}{2}$$

Proof. The case $k=0$ is easy, so we shall assume that $k\geq 1$. First suppose that $|z|\leq\frac{1}{2}$. Then

$$E_k(z)=\exp\left[\log(1-z)+\sum_{n=1}^{k}\frac{z^n}{n}\right]=\exp\left[-\sum_{n=k+1}^{\infty}\frac{z^n}{n}\right]=e^w$$

where

$$w=\sum_{n=k+1}^{\infty}\frac{z^n}{n}=z^{k+1}\left[\frac{1}{k+1}+\frac{z}{k+2}+\cdots+\frac{z^n}{n+k+1}+\cdots\right]$$

For $|z|\leq\frac{1}{2}$ we thus have

$$|w|\leq|z|^{k+1}\big[1+|z|+|z|^2+\cdots+|z|^n+\cdots\big]\leq 2|z|^{k+1}.$$

Thus for $|z|\leq\frac{1}{2}$

$$|E_k(z)|=\exp\big[\Re\text{e}[w]\big]\geq\exp\big[-|w|\big]\geq e^{-2|z|^{k+1}}.$$

To establish the second inequality, note that it suffices to show that

$$\left|\exp\left[z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right]\right| \geq \exp\left[- c'_\epsilon |z|^k\right] \tag{10.1}$$

if $|z| \geq \frac{1}{2}$. As in the text, we have

$$\left|\exp\left[z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right]\right| \geq \exp\left[-\left|z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right|\right]$$

and if $|z| \geq 2$ we have

$$\begin{aligned}\left|z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}\right| &\leq |z|^k \left[\frac{1}{k} + \frac{1}{k-1}\left(\frac{1}{2}\right) + \frac{1}{k-2}\left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^k\right] \\ &\leq |z|^k \left[1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^k\right] \\ &< 2|z|^k.\end{aligned}$$

Thus if $|z| \geq 2$ we have

$$|E_k(z)| \geq |1-z|e^{-2|z|^k}.$$

Now suppose that $\frac{1}{2} \leq |z| \leq 2$. We want to show that there is a constant $c' > 0$ so that

$$\Re\mathrm{e}\left[z + \frac{z^2}{2} + \cdots + \frac{z^N}{N}\right] \geq -c'|z|^k.$$

We write $z = re^{i\theta}$.

□

CASE 1: $N|\theta| \leq \frac{\pi}{4}$**.** Then for $1 \leq n \leq N$ we have

$$\Re\mathrm{e}\left(e^{in\theta}\right) = \cos(n\theta) \geq \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Thus for $r \geq 0$

$$RE\left[\sum_{n=1}^{N} r^n \frac{e^{in\theta}}{n}\right] \geq \sum_{n=1}^{N} r^n \frac{\cos(n\theta)}{n} \geq \frac{1}{\sqrt{2}} \sum_{n=1}^{N} \frac{r^n}{n} \geq 0. \tag{10.2}$$

CASE 2: $N|\theta| \geq \pi/4$ **and** $r \leq 1$**.** Let

$$F_N(r,\theta) = re^{i\theta} + \frac{1}{2}f^2e^{2i\theta} + \cdots + \frac{1}{N}r^N e^{iN\theta}.$$

Then if $r \neq 1$ or $\theta \neq 0$,

$$\begin{aligned}\frac{\partial F_n}{\partial r}(r,\theta) &= e^{i\theta} + re^{2i\theta} + r^2e^{3i\theta} + \cdots + r^{n-1}e^{in\theta} \\ &= e^{i\theta}\sum_{k=0}^{n-1} r^k e^{ik\theta} \\ &= e^{i\theta}\,\frac{1 - r^n e^{in\theta}}{1 - re^{i\theta}}.\end{aligned}$$

It follows that for $\theta \neq 0$,

$$\begin{aligned} F_N(r,\theta) &= e^{i\theta} \int_0^r \frac{1-s^N e^{iN\theta}}{1-se^{i\theta}}\, ds \\ &= \int_0^r \frac{e^{i\theta}\, ds}{1-se^{i\theta}} - e^{iN\theta} \int_0^r \frac{s^N}{1-se^{i\theta}}\, ds \\ &= -\log\left(1-re^{i\theta}\right) - e^{iN\theta}\int_0^r \frac{s^N}{1-se^{i\theta}}\, ds. \end{aligned}$$

It follows that

$$\begin{aligned} \Re\mathrm{e}\left[re^{i\theta} + \frac{1}{2}f^2 e^{2i\theta} + \cdots + \frac{1}{N} r^N e^{iN\theta}\right] &= \log\left[\frac{1}{|1-re^{i\theta}|}\right] - \Re\mathrm{e}\left[e^{iN\theta}\int_0^r \frac{s^N}{1-se^{i\theta}}\, ds\right] \\ &\geq \log\left[\frac{1}{|1-re^{i\theta}|}\right] - \int_0^r \frac{s^N}{\left|1-se^{i\theta}\right|}\, ds \\ &\geq \log\left[\frac{1}{|1-re^{i\theta}|}\right] - |\sin(\theta)|^{-1}\int_0^r s^N\, ds \\ &\geq \log\left[\frac{1}{|1-re^{i\theta}|}\right] - |\sin(\theta)|^{-1}(N+1)^{-1} r^{N+1} \\ &\geq \log\left[\frac{1}{|1-re^{i\theta}|}\right] - \frac{1}{N|\sin(\theta)|}\, r^{N+1} \\ &\geq \log\left[\frac{1}{|1-re^{i\theta}|}\right] - \frac{4C}{\pi}\, r^{N+1} \end{aligned}$$

Note that in the third inequality, we needed a bound from below on $|1-se^{i\theta}|$. We have

$$\begin{aligned} \left|1-se^{i\theta}\right|^2 &= |(1-s\cos(\theta)) - is\sin(\theta)|^2 \\ &= 1-2s\cos(\theta) + s^2\cos^2(\theta) + s^2\sin^2(\theta) \\ &= 1-2s\cos(\theta)+s^2. \end{aligned}$$

As a function of s, this has a minimum when $s = \cos(\theta)$. thus

$$\left|1-se^{i\theta}\right|^2 \geq 1-\cos^2(\theta) = \sin^2(\theta)$$

so

$$|1-se^{i\theta}| \geq |\sin(\theta)|.$$

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10.9. **Completion of the proof of the theorem of Hadamard.**

Recall we have the following estimates:

$$\begin{aligned} |E_k(z)| &\geq e^{-C|z|^{k+1}} & &\text{if } |z| \leq \frac{1}{2}, \\ |E_k(z)| &\geq e^{-C_k|z|^k} & &\text{if } |z| \geq \frac{1}{2}. \end{aligned}$$

Also if F is an entire function of precise order of growth ρ_F, if k is the unique nonnegative integer such that $k \leq \rho_F < k+1$, and if the zeros of F, listed with multiplicity, are $\{a_1, a_2, \ldots, a_n, \ldots\}$, then for any $s > \rho_F$ we have

$$\sum_{n=1}^{\infty} |a_n|^{-s} < +\infty.$$

If $\mathfrak{n}_F(r)$ is the number of zeros of F in the disk $\{|z| < r\}$, then

$$\mathfrak{n}_F(r) \le C\, r^s.$$

Lemma 10.18. *Let F be an entire function with precise order of growth ρ_F, and suppose k is the unique non-negative integer such that $k \le \rho_F < k+1$. Let the zeros of F be given (with multiplicity) by $\{a_1, a_2, \dots, \}$. Choose a real number s so that $\rho_F < s < k+1$. Then there is a constant C (depending on k and s and F but independent of $|z|$) so that*

$$\left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \ge \exp\left[-C|z|^s\right],$$

for all $z \in \mathbb{C}$ such that $|z - a_n| > |a_n|^{-k-1}$.

Proof. Write

$$\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) = \prod_{|a_n| \le 2|z|} E_k\left(\frac{z}{a_n}\right) \prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right) = \Pi_1\, \Pi_2.$$

Now our first estimate

$$\begin{aligned} \left|\Pi_2\right| &= \prod_{\left|\frac{z}{a_n}\right| < \frac{1}{2}} \left| E_k\left(\frac{z}{a_n}\right) \right| \\ &\ge \prod_{\left|\frac{z}{a_n}\right| < \frac{1}{2}} e^{-c|z/a_n|^{k+1}} \\ &= e^{-c|z|^{k+1} \sum_{|a_n| > 2|z|} |a_n|^{-k-1}}. \end{aligned}$$

But since each $|a_n| > 2|z|$ and $s - k - 1 < 0$ we have

$$|a_n|^{-k-1} = |a_n|^{-s} |a_n|^{s-k-1} \le C_k |a_n|^{-s} |z|^{s-k-1}.$$

Thus

$$\begin{aligned} \left|\Pi_2\right| &\ge e^{-C_k |z|^{k+1}\, |z|^{s-k-1}\, \sum_{|a_n > 2|z|} |a_n|^{-s}} \\ &\ge e^{-C_k |z|^{k+1}\, |z|^{s-k-1}\, \sum_{n=1}^{\infty} |a_n|^{-s}} \\ &\ge e^{-C_k |z|^{-s}}. \end{aligned}$$

To estimate Π_1, we use the second estimate, and have

$$\begin{aligned} \left|\Pi_1\right| &\le \prod_{|a_n| < 2|z|} \left| 1 - \frac{z}{a_n} \right| \prod_{|a_n| < 2|z|} e^{-C_k |z/a_n|^k} \\ &= \prod_{|a_n| < 2|z|} \frac{|a_n - z|}{|a_n|} \exp\left[-C_k \sum_{|a_n| < 2|z|} \left| \frac{z}{a_n} \right|^k \right] \\ &\ge \left(\prod_{|a_n| < 2|z|} |a_n|^{-k-2} \right) \exp\left[-C_k |z|^k \sum_{|a_n| < 2|z|} |a_n|^{-k} \right]. \end{aligned}$$

Now since $s - k > 0$,

$$|a_n|^{-k} = |a_n|^{-s} |a_n|^{s-k} \le C_k\, |a_n|^{-s} |z|^{s-k},$$

and so

$$\begin{aligned}\exp\left[-C_k|z|^k \sum_{|a_n|<2|z|} |a_n|^{-k}\right] &\geq \exp\left[-C_k|z|^s \sum_{|a_n|<2|z|} |a_n|^{-s}\right] \\ &\geq \exp\left[-C_k|z|^s \sum_{n=1}^{\infty} |a_n|^{-s}\right] \\ &\geq \exp\left[-C_k'|z|^s\right].\end{aligned}$$

On the other hand, 3/20/12

$$\begin{aligned}\log\left(\prod_{|a_n|<2|z|} |a_n|^{-k-2}\right) &= -(k+2) \sum_{|a_n|<2|z|} \log(|a_n|) \\ &\geq -(k+2)\mathfrak{n}(2|z|)\,\log(2|z|) \\ &\geq -C_k|z|^s \log(|z|) \\ &\geq -C_k'|z|^{s+\epsilon}\end{aligned}$$

for any $\epsilon > 0$. This completes the proof. □

Corollary 10.19. *There exists a sequence of radii $\{r_n\}$ with $\lim_{n\to\infty} r_n = +\infty$ so that*

$$\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \geq e^{-C_k|z|^s}$$

whenever $|z| \in \{r_1, r_2, \ldots, r_n, \ldots\}$.

The proof of Hadamard's theorem now follows from the following easy lemma.

Lemma 10.20. *Let g be an entire function, and let $u = \Re\mathrm{e}[g]$. Suppose that*

$$u(z) \leq C\, r_n^s$$

when $|z| = r_n$ for a sequence of real numbers $\{r_n\}$ tending to positive infinity. Then g is a polynomial of degree at most s.

10.10. **An example.**

Let n be a positive integer, and put

$$F_n(z) = \int_{-\infty}^{+\infty} e^{tz-t^{2n}}\, dt.$$

Then when $n = 1$, the equation $F_1(z) = 0$ has no solutions, but for $n > 1$ each equation $F_n(z) = 0$ has infinitely many solutions.

11. The Fourier Transform

Let f be a Lebesgue integrable function on $\mathbb{R}$. Then the Fourier transform of f is the function

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\, dx. \tag{11.1}$$

11.1. **Basic Properties.** We have the following properties of the Fourier transform:

A) The function $\widehat{f}$ is bounded. In fact

$$||\widehat{f}||_\infty \equiv \sup_{\xi\in\mathbb{R}^n} |\widehat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x)|\,dx \equiv ||f||_1. \tag{11.2}$$

B) The function $\widehat{f}$ is continuous. This follows from the Lebesgue dominated convergence theorem.

C) The rate of decay of the Fourier transform of f at infinity is related to the smoothness of f. Thus, for example, if f is continuously differentiable and if both f and f' are integrable, then for $\xi \neq 0$,

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\,dx \\ &= \frac{-1}{2\pi\xi}\int_{\mathbb{R}} \frac{d}{dx}\left(e^{-2\pi i x\xi}\right) f(x)\,dx \\ &= \frac{-1}{2\pi\xi}\left(e^{-2\pi i x\xi} f(x)\right)\Big|_{-\infty}^{+\infty} + \frac{1}{2\pi i\xi}\int_{\mathbb{R}} e^{-2\pi i x\xi} f'(x)\,dx \\ &= \frac{1}{2\pi i\xi}\int_{\mathbb{R}} e^{-2\pi i x\xi} f'(x)\,dx \\ &= \frac{1}{2\pi i\xi}\widehat{f'}(\xi).\end{aligned}$$

It follows that

$$|\widehat{f}(\xi)| \le C(1+|\xi|)^{-1}||f'||_{L^1(\mathbb{R})}.$$

More generally, if f is n-times continuously differentiable, will all derivatives integrable, then

$$|\widehat{f}(\xi)| \le C(1+|\xi|)^{-n}||f'^{(n)}||_{L^1(\mathbb{R})}.$$

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D) (Riemann-Lebesgue Lemma) If $f \in L^1(\mathbb{R})$, then $\lim_{|\xi|\to\pm\infty} \widehat{f}(\xi) = 0$.

Proof. Let $\epsilon > 0$. The space of infinitely differentiable functions with compact support is dense in $L^1(\mathbb{R})$, and so we can find $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ so that $||f-\varphi||_1 < \frac{1}{2}\epsilon$. Then

$$\begin{aligned}|\widehat{f}(\xi)| &\le |\widehat{\varphi}(\xi)| + |\widehat{(f-\varphi)}(\xi)| \\ &\le |\widehat{\varphi}(\xi)| + ||f-\varphi||_{L^1(\mathbb{R})} \\ &\le C(1+|\xi|)^{-1}||\varphi'||_{L^1} + \frac{1}{2}\epsilon < \epsilon\end{aligned}$$

if $|\xi|$ is sufficiently large. This completes the proof. ∎

E) Let $f \in L^1(\mathbb{R})$, and for $x \in \mathbb{R}$ let $f_x(t) = f(t+x)$. Then

$$\begin{aligned}\widehat{f_x}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i t\xi} f_x(t)\,dt \\ &= \int_{\mathbb{R}} e^{-2\pi i x\xi} f(t+x)\,dt \\ &= \int_{\mathbb{R}} e^{-2\pi i (t-x)\xi} f(t)\,dt \\ &= e^{2\pi i x\xi}\widehat{f}(\xi).\end{aligned}$$

Definition 11.1. *If $f, g \in L^1(\mathbb{R})$, the convolution $f * g$ is defined by the integral*

$$f * g(x) = \int_{\mathbb{R}} f(x-t)g(t)\,dt = \int_{\mathbb{R}} f(s)g(x-s)\,ds.$$

It follows from Tonelli's Theorem that

$$\begin{aligned}\left|\int_{\mathbb{R}} f * g(x)\,dx\right| &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-t)|\,|g(t)|\,dt\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-t)|\,|g(t)|\,dx\,dt \\ &= ||f||_{L^1}\,||g||_{L^1}\end{aligned}$$

and so the integral defining the convolution converges absolutely for almost every $x \in \mathbb{R}$, and the result is again in $L^1(\mathbb{R})$.

Proposition 11.2. *If $f,\, g \in L^1(\mathbb{R})$, then*

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\,\widehat{g}(\xi).$$

Proof. We have

$$\begin{aligned}\widehat{f * g}(\xi) &= \int_{\mathbb{R}} e^{-2\pi i x\xi} f * g(x)\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i x\xi} f(x-t)\,g(t)\,dt\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i x\xi} f(x-t)\,g(t)\,dx\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i(x+t)\xi} f(x)\,g(t)\,dx\,dt \\ &= \int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\,dx \;\int_{\mathbb{R}} e^{-2\pi i t\xi} g(t)\,dt \\ &= \widehat{f}(\xi)\,\widehat{g}(\xi).\end{aligned}$$

□

Proposition 11.3. *If $f,\, g \in L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}} f(x)\,\widehat{g}(x)\,dx = \int_{\mathbb{R}} \widehat{f}(\xi)\,g(\xi)\,d\xi.$$

Proof. We just use Fubini's theorem:

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,\widehat{g}(x)\,dx &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} g(\xi)\,d\xi\,dx \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)e^{-2\pi i x\xi} g(\xi)\,dx\,d\xi \\ &= \int_{\mathbb{R}} \widehat{f}(\xi)\,g(\xi)\,d\xi.\end{aligned}$$

□

We now recall Proposition 5.1 which asserted that the function $e^{-\pi x^2}$ is its own Fourier transform. As a result we get

Proposition 11.4.

$$\int_{\mathbb{R}} e^{-2\pi i x\xi} e^{-\pi\lambda x^2}\, dx = \frac{1}{\sqrt{\lambda}} e^{-\pi\xi^2/\lambda},$$
$$\frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} e^{-2\pi i\xi s} e^{-\pi\xi^2/\lambda} d\xi = e^{-\pi\lambda s^2}.$$

Now suppose that $f \in L^1(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$, and that $\widehat{f} \in L^1(\mathbb{R})$ as well. If we put $g(\xi) = \exp[-\pi\lambda|\xi|^2]$, and use Proposition 11.3, we get

$$\begin{aligned}\int_{\mathbb{R}} e^{2\pi i x\xi} \widehat{f}(\xi)\, \exp[-\pi\lambda|\xi|^2]\, d\xi &= \int_{\mathbb{R}} \widehat{f_x}(\xi)\, \exp[-\pi\lambda|\xi|^2]\, d\xi \\ &= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} f_x(t) e^{-\pi\lambda t^2}\, dt \\ &= \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} f(x+t) e^{-\pi\lambda t^2}\, dt\end{aligned}$$

Now since $\widehat{f} \in L^1(\mathbb{R})$, by the Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_{\lambda\to 0} \int_{\mathbb{R}} e^{2\pi i x\xi} \widehat{f}(\xi)\, \exp[-\pi\lambda|\xi|^2]\, d\xi = \int_{\mathbb{R}} e^{2\pi i x\xi} \widehat{f}(\xi)\, d\xi$$

and so

$$\lim_{\lambda\to 0} \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} f(x+t) e^{-\pi\lambda t^2}\, dt$$

exists as well. But since $\frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} e^{-\pi\lambda t^2}\, dt = 1$ we have

$$\int_{\mathbb{R}} f(x+t) e^{-\pi\lambda t^2}\, dt = f(x) + \int_{\mathbb{R}} \left[f(x+t) - f(x)\right] e^{-\pi\lambda t^2}\, dt$$

and standard arguments show that the second integral has limit zero. Thus we have proved:

Theorem 11.5 (Inversion Formula)**.** *Suppose that f is continuous and integrable on $\mathbb{R}$, and that the Fourier transform $\widehat{f}$ is also integrable. Then for all $x \in \mathbb{R}$ we have*

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x\xi} \widehat{f}(\xi)\, d\xi. \tag{11.3}$$

Remark: It is actually true that if f and $\widehat{f}$ are both integrable, then f can be redefined on a set of measure zero (by equation (11.3)) so that it becomes continuous.

Now suppose that f and $\widehat{f}$ are both integrable (and hence continuous). Let

$$\begin{aligned}g(\xi) = \overline{\widehat{f}(\xi)} &= \overline{\int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\, dx} \\ &= \int_{\mathbb{R}} e^{2\pi i x\xi} \overline{f}(x)\, dx \\ &= \widehat{\overline{f}}(-\xi)\end{aligned}$$

Thus

$$\begin{aligned}\widehat{g}(x) &= \int_{\mathbb{R}} e^{-2\pi i x\xi} g(\xi)\, d\xi \\ &= \int_{\mathbb{R}} e^{-2\pi i x\xi} \overline{\widehat{f}}(-\xi)\, d\xi \\ &= \int_{\mathbb{R}} e^{2\pi i x\xi} \overline{\widehat{f}}(\xi)\, d\xi \\ &= \overline{f(x)}.\end{aligned}$$

Then

$$\begin{aligned}\int_{\mathbb{R}} |\widehat{f}(\xi)|^2\, d\xi &= \int_{\mathbb{R}} \widehat{f}(\xi)\, \overline{\widehat{f}(\xi)}\, d\xi \\ &= \int_{\mathbb{R}} \widehat{f}(\xi) g(\xi)\, d\xi \\ &= \int_{\mathbb{R}} f(x)\widehat{g}(x)\, dx \\ &= \int_{\mathbb{R}} f(x)\overline{f(x)}\, dx \\ &= \int_{\mathbb{R}} |f(x)|^2\, dx.\end{aligned}$$

This gives us a version of the Plancherel Theorem:

Theorem 11.6. *suppose that f is continuous and both f and $\widehat{f}$ belong to $L^1(\mathbb{R})$. Then f and $\widehat{f}$ both belong to $L^2(\mathbb{R})$ and*

$$||f||_{L^2}^2 = \int_{\mathbb{R}} |f(x)|^2\, dx = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2\, d\xi = ||\widehat{f}||_{L^2}^2.$$

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11.2. **Fourier transforms and holomorphic functions.**

Definition 11.7. *For $a > 0$ let $\mathcal{F}_a$ denote the class of functions f such that*

(1) *f is a holomorphic function in the horizontal strip $\{z = x + iy \in \mathbb{C} : |y| < a\}$.*

(2) *There exists constants $A > 0$ and $\eta > 1$ so that for all $x \in \mathbb{R}$ and all $|y| < a$*

$$|f(x+iy)| \leq \frac{A}{1+|x|^\eta}.$$

Proposition 11.8. *If $f \in \mathcal{F}_a$ for some $a > 0$, then $f^{(n)} \in \mathcal{F}_b$ for all $n \geq 1$ and all $0 < b < a$.*

Proof. Let $f \in \mathcal{F}_a$, let $b < a$ and let $\epsilon = \frac{1}{2}(a-b) > 0$. Then if $|\Im\text{m}[z_0]| < b$, the closed disk $\overline{D} = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$ is contained in the strip $\{z \in \mathbb{C} : |\Im\text{m}[z]| < a\}$. By the Cauchy integral formula,

$$\begin{aligned}|f^{(n)}(z_0)| &= \left|\frac{n!}{2\pi i}\int_{\partial\overline{D}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right| \\ &\leq \frac{n!\,\epsilon^{-n}}{2\pi}\int_0^{2\pi} |f(z_0 + \epsilon e^{i\theta})|\, d\theta \\ &\leq n!\,\epsilon^{-n} \sup_{0\leq\theta\leq 2\pi} |f(z_0 + \epsilon e^{i\theta})|.\end{aligned}$$

Let $z_0 = x_0 + iy_0$. Then for $|x_0| \geq 2$ and $\epsilon < 1$ we have

$$|\Re\mathrm{e}(z_0 + \epsilon e^{i\theta})| \geq |x_0| - \epsilon \geq \frac{1}{2}|x_0|.$$

It follows that

$$|f^{(n)}(x_0 + iy_0| \leq n!\epsilon^{-n}\frac{A}{1 + 2^{-\eta}|x_0|^\eta} \leq \frac{B}{1 + |\mathbf{x}_0|^\eta}.$$

This completes the proof. □

Note that it follows from this proposition that every derivative of f is integrable on the real line. As we have seen, this implies that for every $N \geq 1$ there is a constant C_N so that

$$|\widehat{f}(\xi)| \leq C_N(1 + |\xi|)^{-N}.$$

The next result shows that in fact much more is true:

Theorem 11.9. *If $a > 0$ and if $f \in \mathcal{F}_a$, then for any $0 \leq b < a$ there is a constant $C = C(a, f)$ so that*

$$|\widehat{f}(\xi)| \leq C \exp[-2\pi b|\xi|].$$

Proof. Let$0 < b < a$. We already know that $|\widehat{f}(\xi)| \leq ||f||_1$, so it suffices to establish the desired inequality for $|\xi| \geq 1$. Suppose first that $\xi > 1$. Let Γ_R be the rectangular contour consisting of four pieces:

$$\begin{aligned}
\Gamma_1(R) &= \{x \in \mathbb{R} : -R \leq x \leq R\},\\
\Gamma_2(R) &= \{R - is \in \mathbb{C} : 0 \leq s \leq b\},\\
\Gamma_3(R) &= \{x - ib \in \mathbb{C} : R \geq x \geq -R\},\\
\Gamma_4(R) &= \{-R - it \in \mathbb{C} : b \geq t \geq 0\}.
\end{aligned}$$

The function $g(z) = f(z)\, e^{-2\pi i z\xi}$ is holomorphic in a neighborhood of Γ_R so we have

$$\int_{\Gamma_R} g(z)\, dz = 0.$$

Now

$$\begin{aligned}
\left|\int_{\Gamma_2(R)} g(z)\, dx\right| &= \left|\int_0^b f(R - it)e^{-2\pi i(R-it)\xi}(-i)\, dt\right|\\
&\leq \int_0^b |f(R - it)|e^{-t|\xi|}\, dt\\
&\leq \frac{A}{1 + R^\eta}\int_0^\infty e^{-t|\xi|}\, dt\\
&= \frac{A|\xi|^{-1}}{1 + R^\eta}\\
&\leq \frac{A}{1 + R^\eta}.
\end{aligned}$$

It follows that

$$\lim_{R\to+\infty}\int_{\Gamma_2(R)} g(z)\, dz = 0,$$

and a similar estimate shows that the same is true for $\lim_{R\to+\infty}\int_{\Gamma_4(R)} g(z)\, dz$.

On the other hand,

$$\lim_{R\to+\infty}\int_{\Gamma_1(R)} = \int_{-\infty}^{+\infty} g(x)\,dx = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i x\xi}\,dx = \widehat{f}(\xi).$$

Also

$$\begin{aligned}\lim_{R\to+\infty}\int_{\Gamma_3(R)} &= \int_{+\infty}^{-\infty} g(x-ib)\,dx\\ &= \int_{+\infty}^{-\infty} f(x-ib)e^{-2\pi i(x-ib)\xi}\,dx\\ &= -e^{-b\xi}\int_{-\infty}^{+\infty} f(x-ib)\,e^{-2\pi i x\xi}\,dx\\ &= -e^{-b\xi}\,\widehat{f_{-b}}(\xi)\end{aligned}$$

where $f_{-b}(x) = f(x-ib)$. By hypothesis $f_b \in L^1(\mathbb{R})$, and so if $\xi > 1$ we get

$$|\widehat{f}(\xi)| \le A\,e^{-b\xi} = A\,e^{-b|\xi|}.$$

If $\xi < -1$, we use a similar argument where we move the contour into the upper half plane instead of the lower half plane. This completes the proof. □

Theorem 11.10 (Poisson Summation Formula)**.** *Suppose that $a > 0$ and that $f \in \mathcal{F}_a$. Then*

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \widehat{f}(n).$$

Proof. Let $f \in \mathcal{F}_a$, let $0 < b < a$, let $N \ge 1$ be a (large) positive integer, and let Γ_N be the rectangular contour consisting of four pieces:

$$\begin{aligned}\Gamma_1(N) &= \{x - ib \in \mathbb{C} : -N - \tfrac{1}{2} \le x \le N + \tfrac{1}{2}\},\\ \Gamma_2(N) &= \{N + \tfrac{1}{2} + is \in \mathbb{C} : -b \le s \le b\},\\ \Gamma_3(N) &= \{x + ib \in \mathbb{C} : N + \tfrac{1}{2} \ge x \ge -N - \tfrac{1}{2}\},\\ \Gamma_4(N) &= \{-N - \tfrac{1}{2} + it \in \mathbb{C} : b \ge t \ge -b\}.\end{aligned}$$

Consider the meromorpphic function g defined on the strip $\{|\Im\mathrm{m}[z]| < a\}$ by

$$g(z) = \frac{f(z)}{e^{2\pi i z} - 1}.$$

Note that

$$e^{2\pi i z} - 1 = \sum_{n=1}^{\infty} \frac{1}{n!}(2\pi i z)^n = (2\pi i z)\left[1 + O(z)\right].$$

Since this function is period with period 1, it follows that g has simple poles at the integers, and the residue at $z = n$ is $2\pi i f(n)$. It now follows that

$$\int_{\Gamma(N)} \frac{f(z)}{e^{2\pi i z} - 1}\, dz = \sum_{n=-N}^{+N} f(n).$$

When $z = N + \frac{1}{2} + it$,

$$|e^{2\pi i z} - 1| = |e^{\pi i - 2\pi t} - 1| = 1 + e^{-2\pi t} \geq 1.$$

Because of the decay of $|f(x+iy)|$ as $|x| \to \infty$, it then follows as in the earlier computation that

$$\lim_{N\to\infty} \int_{\Gamma_2(N)} \frac{f(z)}{e^{2\pi i z} - 1}\, dz = \lim_{N\to\infty} \int_{\Gamma_4(N)} \frac{f(z)}{e^{2\pi i z} - 1}\, dz = 0.$$

Also, because of the same decay estimates,

$$\lim_{N\to\infty} \sum_{n=-N}^{+N} f(n) = \sum_{-\infty}^{+\infty} f(n)$$

converges. It follows that

$$\sum_{-\infty}^{+\infty} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z} - 1}\, dz - \int_{L_2} \frac{f(z)}{e^{2\pi i z} - 1}\, dz$$

where L_1 is the real axis shifted down by b and L_2 is the real axis shifted up by b.

Observe that if $z = x + iy$, then

$$|e^{2\pi i z}| = \begin{cases} e^{-2\pi y} > 0 & \text{if } z \in L_1 \text{ so } y < 0, \\ \\ e^{-2\pi y} < 0 & \text{if } z \in L_2 \text{ so } y > 0. \end{cases}$$

Thus on L_1 we have

$$\frac{1}{e^{2\pi i z} - 1} = e^{-2\pi i z} \sum_{n=0}^{\infty} e^{-2\pi i n z} = \sum_{n=1}^{\infty} e^{-2\pi i n z},$$

while on L_2 we have

$$\frac{1}{e^{2\pi i z} - 1} = \sum_{n=-\infty}^{0} e^{-2\pi i n z},$$

Thus

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{0} \int_{L_1} f(z) e^{-2\pi i n z}\, dz + \sum_{n=1}^{\infty} \int_{L_2} f(z) e^{-2\pi i n z}\, dz.$$

However, we can now shift each of the integrals back to the real axis, and we obtain

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \int_{\mathbb{R}} f(x) e^{-2\pi i n x} = \sum_{n=-\infty}^{+\infty} \widehat{f}(\xi).$$

This completes the proof. □

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Corollary 11.11. *Let $f \in \mathcal{F}_a$ and let $t \in \mathbb{R}$. Then*

$$\sum_{n=-\infty}^{+\infty} f(n+t) = \sum_{n=-\infty}^{+\infty} e^{2\pi i n t} \widehat{f}(n).$$

Proof. We let $f_t(z) = f(z+t)$. The Poisson summation formula then gives

$$\sum_{n=-\infty}^{+\infty} f(n+t) = \sum_{n=-\infty}^{+\infty} f_t(n) = \sum_{n=-\infty}^{+\infty} \widehat{f_t}(n) = \sum_{n=-\infty}^{+\infty} e^{2\pi int}\widehat{f}(n).$$

□

Corollary 11.12. *For $\lambda > 0$ we have*

$$\sum_{n=-\infty}^{+\infty} e^{-\pi\lambda n^2} = \frac{1}{\sqrt{\lambda}}\sum_{n=-\infty}^{+\infty} e^{-\pi n^2/\lambda}.$$

Proof. The Fourier transform of $e^{-\pi\lambda x^2}$ is $\frac{1}{\sqrt{\lambda}}e^{-\pi\xi^2/\lambda}$. □

Definition 11.13. *The **theta function** $\vartheta(z)$ is defined for $\Re\mathrm{e}[z] > 0$ by*

$$\vartheta(z) = \sum_{n=-\infty}^{+\infty} e^{-n^2\pi z} = 1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi z}. \tag{11.4}$$

It follows from Corollary 11.12 that

$$\vartheta(z) = z^{-\frac{1}{2}}\,\vartheta\left(\frac{1}{z}\right). \tag{11.5}$$

Here we choose the branch of $z^{-\frac{1}{2}}$ in the right half plane which is positive on the real axis.

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12. An application of the Poisson Summation Formula to lattice point problems

Our objective is to count the number $N(R)$ of points $(m,n) \in \mathbb{R}^2$ with integer entries (*i.e. lattice points*) inside the closed disk of radius R, $D(R) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\}$. One might expect that a reasonable approximation to $N(R)$ is the area, πR^2 of the disk $D(R)$, and it is not hard to prove that there is a constant C so that

$$N(R) = \pi R^2 + E(R) \qquad \text{where} \qquad |E(R)| \le C\,R.$$

In fact, for every lattice point (m,n), consider the square

$$Q(m,n) = \{(x,y) \in \mathbb{R}^2 : m \le x < m+1,\ n \le y < n+1\},$$

and consider

$$Q(R) = \bigcup_{(m,n)\in D(R)} Q(m,n).$$

Then $N(R)$ is the area of $Q(R)$. However, it is clear that

$$\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < (R-\sqrt{2})^2\} \subset Q(R) \subset \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < (R+\sqrt{2})^2\},$$

and hence

$$\pi(R-\sqrt{2})^2 \le N(R) \le \pi(R+\sqrt{2})^2.$$

It follows that

$$-2\sqrt{2}R + 2 \le N(R) - \pi R^2 \le 2\sqrt{2}R + 2,$$

and hence for $R \ge 1$ we have

$$|E(R)| = |N(R) - \pi R^2| \le 4R.$$

It is a remarkable fact that the error is in fact much smaller than this. It is conjectured that for any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ so that for $R \ge 1$

$$|E(R)| = |N(R) - \pi R^2| \le C_\epsilon R^{\frac{1}{2}+\epsilon}.$$

As an application of the Poisson summation formula we prove perhaps the easiest non-trivial estimate

Theorem 12.1. *There is a constant $C > 0$ so that for $R \geq 1$, $|E(R)| = |N(R) - \pi R^2| < CR^{\frac{2}{3}}$.*

12.1. The set-up.

Let χ be the characteristic function of the unit disk:

$$\chi(x,y) = \begin{cases} 1 & \text{if } |(x,y)| \leq 1, \\ \\ 0 & \text{if } |(x,y)| > 1. \end{cases}$$

Also, for $R > 1$ put

$$\chi_R(x,y) = \chi(R^{-1}x, R^{-1}y)$$

so that χ_R is the characteristic function of the disk of radius R.

Next, let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ with $\varphi(x,y) \geq 0$ have support in the unit disk and $\iint_{\mathbb{R}^2} \varphi(x,y)\,dx\,dy = 1$. For $\epsilon > 0$ let

$$\varphi_\epsilon(x,y) = \epsilon^{-2}\varphi(\epsilon^{-1}x, \epsilon^{-1}y).$$

Then φ_ϵ has support in the disk of radius ϵ and $\iint_{\mathbb{R}^2} \varphi_\epsilon(x,y)\,dx\,dy = 1$.

We consider the function

$$\chi_R * \varphi_\epsilon(x,y) = \iint_{\mathbb{R}^2} \chi_R(x-s, y-t)\varphi_\epsilon(s,t)\,ds\,dt = \iint_{\mathbb{R}^2} \chi_R(s,t)\varphi_\epsilon(x-s, y-t)\,ds\,dt.$$

It is not hard to check the following properties:

(a) $\chi_R * \varphi_\epsilon$ is infinitely differentiable with compact support.

(b) For all $(x,y) \in \mathbb{R}^2$ we have

$$0 \leq \chi_R * \varphi_\epsilon(x,y) \leq 1.$$

(c)

$$\chi_R * \varphi_\epsilon(x,y) = \begin{cases} 1 & \text{if } x^2 + y^2 < (R-\epsilon)^2, \\ \\ 0 & \text{if } x^2 + y^2 > (R+\epsilon)^2. \end{cases}$$

Let $\mathbb{Z}^2$ denote the set of lattice points in $\mathbb{R}^2$. Then it follows from (b) and (c) that

$$N(R-\epsilon) \leq \sum_{(m,n)\in\mathbb{Z}^2} \chi_R * \varphi_\epsilon(m,n) \leq N(R+\epsilon). \tag{12.1}$$

12.2. Using the Poisson summation formula.

Recall that if f is an integrable function of two variables, its Fourier transform is given by

$$\widehat{f}(\xi,\eta) = \iint_{\mathbb{R}^2} e^{-2\pi i(x\xi + y\eta)} f(x,y)\,dx\,dy.$$

We will use the following version of the Poisson summation formula.

Lemma 12.2. *Suppose that f is infinitely differentiable with compact support. Then $\widehat{f}$ is infinitely differentiable, has rapid decay at infinity, and*

$$\sum_{(m,n)\in\mathbb{Z}^2} f(m,n) = \sum_{(m,n)\in\mathbb{Z}^2} \widehat{f}(m,n).$$

We apply this to the function $f(x,y) = \chi_R * \varphi_\epsilon(x,y)$. Then

$$\widehat{f}(\xi,\eta) = \widehat{\chi_R}(\xi,\eta)\,\widehat{\varphi_\epsilon}(\xi,\eta),$$

and it is easy to check that

$$\begin{aligned} \widehat{\chi_R}(\xi,\eta) &= R^2\widehat{\chi}(R\xi,R\eta), \\ \widehat{\varphi_\epsilon}(\xi,\eta) &= \widehat{\varphi}(\epsilon\xi,\epsilon\eta). \end{aligned} \tag{12.2}$$

Note also from the definition of the Fourier transform and the functions χ and φ we have

$$\begin{aligned} \widehat{\chi}(0,0) &= \pi, \\ \widehat{\varphi}(0,0) &= 1. \end{aligned} \tag{12.3}$$

Key Facts:

A) The Fourier transform $\widehat{\chi}$ has extra decay at infinity due to the curvature of the boundary of the disk. In fact

$$|\widehat{\chi}(\xi,\eta)| \le C(1+|\xi|^2+|\eta|^2)^{-\frac{3}{4}}. \tag{12.4}$$

In particular, if $(m,n) \neq (0,0)$ it follows that

$$|\widehat{\chi_R}(m,n)| \le C\,R^{\frac{1}{2}}(m^2+n^2)^{-\frac{3}{4}}.$$

B) Since φ is infinitely differential with compact support, for every positive N there is a constant $C_N > 0$ so that $|\widehat{\varphi}(\xi,\eta)| \le C_N(1+|\xi|^2+|\eta|^2)^{-N}$. In particular, if $(m,n) \neq (0,0)$ it follows that

$$|\widehat{\varphi_\epsilon}(m,n)| \le C_N(1+\epsilon^2(|\xi|^2+|\eta|^2))^{-N}. \tag{12.5}$$

Now it follows from Lemma 12.2 that

$$\begin{aligned} \sum_{(m,n)\in\mathbb{Z}^2} \chi_R * \varphi_\epsilon(m,n) &= \sum_{(m,n)\in\mathbb{Z}^2} \widehat{\chi_R * \varphi_\epsilon}(m,n) \\ &= \sum_{(m,n)\in\mathbb{Z}^2} \widehat{\chi_R}(m,n)\widehat{\varphi_\epsilon}(m,n) \\ &= R^2 \sum_{(m,n)\in\mathbb{Z}^2} \widehat{\chi}(Rm,Rn)\widehat{\varphi}(\epsilon m,\epsilon n) \\ &= R^2\widehat{\chi}(0,0)\widehat{\varphi}(0,0) + \sum_{(m,n)\neq(0,0)} \widehat{\chi}(Rm,Rn)\widehat{\varphi}(\epsilon m,\epsilon n) \\ &= \pi R^2 + \sum_{(m,n)\neq(0,0)} \widehat{\chi}(Rm,Rn)\widehat{\varphi}(\epsilon m,\epsilon n) \\ &\equiv: \pi R^2 + E(R,\epsilon). \end{aligned}$$

We now estimate the error $E(R,\epsilon)$. We have

$$\begin{aligned}
\left|E(r,\epsilon)\right| &= \left|\sum_{(m,n)\neq(0,0)} \widehat{\chi}(Rm,Rn)\widehat{\varphi}(\epsilon m,\epsilon n)\right| \\
&\leq \sum_{(m,n)\neq(0,0)} \left|\widehat{\chi}(Rm,Rn)\right| \left|\widehat{\varphi}(\epsilon m,\epsilon n)\right| \\
&\leq C_N R^{\frac{1}{2}} \sum_{(m,n)\neq(0,0)} (m^2+n^2)^{-\frac{3}{4}}\big(1+\epsilon^2(m^2+n^2)\big)^{-N} \\
&\leq C_N R^{\frac{1}{2}} \sum_{0<m^2+n^2\leq\epsilon^{-2}} (m^2+n^2)^{-\frac{3}{4}} + C_N R^{\frac{1}{2}}\epsilon^{-2N} \sum_{m^2+n^2>\epsilon^{-2}} (m^2+n^2)^{-\frac{3}{4}-N} \\
&= I + II.
\end{aligned}$$

Exercise: Using comparisons with double integrals show there is a constant C so that

(a) If $0<p<1$, $\displaystyle\sum_{0<m^2+n^2<\epsilon^{-2}} (m^2+n^2)^{-p} \leq C\,\epsilon^{2p-2}$.

(b) If $p>1$, $\displaystyle\sum_{0<m^2+n^2>\epsilon^{-2}} (m^2+n^2)^{-p} \leq C\,\epsilon^{2p-2}$;

Using (a) it follows that

$$I \leq C_N R^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$$

and using (b) it follows that for $N>\frac{1}{4}$,

$$II \leq C_N R^{\frac{1}{2}}\epsilon^{-2N}\epsilon^{2(N+\frac{3}{4})-2} = C_N R^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}.$$

It thus follows that $\displaystyle\sum_{(m,n)\in\mathbb{Z}^2} \chi_R * \varphi_\epsilon(m,n) = \pi R^2 + E(R,\epsilon)$ where $|E(R,\epsilon)| \leq CR^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$.

12.3. **Putting the two sides together.**

It now follows from equation (12.1) that

$$N(R-\epsilon) \leq \pi\mathbb{R}^2 + E(R,\epsilon) \leq N(R+\epsilon).$$

The left hand inequality gives

$$N(R) \leq \pi(R+\epsilon)^2 + E(R+\epsilon,\epsilon) \leq \pi R^2 + E_1(R,\epsilon)$$

where

$$|E_1(R,\epsilon)| \leq C\left[R\epsilon + R^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}\right].$$

The right hand inequality gives

$$N(R) \geq \pi(R-\epsilon)^2 + E(R-\epsilon,\epsilon) \geq \pi R^2 + E_2(R,\epsilon),$$

where

$$|E_2(R,\epsilon)| \leq C\left[R\epsilon + R^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}\right].$$

We now choose the relationship between ϵ and R so that $R\epsilon = R^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$. In other words, we choose $\epsilon = R^{-\frac{1}{3}}$, in which case we see that

$$\left|N(R) - \pi R^2\right| \leq CR^{\frac{2}{3}}.$$

13. The Riemann Zeta Function

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In discussion the Riemann zeta function, it is traditional to use the notation $s = \sigma + it \in \mathbb{C}$ for the complex variable. Then for $\Re\text{e}[s] > 1$, set

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-s\,\log(n)}. \tag{13.1}$$

Note that

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}.$$

and so the series in (13.1) converges absolutely and uniformly on any half plane $\Re\text{e}[s] \geq \eta > 1$. It follows that the Riemann zeta function ζ defines a holomorphic function for $\Re\text{e}[s] > 1$. Our first objective is to show that ζ extends to a meromorphic function on $\mathbb{C}$ with a single simple pole at $s = 1$ where the residue is $+1$.

13.1. **A relationship between ζ, Γ and ϑ.**

Let $\sigma > 0$. Making the change of variables $u = \pi^{-1} n^{-2} t$, we have

$$\int_0^\infty e^{-\pi n^2 u}\, u^{\frac{\sigma}{2}}\, \frac{du}{u} = \pi^{-\frac{\sigma}{2}}\, n^{-\sigma} \int_0^\infty e^{-t} t^{\frac{\sigma}{2}}\, \frac{dt}{t} = \pi^{-\frac{\sigma}{2}}\, n^{-\sigma}\, \Gamma\left(\frac{\sigma}{2}\right),$$

and summing over n it follows that

$$\begin{aligned}
\pi^{-\frac{\sigma}{2}}\, \Gamma\left(\frac{\sigma}{2}\right) \zeta(\sigma) &= \sum_{n=1}^{\infty} \int_0^\infty e^{-\pi n^2 u}\, u^{\frac{\sigma}{2}}\, \frac{du}{u} \\
&= \int_0^\infty u^{\frac{\sigma}{2}} \sum_{n=1} e^{-\pi n^2 u}\, \frac{du}{u} \\
&= \frac{1}{2} \int_0^\infty u^{\frac{\sigma}{2}}\, [\vartheta(u) - 1]\, \frac{du}{u}
\end{aligned}$$

But since ζ, φ and Γ are all holomorphic in the half-plane $\Re\text{e}[s] > 1$, we have established:

Lemma 13.1. *Let $s \in \mathbb{C}$ with $\Re\text{e}[s] > 1$. Then*

$$\pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \int_0^\infty u^{\frac{s}{2}} \big[\vartheta(u) - 1\big]\, \frac{du}{u}.$$

Next observe that

$$\begin{aligned}
\frac{1}{2}\big[\vartheta(u) - 1\big] &= \frac{1}{2}\left[\frac{1}{\sqrt{u}}\, \vartheta\left(\frac{1}{u}\right) - 1\right] \\
&= \frac{1}{2\sqrt{u}}\left[\vartheta\left(\frac{1}{u}\right) - 1\right] + \frac{1}{2\sqrt{u}} - \frac{1}{2}.
\end{aligned}$$

Hence if we set

$$\psi(u) = \frac{1}{2}\big[\vartheta(u) - 1\big],$$

it follows that

$$\psi(u) = \frac{1}{\sqrt{u}} \psi\left(\frac{1}{u}\right) + \frac{1}{2\sqrt{u}} - \frac{1}{2}. \tag{13.2}$$

Proposition 13.2. *There is a constant $C > 0$ so that*

$$\vartheta(t) \le Ct^{-\frac{1}{2}} \qquad \text{for } t < 1,$$
$$\vartheta(t) - 1 \le Ce^{-\pi t} \qquad \text{for } t \ge 1,$$
$$\psi(t) \le e^{-\pi t} \qquad \text{for } t \ge 1.$$

Proof. For $t \ge 1$ we have

$$2\psi(t) = \vartheta(t) - 1 = 2\sum_{n=1}^{+\infty} e^{-\pi n^2 t} \le 2\sum_{n=1}^{\infty} e^{-\pi n t} < 2e^{-\pi t}.$$

This gives the second and third inequalities, and the first then follows from the functional equation for the theta function. This completes the proof. □

For $\Re\mathrm{e}[s] > 1$ put

$$\xi(s) = \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) \zeta(s). \tag{13.3}$$

Theorem 13.3 (Functional Equation)**.** *The function ξ defined initially for $\Re\mathrm{e}[s] > 1$ has a meromorphic extension to the whole complex plane with simple poles at 1 and 0 with residues $+1$ and -1. Moreover*

$$\xi(s) = \xi(1 - s);$$

that is

$$\pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}}\, \Gamma\left(\frac{1-s}{2}\right) \zeta(1 - s).$$

In particular,

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(s/2)}\, \zeta(1 - s).$$

Proof. From Lemma 13.1 we have

$$\begin{aligned}
\xi(s) &= \int_0^\infty u^{\frac{s}{2}}\psi(u)\,\frac{du}{u}\\
&= \int_0^1 u^{\frac{s}{2}}\psi(u)\,\frac{du}{u} + \int_1^\infty u^{\frac{s}{2}}\psi(u)\,\frac{du}{u}\\
&= \int_1^\infty u^{-\frac{s}{2}}\psi\left(\frac{1}{u}\right)\frac{du}{u} + \int_1^\infty u^{\frac{s}{2}}\psi(u)\,\frac{du}{u}\\
&= \int_0^1 u^{-\frac{s}{2}}\left[\sqrt{u}\psi(u) + \frac{1}{2}\sqrt{u} - \frac{1}{2}\right]\frac{du}{u} + \int_1^\infty u^{\frac{s}{2}}\psi(u)\,\frac{du}{u}\\
&= \int_1^\infty \left[u^{\frac{1-s}{2}} + u^{\frac{s}{2}}\right]\psi(u)\,\frac{du}{u} + \frac{1}{2}\int_0^1 u^{\frac{1-s}{2}-1}\,du - \frac{1}{2}\int_0^1 u^{-\frac{s}{2}-1}\,du\\
&= \int_1^\infty \left[u^{\frac{1-s}{2}} + u^{\frac{s}{2}}\right]\psi(u)\,\frac{du}{u} + \frac{1}{1-s} + \frac{1}{s}.
\end{aligned}$$

The first integral defines an entire function of s because of the decay estimate $0 < \psi(u) < Ce^{-\pi u}$, and thus $\xi(s)$ is a meromorphic function with simple poles only at 0 and 1. Moreover, the last expression is unchanged if we replace s by $1 - s$. This completes the proof. □

Corollary 13.4. *The Riemann zeta function has a meromorphic extension to the entire complex plane, with only one simple pole at $s = 1$ where the residue is 1. The equation $\zeta(s) = 0$ has real solutions at $s = -2, -4, \ldots, -2m, \ldots$.*

Proof. We have

$$\zeta(s) = \pi^{\frac{s}{2}} \xi(s) \, \Gamma\left(\frac{s}{2}\right)^{-1},$$

and since Γ^{-1} is an entire function, it follows that $\zeta(s)$ has a meromorphic extension with poles only possible at the poles of $\xi(s)$. However $\Gamma\left(\frac{s}{2}\right)^{-1}$ has simple zeros at $s = 0, -2, -4, \ldots$, and so the pole of ξ at $s = 0$ is cancelled by the zero of $\Gamma\left(\frac{s}{2}\right)^{-1}$. Since the residue of ξ at the point $s = 1$ is equal to 1, the residue of ζ at $s = 1$ is $\pi^{\frac{1}{2}} \Gamma\left(\frac{1}{2}\right)^{-1} = 1$. This completes the proof. □

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13.2. **The connection between primes and the Riemann zeta function.**

The key connection between primes and the zeta function was established by Euler.

Theorem 13.5. *For $\Re\mathrm{e}[s] > 1$, we have*

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1},$$

where the product is taken over all primes $p = 2, 3, 5, 7, 11, \ldots$.

Proof. First observe that

$$1 - \left(1 - p^{-s}\right)^{-1} = -\frac{p^{-s}}{1 - p^{-s}} = \frac{1}{p^s - 1}$$

so

$$\left|1 - \left(1 - p^{-s}\right)^{-1}\right| = \frac{1}{|p^s - 1|} \leq \frac{1}{p^{\Re\mathrm{e}[s]} - 1} < \frac{1}{p^{\Re\mathrm{e}[s]}}$$

and thus

$$\sum_{p \text{ prime}} \left|1 - \left(1 - p^{-s}\right)^{-1}\right| \leq \sum_{p \text{ prime}} p^{-\Re\mathrm{e}[s]} < \sum_{n=1}^{\infty} n^{-\Re\mathrm{e}[s]} < +\infty,$$

provided that $\Re\mathrm{e}[s] > 1$. Thus the infinite product $\prod_p \left(1 - p^{-s}\right)^{-1}$ converges absolutely and uniformly on any half-plane $RE[s] \geq 1 + \epsilon$ and defines there a holomorphic function. To prove the identity in the theorem, it suffices to show that we have equality for $s = \sigma \in \mathbb{R}$ and $\sigma > 1$.

For any prime p and $\sigma > 1$ we have $p^{-\sigma} < 1$, and so

$$\left(1 - p^{-\sigma}\right)^{-1} = 1 + p^{-\sigma} + p^{-2\sigma} + \cdots + p^{-m\sigma} + \cdots$$

where the geometric series on the right converges absolutely and uniformly on any half plane $\sigma = \Re\mathrm{e}[s] \geq 1 + \epsilon$. Formally, when we multiply these series together, we get a sum of terms of the form $N^{-\sigma}$ where N is a product of primes to various powers. But by the fundamental theorem of arithmetic, every integer N can be written uniquely as a product of powers of primes. Thus the equality in Euler's Theorem is a restatement of the Fundamental Theorem of Arithmetic.

To give a complete proof, we first observe the following algebraic identity. Let $\mathcal{P} = \{p_1, \ldots, p_M\}$ be a set of M distinct primes, let m be a positive integer, and let $\sigma > 0$.Then

$$\prod_{j=1}^{M}\left[\sum_{k=0}^{m} p_j^{-k\sigma}\right] = \prod_{j=1}^{M}\left(1 + p_j^{-\sigma} + p_j^{-2\sigma} + \ldots + p_j^{-m\sigma}\right)$$
$$= \sum_{\substack{0\le\alpha_j\le m\\ 1\le j\le M}} \left(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_M^{\alpha_M}\right)^{-\sigma}$$
$$= \sum_{n\in\Xi(\mathcal{P},m)} n^{-\sigma}$$

where $\Xi(\mathcal{P}, m)$ is the set of integers whose only prime factors belong to the set $\mathcal{P}$, and the prime p_j appears to at most the power m.

By the fundamental theorem of arithmetic, if $M > N$ are positive integers,

$$\{1,\, 2,\, \ldots, N\} \subset \Xi(\mathcal{P}, M\}$$

where $\mathcal{P}$ consists of all primes less than or equal to N. Hence

$$\sum_{n=1}^{N}\frac{1}{n^\sigma} \le \prod_{p\le N}\left(1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} + \cdots + \frac{1}{p^{M\sigma}}\right)$$
$$\le \prod_{p\le N}\left(\frac{1}{1-p^{-\sigma}}\right)$$
$$\le \prod_{p}\left(\frac{1}{1-p^{-\sigma}}\right).$$

We now let $N \to \infty$ and conclude that

$$\sum_{n=1}^{\infty} \le \prod_{p}\left(\frac{1}{1-p^{-\sigma}}\right). \tag{13.4}$$

To obtain the reverse inequality, we observe that

$$\prod_{p\le N}\left(1 + \frac{1}{p^\sigma} + \frac{1}{p^{2\sigma}} + \cdots + \frac{1}{p^{M\sigma}}\right) \le \sum_{n=1}^{\infty}\frac{1}{n^\sigma}.$$

Letting $M \to \infty$ it follows that

$$\prod_{p\le N}\left(\frac{1}{1-p^{-\sigma}}\right) = \prod_{p\le N}\sum_{k=0}^{\infty} p^{-k\sigma} \le \sum_{n=1}^{\infty}\frac{1}{n^\sigma},$$

and finally letting $N \to \infty$ we obtain

$$\prod_{p}\left(\frac{1}{1-p^{-\sigma}}\right) \le \sum_{n=1}^{\infty}\frac{1}{n^\sigma}. \tag{13.5}$$

This completes the proof. □

Corollary 13.6. *There are infinitely many primes.*

Proof. If there were only finitely many primes, the product $\prod_p (1-p^{-s})^{-1}$ would define a holomorphic function for $\Re\mathrm{e}[s] > 0$. Since $\lim_{\sigma\to 0^+}\zeta(\sigma) = +\infty$, this would lead to a contradiction. □

Corollary 13.7. *If* $\Re e[s] > 1$

$$\log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-s}$$

where

$$c_n = \begin{cases} \frac{1}{m} & \text{if } n = p^m \text{ for some prime } p, \\ \\ 0 & \text{if } n \text{ is not a power of a single prime.} \end{cases}$$

Proof. For $\Re e[s] > 1$ we have

$$\begin{aligned} \log\big(\zeta(s)\big) &= -\sum_{p \text{ prime}} \log(1 - p^{-s}) \\ &= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{1}{m} p^{-ms}. \end{aligned}$$

□

Corollary 13.8. *The infinite series* $\displaystyle\sum_{p \text{ prime}} \frac{1}{p}$ *diverges.*

Proof. For $\sigma > 1$ we have

$$\begin{aligned} \log\big(\zeta(\sigma)\big) &= \sum_{p \text{ prime}} \sum_{m=1}^{\infty} \frac{1}{m} p^{-m\sigma} \\ &= \sum_{p \text{ prime}} p^{-\sigma} + \sum_{p \text{ prime}} \sum_{m=2}^{\infty} \frac{1}{m} p^{-m\sigma}. \end{aligned}$$

Now since $p \geq 2$ we have

$$\sum_{m=2}^{\infty} \frac{1}{m} p^{-m\sigma} < \frac{1}{2} p^{-2\sigma} \sum_{n=0}^{\infty} p^{-n\sigma} = \frac{1}{2} \frac{p^{-2\sigma}}{1 - p^{-\sigma}} \leq p^{-2\sigma},$$

and thus

$$\sum_{p \text{ prime}} \sum_{m=2}^{\infty} \frac{1}{m} p^{-m\sigma} < \sum_{p} p^{-2\sigma} < \sum_{n=1}^{\infty} n^{-2} < +\infty.$$

Thus as $\sigma \to 1^+$,

$$\sum_{p} p^{-\sigma} = \log\big(\zeta(\sigma)\big) + O(1).$$

Since $\log\big(\zeta(\sigma)\big) \to +\infty$ as $\sigma \to 1^+$, it follows that $\sum_p \frac{1}{p}$ diverges. □

Corollary 13.9. *The only solutions of the equation* $\zeta(s) = 0$ *outside the strip* $0 \leq \Re e[s] \leq 1$ *are at the points* $s = -2, -4, \ldots$.

Proof. None of the factors in the Euler product equals zero in the half-plane $\Re e[s] > 1$, and hence the product $\zeta(s)$ has no zeros in this half plane. The rest of the result follows from the functional equation for the zeta function. □

Theorem 13.10. *If* $\Re e[s] = 1$, *then* $\zeta(s) \neq 0$, *and by the functional equation, the same is true if* $\Re e[s] = 0$..

Lemma 13.11. *If* $\theta \in \mathbb{R}$, *then*

$$3 + 4\cos(\theta) + \cos(2\theta) \geq 0.$$

Proof. We have

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1.$$

Thus

$$\begin{aligned} 3 + 4\cos(\theta) + \cos(2\theta) &= 3 + 4\cos(\theta) + 2\cos^2(\theta) - 1 \\ &= 2\big(1 + 2\cos(\theta) + \cos^2(\theta)\big) \\ &= 2\big(1 + \cos(\theta)\big)^2 \geq 0. \end{aligned}$$

□

Lemma 13.12. *If $\sigma > 1$ and $t \in \mathbb{R}$, then*

$$\log\left|\zeta^3(\sigma)\,\zeta^4(\sigma + it)\,\zeta(\sigma + 2it)\right| \geq 0.$$

Proof. First note that

$$\Re\mathrm{e}[n^{\sigma+it}] = \Re\mathrm{e}[e^{(\sigma+it)\log(n)}] = n^\sigma \cos(t\log(n)).$$

Then

$$\begin{aligned} &\log\left|\zeta^3(\sigma)\,\zeta^4(\sigma + it)\,\zeta(\sigma + 2it)\right| \\ &\qquad = 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)| \\ &\qquad = 3\Re\mathrm{e}[\log\zeta(\sigma)] + 4\Re\mathrm{e}[\log\zeta(\sigma + it)] + \Re\mathrm{e}[\log\zeta(\sigma + 2it)] \\ &\qquad = \sum_{n=1}^{\infty} c_n n^{-\sigma}\big(3 + 4\cos(\theta_n) + \cos(2\theta_n)\big) \geq 0 \end{aligned}$$

where c_n is defined in Corollary 13.7

□

Proof of Theorem 13.10. Suppose that $\zeta(1 + it_0) = 0$ for some $t_0 \neq 0$. Then *zeta* must vanish to at least first order at this point, and it follows that for $\sigma > 1$,

$$|\zeta(\sigma + it_0)|^4 \leq C_1(\sigma - 1)^4$$

Also, the point $s = 1$ is a simple pole of the zeta function, and so for $\sigma > 1$,

$$|\zeta(\sigma)|^3 \leq C_2(\sigma - 1)^{-3}.$$

Finally, ζ is holomorphic at $1 + 2i\sigma$, and hence for $\sigma > 1$,

$$|\zeta(\sigma + 2it_0)| \leq C_3.$$

It thus follows that

$$|\zeta^3(\sigma)\zeta^4(\sigma + it_0)\zeta(\sigma + 2it_0)| \to 0$$

as $\sigma \to 1$. But this contradicts Lemma 13.12, and completes the proof. □