Math 722 **Assignment # 2** **Due Monday, February 20, 2012**

NOTE: IN THESE PROBLEMS, WE USE THE NOTATION $D(a;R) = \{z \in \mathbb{C} : |z-a| < R\}$ FOR THE OPEN DISK OF RADIUS R CENTERED AT $a \in \mathbb{C}$, AND $D(a;R)^* = \{z \in \mathbb{C} : 0 < |z-a| < R\}$ FOR THE PUNCTURED OPEN DISK OF RADIUS R CENTERED AT $a \in \mathbb{C}$.

Problem 1. Consider the following three power series:

$$E(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n; \qquad S(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}; \qquad C(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}.$$

(a) Show that the radius of convergence of all three series is $+\infty$ so that E, S, and C are entire functions. (Of course, E, S, and C stand for 'exponential', 'sine', and 'cosine'.)

(b) Show that for all $z \in \mathbb{C}$,

$$E'(z) = E(z) \qquad S'(z) = C(z) \qquad C'(z) = -S(z)$$
$$E(iz) = C(z) + iS(z) \qquad S^2(z) + C^2(z) = 1.$$

(c) Show that if $a, b \in \mathbb{C}$ then

$$E(a+b) = E(a)E(b), \qquad S(a+b) = S(a)C(b) + S(b)C(a), \qquad C(a+b) = C(a)C(b) - S(a)S(b).$$

(d) Show that $E(z) \neq 0$ for all $z \in \mathbb{C}$, that $|E(iy)| = 1$ for all $y \in \mathbb{R}$, and that $|E(x+iy)| = E(x)$ for all $x + iy \in \mathbb{C}$.

Problem 2. Consider the power series $L(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(z-1)^n$.

(a) Show that the radius of convergence of this series is 1, so L is a holomorphic function in the disk $D(1;1)$. (Of course, L stands for 'logarithm'.)

(b) Show that $L'(z) = z^{-1}$ for all $z \in D(1;1)$.

(c) Show that $E(L(z)) \equiv z$ for all $z \in D(1;1)$ and that there exists $\epsilon > 0$ so that if $|z| < \epsilon$ then $L(E(z)) \equiv z$.

(d) Show that there exists $0 < \delta_n < 1$ so that if $a_j \in D(1;\delta_n)$ for $1 \leq j \leq n$ then $\prod_{j=1}^n a_j \in D(1;1)$. For this δ_n, show that if $a_j \in D(1;\delta_n)$ for $1 \leq j \leq n$, then $L\big(a_1 \cdots a_n\big) = \sum_{j=1}^n L(a_j)$.

Problem 3. Let f be holomorphic in the open disk $D(a;R)$, suppose that $f(a) \neq 0$, and let $N \geq 2$ be an integer. Prove that there exists $0 < \delta \leq R$ and a holomorphic function g in the disk $D(a;\delta)$ such that $f(z) = g(z)^N$ for all $z \in D(a;\delta)$. [*Hint: Use the properties of the exponential and logarithm functions established in Problems 1 and 2.*]

Problem 4. Let f be holomorphic in the open disk $D(a;R)$ with $f(a) = b$. Suppose that $f'(a) \neq 0$. Use the open mapping theorem[1] to show that there exists $\epsilon > 0$ and an open neighborhood $U \subset D(a;R)$ of the point a so that $f : U \to D(b;\epsilon)$ is one-to-one and onto. Show that the inverse mapping $f^{-1} : D(b;\epsilon) \to U$ is also holomorphic.

Problem 5. Let $\Omega \subset \mathbb{C}$ be a connected open set and let f be a holomorphic function defined on Ω. Show that if f is not a constant function, then f is an open mapping: for each $z_0 \in \Omega$ there exists $\epsilon > 0$ so that if $|w - f(z_0)| < \epsilon$, there exists $z \in \Omega$ with $f(z) = w$. [*Hint: If f is not constant, for z close to z_0 we can write $f(z) = a_N(z-z_0)^N h(z)$ with $h(z_0) = 1$. For such z use the result of Problem 3 to show that we can write $f(z) = g(z)^N$ with $g'(z_0) \neq 0$. Then use Problem 4.*]

Problem 6. Let F be an entire function, and suppose that F is one-to-one. Prove that there exist complex numbers a, b with $a \neq 0$ so that $F(z) = az + b$. [*Hint: What can you say about the isolated singularity at* 0 *of the function $G(z) = F(z^{-1})$?*]

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[1] Let $U \subset \mathbb{R}^n$ be open, let $F : U \to \mathbb{R}^n$ be a continuously differentiable mapping, let $a \in U$, and assume that $\det[JF(a)] \neq 0$ where JF is the Jacobian matrix of the mapping F. Then if $F(a) = b \in \mathbb{R}^n$, there exists $\epsilon > 0$ and an open neighborhood U of the point a so that $F : U \to \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{b}\} < \epsilon\}$ is one-to-one and onto, and the inverse mapping is also continuously differentiable.

Problem 7. Use contour integration and the residue formula to evaluate the following definite integrals:

$$(a)\quad \int_{-\infty}^{+\infty} \frac{1}{1+x^4}\,dx; \qquad (b)\quad \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^3}\,dx; \qquad (c)\quad \int_{-\infty}^{+\infty} \frac{\cos(x)}{x^2+9}\,dx$$

$$(d)\quad \int_{-\infty}^{+\infty} \frac{x\sin(x)}{x^2+25}\,dx; \qquad (e)\quad \int_{0}^{2\pi} \frac{1}{9+\cos(\theta)}\,d\theta; \qquad (f)\quad \int_{0}^{2\pi} \frac{1}{1+\cos^2(\theta)}\,d\theta;.$$

Problem 8. Suppose that f is holomorphic in the punctured disk $D(a,R)^*$ and that there are constants $M>0$ and $0<\epsilon<1$ so that $|f(z)| \leq M|z-a|^{-1+\epsilon}$. Prove that a is a removable singularity for f.

Problem 9. Let $\Omega \subset \mathbb{C}$ be a connected open set and let f be holomorphic on Ω.

(a) Suppose that $a \in D(a,R) \subset \Omega$. Prove that for $0<r<R$ we have

$$f(a) = \frac{1}{\pi r^2} \iint_{D(a,r)} f(x+iy)\,dx\,dy.$$

(b) Suppose that $a \in D(a,R) \subset \Omega$. Prove that for $0<r<R$ and $1 \leq p < \infty$ we have

$$|f(a)| \leq \Big[\frac{1}{\pi r^2} \iint_{D(a,r)} |f(x+iy)|^p\,dx\,dy\Big]^{\frac{1}{p}}.$$

(c) Let $K \subset \Omega$ be compact, and let $1 \leq p < \infty$. Prove that there is constant $C(p,K)$ depending on p and K but not on f so that

$$\sup_{z\in K} |f(z)| \leq C(p,K)\,\Big[\iint_{\Omega} |f(x+iy)|^p\,dx\,dy\Big]^{\frac{1}{p}}.$$

Problem 10. Let f be holomorphic in the unit disk $\mathbb{D} = D(0;1)$, and suppose that the Taylor expansion of f is given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z|<1$.

(a) Prove that for $0<r<1$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f\big(re^{i\theta}\big)|^2\,d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

(b) Prove that for $0<r<1$,

$$\frac{1}{\pi}\iint_{D(0;r)} |f\big(x+iy\big)|^2\,dx\,dy = \sum_{n=0}^{\infty} \frac{|a_n|^2 r^{2n}}{n+1}.$$