1. The Siegel upper half space

1.1. The Siegel upper half space and its Bergman kernel.

The Siegel upper half space is the domain

\[ \mathcal{U}^{n+1} = \left\{ z \in \mathbb{C}^{n+1} \left| \Im[z_{n+1}] > \sum_{j=1}^{n} |z_j|^2 \right. \right\}. \]  \hspace{1cm} (1.1)

The unit ball in \( \mathbb{C}^{n+1} \) is the domain

\[ B^{n+1} = \left\{ z \in \mathbb{C}^{n+1} \left| \sum_{j=1}^{n+1} |z_j|^2 < 1 \right. \right\}. \]  \hspace{1cm} (1.2)

**Proposition 1.1.** \( \mathcal{U}^{n+1} \) is biholomorphic to \( B^{n+1} \) via the mapping

\[ F(z_1, \ldots, z_{n+1}) = \left( \frac{2z_1}{i + z_{n+1}}, \ldots, \frac{2z_n}{i + z_{n+1}}, \frac{i - z_{n+1}}{i + z_{n+1}} \right) = (w_1, \ldots, w_n, w_{n+1}). \]

**Proof.** Write \( z_{n+1} = x_{n+1} + iy_{n+1} \), so that

\[ |z - z_{n+1}|^2 = x_{n+1}^2 + y_{n+1}^2 + 1 - 2y_{n+1} = |z_{n+1}|^2 + 1 - 2\Im[z_{n+1}], \]

\[ |z + z_{n+1}|^2 = x_{n+1}^2 + y_{n+1}^2 + 1 + 2y_{n+1} = |z_{n+1}|^2 + 1 + 2\Im[z_{n+1}]. \]

Then

\[ \sum_{j=1}^{n+1} |w_j|^2 = 4 \sum_{j=1}^{n} |z_j|^2 + |z_{n+1}|^2 + 1 - 2\Im[z_{n+1}]. \]

Thus \( \sum_{j=1}^{n+1} |w_j|^2 < 1 \) if and only if \( 3\Im[z_{n+1}] > \sum_{j=1}^{n} |z_j|^2 \), so \( F : \mathcal{U}^{n+1} \to B_{n+1} \).

The two domains are actually biholomorphic since the inverse mapping is given by

\[ F^{-1}(w_1, \ldots, w_{n+1}) = \left( \frac{iw_1}{1 + w_{n+1}}, \ldots, \frac{iw_n}{1 + w_{n+1}}, \frac{1 - w_{n+1}}{1 + w_{n+1}} \right). \]

This completes the proof. \( \square \)

**Proposition 1.2.** The determinant of the Jacobian matrix of the mapping \( F \) is given by

\[ JF(z_1, \ldots, z_{n+1}) \equiv \det \frac{\partial F_j}{\partial z_k}(z_1, \ldots, z_{n+1}) = -\frac{2^{n+1}i}{(i + z_{n+1})^{n+2}}. \]

**Proof.** This is an easy calculation. \( \square \)

**Lemma 1.3.** The Bergman kernel for the domain \( \mathcal{U}^{n+1} \) is given by

\[ B_{\mathcal{U}^{n+1}}(z, w) = \frac{(n + 1)!}{4\pi^{n+1}} \left( \frac{i}{2} \left( \frac{w_{n+1} - z_{n+1}}{w_{n+1} - z_{n+1}} \right) - \sum_{j=1}^{n} z_j w_j \right)^{-n-2}. \]

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Proof. The Bergman kernel for the unit ball is

\[ B_{B^{n+1}}(z, w) = \frac{(n+1)!}{\pi^{n+1}} \frac{1}{(1 - \langle z, w \rangle)^{n+2}}. \]

Therefore, by the transformation rules for the Bergman kernel, we get

\[ B_{U^{n+1}}(z, w) = JF(z)B_{B^{n+1}}(F(z), F(w))\overline{JF(w)} \]

\[ = \frac{(n+1)!}{\pi^{n+1}} \frac{JF(z)\overline{JF(w)}}{(1 - \langle F(z), F(w) \rangle)^{n+2}} \]

\[ = \frac{1}{\pi^{n+1}} \frac{4^{n+1}(n+1)\left|\left(1 + i + w_{n+1} \right) - \left[4\sum_{j=1}^{n} z_j \overline{w}_j + (i - z_{n+1})(i - w_{n+1})\right]\right|^n}{\left(1 - (i + z_{n+1})(i + w_{n+1})\right)^{n-2}} \]

\[ = \frac{1}{\pi^{n+1}} \frac{4^{n+1}(n+1)!}{\left(1 - (i + z_{n+1})(i + w_{n+1})\right)^{n+2}} \]

\[ = \frac{1}{\pi^{n+1}} \frac{4^{n+1}(n+1)!}{\left[2i(w_{n+1} - z_{n+1}) - 4\sum_{j=1}^{n} z_j \overline{w}_j\right]^{n+2}} \]

\[ = \frac{(n+1)!}{4\pi^{n+1}} \left[\frac{i}{2}(w_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \overline{w}_j\right]^{-n-2}. \]

This completes the proof. \(\square\)

1.2. The Heisenberg group.

We define a family of biholomorphic mappings of \(\mathbb{C}^{n+1}\) which carry \(U^{n+1}\) into itself. These are the analogues of translation \(z \rightarrow z + a\) for \(a \in \mathbb{R}\) when \(z \in \mathbb{C}\).

For \(z \in \mathbb{C}^{n+1}\), we use the notation \(z = (z', z_{n+1})\) where \(z' \in \mathbb{C}^n\). For \(w \in \mathbb{C}^{n+1}\), put

\[ T_w(z) = (z' + w', z_{n+1} + w_{n+1} + 2i(z', w')). \]

(1.3)

\(T_w\) is a holomorphic mapping, and \(T_0\) is the identity map. If \(u, w \in \mathbb{C}^{n+1}\), we have

\[ T_u(T_w(z)) = T_u(z' + w', z_{n+1} + w_{n+1} + 2i(z', w')) \]

\[ = (z' + u' + w', z_{n+1} + w_{n+1} + 2i(z', u') + u_{n+1} + 2i(z', w')) \]

\[ = (z' + u' + w', z_{n+1} + w_{n+1} + u_{n+1} + 2i(w', u') + 2i(z', w' + u')) \]

\[ = T_\zeta(z), \]

where \(\zeta = (w' + u', w_{n+1} + u_{n+1} + 2i(w', u')).\)

It follows that if we define a product \((z, w) \rightarrow z \cdot w\) on \(\mathbb{C}^{n+1}\) by

\[ z \cdot w = (z' + w', z_{n+1} + w_{n+1} + 2i(w', z')), \]

(1.4)

then

\[ T_{z \cdot w} = T_z \circ T_w. \]

(1.5)

Since composition of mappings is always associative, we have \((z \cdot w) \cdot u = z \cdot (w \cdot u)\) for all \(z, w, u \in C^{n+1}\). Clearly \(0 \cdot z = z \cdot 0 = z\), so 0 acts as an identity. Finally, if we
Proposition 1.4. The product
\[ z \cdot w = (z' + w', z_{n+1} + w_{n+1} + 2i\langle w', z' \rangle) \]
makes \( \mathbb{C}^{n+1} \) is a group with identity \( e = 0 \), and inverse defined by
\[ w^{-1} = (-w', -w_{n+1} + 2i|w'|^2). \]

Note that if \( dz \) denotes Lebesgue measure on \( \mathbb{C}^{n+1} \), and if \( f \in L^1(\mathbb{C}^{n+1}) \), then
\[
\int_{\mathbb{C}^{n+1}} f(z \cdot w) \, dz = \int_{\mathbb{C}^{n+1}} f(w \cdot z) \, dz = \int_{\mathbb{C}^{n+1}} f(z^{-1}) \, dz = \int_{\mathbb{C}^{n+1}} f(z) \, dz.
\]
Thus the (left and right) Haar measure on this group is just Lebesgue measure.

Define \( \rho : \mathbb{C}^{n+1} \rightarrow \mathbb{R} \) by
\[
\rho(z) = \Im m[z_{n+1}] - \sum_{j=1}^{n} |z_j|^2,
\]
so that \( U^{n+1} = \{ z \in \mathbb{C}^{n+1} \mid \rho(z) > 0 \} \). Thus \( \rho \) is a “height” function on \( \mathbb{C}^{n+1} \).

Proposition 1.5. If \( z, w \in \mathbb{C}^{n+1} \), then \( \rho(z \cdot w) = \rho(z) + \rho(w) \).

Proof. We make the following computation:
\[
\rho(w \cdot z) = \Im m[z_{n+1} + w_{n+1} + 2i\langle z', w' \rangle] - \sum_{j=1}^{n} |z_j + w_j|^2
\]
\[= \Im m[z_{n+1}] + \Im m[w_{n+1}] + 2\Re \langle z', w' \rangle - \sum_{j=1}^{n} (|z_j|^2 + |w_j|^2) - 2\Re \langle z', w' \rangle
\]
\[= \rho(z) + \rho(w).
\]

It follows that if \( \rho(w) = 0 \), the mapping \( T_w \) is a biholomorphic mapping of \( U^{n+1} \)
to itself. However, the set \( \{ w \in \mathbb{C}^{n+1} \mid \rho(w) = 0 \} \) is just the boundary of \( U^{n+1} \).
Identify \( \partial U^{n+1} \) with \( \mathbb{C}^n \times \mathbb{R} \) via the correspondence
\[
\mathbb{C}^n \times \mathbb{R} \ni (z_1, \ldots, z_n, t) \longmapsto (z_1, \ldots, z_n, t + i \sum_{j=1}^{n} |z_j|^2) \in \partial U^{n+1}.
\]

Then the point \( (w', s) \in \mathbb{C}^n \times \mathbb{R} \) is identified with the biholomorphic mapping
\( T_{w',s} : U^{n+1} \rightarrow U^{n+1} \) where
\[ T_{w',s}(z) = (z' + w', z_{n+1} + s + i|w'|^2 + 2i\langle z', w' \rangle). \]

Moreover, according to (1.5) and Proposition 1.5, the composition of two such maps is a map of the same form. Thus the set of mapping \( \{ T_w \} \) with \( \rho(w) = 0 \)
is a subgroup of the group structure we have put on \( \mathbb{C}^{n+1} \). Explicitly, if \( z = z', t + i|z'|^2, w = (w', s + i|w'|^2) \in \partial U^{n+1} \), the analogue of (1.4) is
\[
(z', t + i|z'|^2) \cdot (w', s + i|w'|^2) = (z' + w', t + s + 2\Re \langle z', w' \rangle + i|z' + w'|^2).
\]
**Def 1.6** **Definition 1.6.** The set $\mathbb{C}^n \times \mathbb{R}$ with the product given by

$$(z, t) \cdot (w, s) = (z + w, t + s + 2\text{Im}(z, w))$$

is called the $n$-dimensional Heisenberg group $\mathbb{H}^n$. The identity is $e = (0, 0)$, and the inverse is given by

$$(z, t)^{-1} = (-z, -t).$$

We now want to connect the new product on $\mathbb{C}^n+1$ with the Bergman kernel. For any $z = (z', z_{n+1}) \in \mathbb{C}^{n+1}$, put

$$\tilde{z} = (z', z_{n+1} - 2i\rho(z)).$$

Then

$$\rho(\tilde{z}) = 3\text{m}[z_{n+1}] - 2\rho(z) - |z'|^2 = \rho(z) - 2\rho(z) = -\rho(z).$$

Thus the operation $z \to \tilde{z}$ just changes the sign of the height of $z$. We have

$$\tilde{z} \cdot w = z \cdot w - 2i(0, \rho(z \cdot w)) = z \cdot w - 2i(0, \rho(z) + \rho(w))
= (z' + w', z_{n+1} + w_{n+1} + 2i[(w', z') - \rho(z) - \rho(w)]),$$

while

$$\tilde{z} \cdot \tilde{w} = (z', z_{n+1} - 2i\rho(z)) \cdot (w', w_{n+1} - 2i\rho(w))
= (z' + w', z_{n+1} - 2i\rho(z) + w_{n+1} - 2i\rho(w) + 2i(w', z')).$$

Thus $\tilde{z} \cdot \tilde{w} = \tilde{z} \cdot \tilde{w}$, so $z \to \tilde{z}$ is a period 2 automorphism of $\mathbb{C}^{n+1}$ which leaves the subgroup $\mathbb{H}^n$ invariant.

**Prop 1.7** **Proposition 1.7.** Let $z = (z', z_{n+1}) = (z', t + i|z'|^2 + i\rho(z))$ and $w = (w', w_{n+1}) = (w', s + i|w'|^2 + i\rho(w))$. Then

$$\tilde{w}^{-1} \cdot z = (z' - w', z_{n+1} - \overline{w_{n+1}} - 2i(z', w')).$$

**Proof.** According to (1.8), $\tilde{w} = (w', w_{n+1} - 2i\rho(w))$, and so by Proposition 1.4,

$$\tilde{w}^{-1} = (-w', -w_{n+1} + 2i\rho(w) + 2i|w'|^2) = (-w', -\overline{w_{n+1}}).$$

Thus

$$\tilde{w}^{-1} \cdot z = (-w', -\overline{w_{n+1}}) \cdot (z', z_{n+1}) = (z' - w', z_{n+1} - \overline{w_{n+1}} - 2i(z', w'))$$

as asserted. □

Lemma 1.3 and Proposition 1.7 now show the following connection between the Bergman kernel and the group structure on $\mathbb{C}^{n+1}$.

**Prop 1.8** **Proposition 1.8.** Let $F(z) = F(z_1, \ldots, z_n, z_{n+1}) = z_{n+1}^{-n-2}$. Then

$$B_{\mathbb{H}^{n+1}}(z, w) = \frac{(n + 1)!}{4\pi^{n+1}} \left(-\frac{i}{2}\right)^{-n-2} F(\tilde{w}^{-1} \cdot z).$$
1.3. The Szegő kernel.

**Definition 1.9.** The function

\[ S(z, w) = \frac{n!}{4\pi^{n+1}} \left[ \frac{i}{2} (w_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \bar{w}_j \right]^{-n-1} \]

is called the Szegő kernel for \( U^{n+1} \).

We show next that there is a reproducing formula for holomorphic functions using the Szegő kernel, but instead of integrating over all of \( U^{n+1} \), we only integrate over the boundary \( \partial U^{n+1} \). Recall that we can identify \( \partial U^{n+1} \) with \( \mathbb{C}^n \times \mathbb{R} \). With this identification, \( \int_{\partial U^{n+1}} f(w) \, d\sigma(w) \) just means integrating with respect to Lebesgue measure on \( \mathbb{C}^n \times \mathbb{R} \).

**Theorem 1.10.** Let \( f \) be holomorphic in a neighborhood of the closure of \( U^{n+1} \), and belong to the Bergman space \( B^2(U^{n+1}) \). Then for \( z \in U^{n+1} \),

\[ f(z) = \frac{n!}{4\pi^{n+1}} \int_{\partial U^{n+1}} \left[ \frac{i}{2} (w_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \bar{w}_j \right]^{-n-1} f(w_1, \ldots, w_n, w_{n+1}) \, d\sigma(w) \]

**Proof.** Let

\[ \omega_{n+1} = dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n \wedge dw_{n+1}, \]
\[ \omega = dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_n \wedge d\bar{w}_n \wedge dw_{n+1} \wedge d\bar{w}_{n+1}. \]

Note that \( d\bar{w}_{n+1} \wedge \omega_{n+1} = -\omega \). Also, \( \omega = (-2i)^{n+1} \, dw \) where \( dw \) is Lebesgue measure on \( \mathbb{C}^n \times \mathbb{R} \).

Then using the generalized Stokes theorem, we have

\[ \int_{\partial U^{n+1}} \left[ \frac{i}{2} (w_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \bar{w}_j \right]^{-n-1} f(w) \, d\omega(w) \]
\[ = - \int_{U^{n+1}} \frac{\partial}{\partial w_{n+1}} \left[ \frac{i}{2} (w_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \bar{w}_j \right]^{-n-1} f(w) \, \omega \]
\[ = \frac{(n+1)i}{2} (-2i)^{n+1} \int_{U^{n+1}} \left[ \frac{i}{2} (w_{n+1} - z_{n+1}) - \sum_{j=1}^{n} z_j \bar{w}_j \right]^{-n-2} f(w) \, dw \]
\[ = \frac{(n+1)i}{2} (-2i)^{n+1} \left( \frac{4\pi^{n+1}}{(n+1)!} \right) F(z) \]

where the last equality follows from the reproducing property of the Bergman kernel.

On the other hand, if we parameterize \( \partial U^{n+1} \) as usual by \( \mathbb{C}^n \times \mathbb{R} \),

\[ \int_{\partial U^{n+1}} G(w) \omega_{n+1}(w) = (-2i)^n \int_{\partial U^{n+1}} G(w) \, dw \, dt. \]

This gives the desired equality. □
We compute
\[
\left( \frac{n!}{\pi^{n+1}} \right)^{-2} \int_{\partial U^{n+1}} |S((z', t + i|z'|^2 + iρ_1), (w', s + i|w'|^2 + iρ_2))|^2 \, dz' \, dt
\]
\[
= \iint_{\mathbb{C}^n \times \mathbb{R}} \left| \frac{i}{2} (s - i|w'|^2 - iρ(w) - t - i|z'|^2 - iρ_1) - \sum_{j=1}^{n} \frac{z_j w_j}{|z_j|^2} \right|^{-2n-2} \, dz' \, dt
\]
\[
= (-4)^{n+1} \iint_{\mathbb{C}^n \times \mathbb{R}} \left| (t - s + 2\Re(z', w') + i||z' - w'||^2 + ρ_1 + ρ_2) \right|^{-2n-2} \, dz' \, dt
\]

1.4. Informal discussion of $H^2$-spaces.

The Szegö kernel arises as the integral kernel for the orthogonal projection from $L^2(\partial U^{n+1})$ to a closed subspace denoted by $H^2(U^{n+1})$. This can be thought of as the space of $L^2$ functions on the boundary which are the boundary values of a holomorphic function on $U^{n+1}$. The following discussion is an informal introduction to this topic.

We first consider the classical case of the unit disk $\mathbb{D}$ or the upper half plane $\mathbb{U}$ in $\mathbb{C}$. Thus if $O(\mathbb{D})$ and $O(\mathbb{U})$ denote the spaces of all holomorphic functions on the disk or the upper half plane, we define

\[
H^2(\mathbb{D}) = \left\{ f \in O(\mathbb{D}) \mid ||f||_{H^2}^2 = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{iθ})|^2 \, dθ < +\infty \right\};
\]

\[
H^2(\mathbb{U}) = \left\{ f \in O(\mathbb{U}) \mid ||f||_{H^2}^2 = \sup_{0 < y < +\infty} \int_{-\infty}^{+\infty} |f(x + iy)|^2 \, dx < +\infty \right\}.
\]

We would like to see that a function $f$ in either of these spaces has boundary values (on the unit circle $\mathbb{T}$ in the case of $H^2(\mathbb{D})$ and on the real line $\mathbb{R}$ in the case of $H^2(\mathbb{U})$). The case of the unit disk is easier, so we start with it.

§1. The case of the unit disk

If $f \in O(\mathbb{D})$, then $f$ is given by a power series

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n,
\]

which converges absolutely and uniformly on any compact subset of $\mathbb{D}$. Thus

\[
\frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{iθ})|^2 \, dθ = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{n=0}^{\infty} a_n r^n e^{inθ} \right) \overline{\left( \sum_{m=0}^{\infty} a_m r^m e^{imθ} \right)} \, dθ
\]

\[
= \sum_{m,n=0}^{\infty} a_n \overline{a_m} r^{m+n} \frac{1}{2\pi} \int_{0}^{2\pi} e^{i(n-m)θ} \, dθ
\]

\[
= \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.
\]

It follows that $f \in H^2(\mathbb{D})$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < +\infty$, and in fact

\[
||f||_{H^2(\mathbb{D})}^2 = \sum_{n=0}^{\infty} |a_n|^2.
\]
Now if \( f \in H^2(\mathbb{D}) \), what are the boundary values of \( f \)? Given the representation of \( f \) as a power series, it is natural to guess that the boundary values should be the function \( f^* \) given by

\[
f^*(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}.\]

Of course, this sum need not converge uniformly, or even at any given point. However since \( \sum_{n=0}^{\infty} |a_n|^2 \) is finite, the sum does converge in \( L^2(\mathbb{T}) \), and thus does define an element of \( L^2(\mathbb{T}) \). In what sense is this function the boundary values of the original \( f \)? One can prove the following facts:

1. If \( f \) is holomorphic on \( \mathbb{D} \), let \( f_r(e^{i\theta}) = f(re^{i\theta}) \). Then each \( f_r \) is infinitely differentiable on \( \mathbb{T} \) and hence in \( L^2(\mathbb{T}) \). Moreover, it follows from the definition that if \( f \in H^2(\mathbb{D}) \), then \( \sup_{0 \leq r < 1} ||f_r||_{L^2(\mathbb{T})} < +\infty \). One can prove that if \( f \in H^2(\mathbb{D}) \), then

\[
\lim_{r \to 1} ||f_r - f^*||_{L^2(\mathbb{T})} = 0.
\]

2. One can prove that if \( f \in H^2(\mathbb{D}) \), then for almost every \( \theta \in \mathbb{T} \),

\[
\lim_{r \to 1} f(re^{i\theta}) = f^*(e^{i\theta}).
\]

Thus one has radial convergence to the boundary values for almost all \( \theta \). In fact, one can show that for almost every \( \theta \)

\[
\lim_{z \to e^{i\theta}} f(z) = f^*(e^{i\theta})
\]

provided that \( z \) approaches \( e^{i\theta} \) within a non-tangential approach region.

The reproducing formula for \( H^2(\mathbb{D}) \) is

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{i\theta}) \frac{d\theta}{1 - ze^{-i\theta}}
\]

so the Szegö kernel in this case is

\[
S(z, w) = \frac{1}{2\pi} (1 - z\overline{w})^{-1}.
\]

§2. The case of the upper half plane

Instead of representations by power series, we need the following:

**Theorem:** A function \( f \in H^2(\mathbb{U}) \) if and only if there exists a function \( a \in L^2(\mathbb{R}) \) such that

\[
f(x + iy) = \int_0^{\infty} a(\xi) e^{2\pi i(x+iy)\xi} d\xi.
\]

(1.9) 

(This is one of the “Paley-Wiener” theorems, and a proof is given on pages 369-370 of the book *Real and Complex Analysis* by Walter Rudin).

Let \( f_y(x) = f(x + iy) \). Then (1.9) shows that \( f_y \) is the Fourier transform of the function \( a(\xi) \chi(\xi)e^{2\pi y\xi} \), where \( \chi \) is the characteristic function of the positive real axis. The Plancherel theorem then implies that

\[
\int_{-\infty}^{+\infty} |f(x+iy)|^2 dx = ||f_y||_{L^2(\mathbb{R})} = ||\hat{f}_y||_{L^2(\mathbb{R})} = \int_0^{\infty} |a(\xi)|^2 e^{-2\pi y\xi} d\xi,
\]
and hence that
\[ ||f||^2_{H^2(U)} = \sup_{y > 0} \int_{-\infty}^{+\infty} |f(x + iy)|^2 \, dx = \int_{0}^{\infty} |a(\xi)|^2 \, d\xi. \]

We would expect that the boundary values of \( f \) are given by
\[ f^*(x) = \int_{0}^{\infty} a(\xi) e^{2\pi ix} \, d\xi. \]
which is the Fourier transform of the function \( \chi_a \). Of course, this last integral does not converge absolutely or even pointwise, but the Fourier transform of a function in \( L^2(\mathbb{R}) \) is in \( L^2(\mathbb{R}) \). Then one can prove analogues of the two assertions for \( H^2(\mathbb{D}) \):

(1') \( \lim_{y \searrow 0} ||f_y - f^*||_{L^2(\mathbb{R})} = 0. \)

(2') For almost every \( x \in \mathbb{R} \), \( \lim_{y \searrow 0} f(x + iy) = f^*(x) \). Moreover, for almost every \( x \in \mathbb{R} \), \( \lim_{z \to x} f(z) = f^*(x) \) provided that \( z \) approaches \( x \) within a non-tangential region.

The reproducing formula for \( H^2(U) \) is
\[ f(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f^*(x) \frac{dx}{z - x}, \]
so the Szegő kernel in this case is
\[ S(z, w) = \frac{1}{\pi} (z - \bar{w})^{-1}. \]

The following are some reference for the classical theory of \( H^p \)-spaces:


2. Some analysis on the Heisenberg group \( \mathbb{H}^n \)

The left and right translation operators act on functions on \( \mathbb{H}^n \) as follows. If \( f : \mathbb{H}^n \to \mathbb{C} \), then
\[ L_w[f](z) = f(w^{-1} \cdot z) \quad \text{(left translation)}, \]
\[ R_w[f](z) = f(z \cdot w) \quad \text{(right translation)}. \]

Note that with these definitions, \( L_{w_1} \circ L_{w_2} = L_{w_1 \cdot w_2} \) and \( R_{w_1} \circ R_{w_2} = R_{w_1 \cdot w_2} \).

We say that a space \( X \) of functions on \( \mathbb{H}^n \) is (left or right) translation invariant if whenever \( f \in X \), it follows that \( L_w[f] \) or \( R_w[f] \) belongs to \( X \) for all \( w \in \mathbb{H}^n \). If \( X \) and \( Y \) are (left or right) translation invariant spaces of functions, and if \( T : X \to Y \) is a linear operator, we say \( T \) is (left or right) translation invariant if \( TL_w = L_wT \) or \( TR_w = R_wT \) for all \( w \in \mathbb{H}^n \).

Suppose that \( T : X \to Y \) is a left-invariant linear operator, and consider the following informal argument. Suppose that the action of \( T \) is given by
\[ T[f](z) = \int_{\mathbb{H}^n} K(z, w) f(w) \, dw \]
where $K$ is the “integral kernel” of the operator $T$. (As always, $dw$ is Lebesgue measure on $\mathbb{H}^n$, and hence is invariant under left and right translation.) Then we have

$$L_u T[f](z) = T[f](u^{-1} \cdot z) = \int_{\mathbb{H}^n} K(u^{-1} \cdot z, w) f(w) \, dw.$$  

But $L_u T = TL_u$, so we also have

$$L_u T[f](z) = \int_{\mathbb{H}^n} K(z, w) T_u[f](w) \, dw = \int_{\mathbb{H}^n} K(z, w) f(u^{-1} \cdot w) \, dw$$

and

$$= \int_{\mathbb{H}^n} K(z, u \cdot w) f(w) \, dw$$

where the last equality follows by replacing the variable $w$ by $u \cdot w$, and using the fact that $dw$ is translation invariant. Thus for every $z \in \mathbb{H}^n$ and every $f \in X$ we have

$$\int_{\mathbb{H}^n} K(u^{-1} \cdot z, w) f(w) \, dw = \int_{\mathbb{H}^n} K(z, u \cdot w) f(w) \, dw.$$  

Under reasonable assumptions about the density of the space $X$, it should follow that $K(u^{-1} \cdot z, w) = K(z, u \cdot w)$ for all $u, z, w \in \mathbb{H}^n$, or replacing $z$ by $u \cdot z$, that $K(z, w) = K(u \cdot z, u \cdot w)$. If we let $u = w^{-1}$, we see that $K(z, w) = K(w^{-1} \cdot z, e)$. Thus if we put $k(z) = K(z, e)$, we see that our left-invariant operator $T$ can be written

$$T[f](z) = \int_{\mathbb{H}^n} f(w) k(w^{-1}z) \, dw = \int_{\mathbb{H}^n} f(z \cdot w^{-1}) k(w) \, dw.$$  

Conversely, any operator of this form is left-invariant since

$$L_u T[f](z) = T[f](u^{-1} \cdot z)$$

$$= \int_{\mathbb{H}^n} f(w) k((u \cdot w)^{-1} \cdot z) \, dw$$

$$= \int_{\mathbb{H}^n} f(w^{-1} \cdot w) k(w^{-1} \cdot z) \, dw$$

$$= \int_{\mathbb{H}^n} L_u [f](w) k(w^{-1} \cdot z) \, dw$$

$$= TL_u [f](z).$$

Note that if $T[f](z) = \int_{\mathbb{H}^n} f(z \cdot w^{-1}) k(w) \, dw$, then

$$\int_{\mathbb{H}^n} T[f](z) g(z) \, dz = \int_{\mathbb{H}^n} k(w) \left[ \int_{\mathbb{H}^n} f(z \cdot w^{-1}) g(z) \, dz \right] \, dw$$

$$= \int_{\mathbb{H}^n} k(w) \left[ \int_{\mathbb{H}^n} \tilde{f}(w \cdot z^{-1}) g(z) \, dz \right] \, dw$$

$$= \int_{\mathbb{H}^n} k(w) (\tilde{f} \ast g)(w) \, dw,$$

where $\tilde{f}(z) = f(z^{-1})$, and

$$f \ast g(z) = \int_{\mathbb{H}^n} f(z \cdot w^{-1}) g(w) \, dw = \int_{\mathbb{H}^n} f(w) g(w^{-1} \cdot z) \, dw.$$  

A similar argument suggest that a right-invariant operator $S$ can be written

$$S[f](z) = \int_{\mathbb{H}^n} k(z \cdot w^{-1}) f(w) \, dw = \int_{\mathbb{H}^n} k(w) f(w^{-1} \cdot z) \, dw.$$
To state a precise result, we introduce the following notation. Let $\mathcal{D}(\mathbb{H}^n)$ be the space of $C^\infty$ functions with compact support on $\mathbb{H}^n$, and given the topology such that $\varphi_n \to 0$ if and only if the supports of the $\{\varphi_n\}$ all lie in a fixed compact set, and the functions $\{\varphi_n\}$ and all their derivatives converge uniformly to zero. Let $\mathcal{D}'(\mathbb{H}^n)$ be the space of continuous linear functionals on $\mathcal{D}(\mathbb{H}^n)$. (Thus $\mathcal{D}'(\mathbb{H}^n)$ is the space of distributions on $\mathbb{H}^n$). The action of $T \in \mathcal{D}'(\mathbb{H}^n)$ on $\varphi \in \mathcal{D}(\mathbb{H}^n)$ is denoted by $(T, \varphi)$. We give $\mathcal{D}'(\mathbb{H}^n)$ the topology such that $T_n \to 0$ if and only if $(T_n, \varphi) \to 0$ for every $\varphi \in \mathcal{D}(\mathbb{H}^n)$. In a similar way, we define the space of test functions $\mathcal{D}(\mathbb{H}^n \times \mathbb{H}^n)$ and the space of distributions $\mathcal{D}'(\mathbb{H}^n \times \mathbb{H}^n)$ on $\mathbb{H}^n \times \mathbb{H}^n$.

If $\varphi, \psi \in \mathcal{D}(\mathbb{H}^n)$, define $\varphi \otimes \psi \in \mathcal{D}(\mathbb{H}^n \times \mathbb{H}^n)$ by setting $\varphi \otimes \psi(z, w) = \varphi(z) \psi(w)$. Also define $\varphi \ast \psi \in \mathcal{D}(\mathbb{H}^n)$ by

$$\varphi \ast \psi(z) = \int_{\mathbb{H}^n} \varphi(z \cdot w^{-1}) \psi(w) \, dw = \int_{\mathbb{H}^n} \varphi(w) \psi(w^{-1} \cdot z) \, dw.$$

**Lemma 2.1.** Suppose that $T : \mathcal{D}(\mathbb{H}^n) \to \mathcal{D}'(\mathbb{H}^n)$ is a continuous linear operator. Then there is a distribution $K \in \mathcal{D}'(\mathbb{H}^n \times \mathbb{H}^n)$ such that if $\varphi, \psi \in \mathcal{D}(\mathbb{H}^n)$, then

$$\langle T[\varphi], \psi \rangle = \langle K, \varphi \otimes \psi \rangle.$$

Moreover, if $T$ is left-invariant, there is a distribution $k \in \mathcal{D}'(\mathbb{H}^n)$ so that

$$\langle T[\varphi], \psi \rangle = \langle k, \varphi \ast \psi \rangle.$$  

(2.1)

If $T$ is right-invariant, there is a distribution $k \in \mathcal{D}'(\mathbb{H}^n)$ so that

$$\langle T[\varphi], \psi \rangle = \langle k, \psi \ast \varphi \rangle.$$  

(2.2)

2.1. **Left-invariant vector fields.**

Recall that $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$ with multiplication given by

$$(w, s) \cdot (z, t) = (w + z, s + t + 2\overline{m}(w, z))$$

$$(w, s)^{-1} \cdot (z, t) = (z - w, t - s - 2\overline{m}(w, z)).$$

If we use real coordinates, so that $z_j = x_j + iy_j$, and $w_j = u_j + iv_j$, then

$$w_j \overline{z_j} = (u_j + iv_j)(x_j - iy_j) = (x_j u_j + y_j v_j) + i(x_j v_j - y_j u_j).$$

Thus the multiplication on $\mathbb{R}^{2n+1}$ is given by

$$(u, v, s) \cdot (x, y, t) = (x + u, y + v, t + s + 2 \sum_{j=1}^{n} (x_j v_j - y_j u_j),$$

$$(u, v, s)^{-1} \cdot (x, y, t) = (x - u, y - v, t - s - 2 \sum_{j=1}^{n} (x_j v_j - y_j u_j).$$

Consider the vector fields $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ on $\mathbb{H}^n$ given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad 1 \leq j \leq n,$$

$$Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad 1 \leq j \leq n,$$

$$T = \frac{\partial}{\partial t}.$$
Proposition 2.2. The vector fields \( \{X_1, Y_1, \ldots, X_n, Y_n, T\} \) are left-invariant operators on \( \mathbb{H}^n \), and they generate a finite dimensional Lie algebra with the following commutator identities:

\[
[X_j, Y_k] = -4\delta_{j,k} T, \quad 1 \leq j, k \leq n, \\
[X_j, X_k] = 0, \quad 1 \leq j, k \leq n, \\
[Y_j, Y_k] = 0, \quad 1 \leq j, k \leq n, \\
[X_j, T] = 0, \quad 1 \leq j \leq n, \\
[Y_k, T] = 0, \quad 1 \leq k \leq n.
\]

Proof. Let \((u, v, s) \in \mathbb{H}^n\). Then

\[
L_{(u,v,s)} X_j[f](x, y, t) = L_{(u,v,s)} \left[ \frac{\partial f}{\partial x_j}(x, y, t) + 2y_j \frac{\partial f}{\partial t}(x, y, t) \right]
\]

\[
= \frac{\partial f}{\partial x_j} ((u, v, s)^{-1} \cdot (x, y, t)) + 2(y_j - v_j) \frac{\partial f}{\partial t} ((u, v, s)^{-1} \cdot (x, y, t)).
\]

On the other hand,

\[
X_j L_{(u,v,s)}[f](x, y, t)
\]

\[
= \left[ \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \right] f(x - u, y - v, t - s - 2 \sum_{j=1}^{n} (x_j v_j - y_j u_j)
\]

\[
= \frac{\partial f}{\partial x_j} ((u, v, s)^{-1} \cdot (x, y, t)) - 2v_j \frac{\partial f}{\partial t} ((u, v, s)^{-1} \cdot (x, y, t))
\]

\[
+ 2y_j \frac{\partial f}{\partial t} ((u, v, s)^{-1} \cdot (x, y, t))
\]

\[
= \frac{\partial f}{\partial x_j} ((u, v, s)^{-1} \cdot (x, y, t)) + 2(y_j - v_j) \frac{\partial f}{\partial t} ((u, v, s)^{-1} \cdot (x, y, t)),
\]

so the two are the same. The argument for \( Y_j \) is similar, and the argument for \( T \) is easy. The calculation of commutators is also easy. \( \square \)

2.2. Tangential CR operators on \( \partial U^{n+1} \).

The defining function for \( U^{n+1} \) is

\[
\rho(z', z_{n+1}) = \frac{z_{n+1} - z_{n+1}}{2i} - \sum_{j=1}^{n} z_j z_j.
\]

We look for operators \( T_j, 1 \leq j \leq n \) of the form \( \tilde{L}_j = \frac{\partial}{\partial z_j} + A_j \frac{\partial}{\partial z_{n+1}} \) which are tangential along \( \partial U^{n+1} \). Recall that this means we need \( \tilde{L}_j[\rho] = 0 \) when \( \rho = 0 \).

But

\[
\tilde{L}_j[\rho](z', z_{n+1}) = -z_j - \frac{1}{2i} A_j,
\]

and this is zero if \( A_j = -2iz_j \). Thus if we put

\[
L_j = \frac{\partial}{\partial z_j} - 2i \frac{\partial}{\partial z_{n+1}},
\]

(2.4)
then the collection \{\bar{L}_1, \ldots, \bar{L}_n\} is a basis for the space of operators of type (0, 1) which are tangential along \partial U^{n+1}. In particular, these operators will annihilate the boundary values of functions which are holomorphic on \partial U^{n+1}.

What are the operators \{\bar{L}_j\} in terms of the coordinates \((z, t)\)? Given a function \(f\) on \(\mathbb{C}^n \times \mathbb{R}\), it defines a function \(\tilde{f}\) on \(\partial U^{n+1}\) by setting \(\tilde{f}(z, t + i|z|^2) = f(z, t)\). To see what \(\bar{L}_j\) does to this function \(\tilde{f}\), we extend it to a function on \(\mathbb{C}^{n+1}\) by setting \(F(z, z_{n+1}) = \tilde{f}(z, \Re[z_{n+1}] + i|z|^2)\). But then it is clear that \(\bar{L}_j[F](z, t + i|z|^2) = \partial f/\partial \bar{z}_j(z, t) - iz_j \partial f/\partial t(z, t)\).

Thus on \(\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}\), a basis for the CR differential operators is given by \{\bar{Z}_1, \ldots, \bar{Z}_n\} where

\[
\bar{Z}_j = 2 \frac{\partial}{\partial z_j} - 2i z_j \frac{\partial}{\partial t} = \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} - 2i(x_j + iy_j) \frac{\partial}{\partial t} = \left[ \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \right] + i \left[ \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \right] = X_j + iY_j.
\]

**Corollary 2.3.** If we set \(Z_j = X_j - iY_j\), then the collection of differential operators \{\bar{Z}_1, \ldots, \bar{Z}_n, \bar{z}_1, \ldots, \bar{Z}_n, T\} are a basis for all first order differential operators on \(\mathbb{H}^n\), and the satisfy the commutation relationships \([Z_j, Z_k] = \bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}_k, T] = 0\), and \([Z_j, \bar{Z}_k] = 8i\delta_{j,k} T\).

### 2.3. The \(\bar{\partial}_b\) complex.

In analogy with the \(\bar{\partial}\) problem on a domain in \(\mathbb{C}^{n+1}\), we can now formulate a corresponding problem on the boundary of a domain. In the case of \(\mathbb{H}^n\), we proceed as follows.

**Definition 2.4.** If \(f \in C^1(\mathbb{H}^n)\), define \(\bar{\partial}_b[f] = \sum_{j=1}^n \bar{Z}_j[f] \, d\bar{z}_j\).

If \(\omega\) is a \((0, q)\)-form on \(\mathbb{H}^n\), that is, if

\[
\omega = \sum_{j_1 < \cdots < j_q} \omega_{j_1, \ldots, j_q} \, d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},
\]

then

\[
\bar{\partial}_b[\omega] = \sum_{j_1 < \cdots < j_q} \sum_{j=1}^n \bar{Z}_j[\omega_{j_1, \ldots, j_q}] \, d\bar{z}_j \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.
\]
Proposition 2.5. If \( \omega \in C^2(\mathbb{H}^n)_{(0,q)} \), then \( \bar{\partial}_b^2[\omega] = 0 \).

Proof. Let \( \omega = \sum_{j_1 < \cdots < j_q} \omega_{j_1,\ldots,j_q} \, d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \). Then
\[
\bar{\partial}_b^2[\omega] = \bar{\partial}_b \left[ \sum_{j_1 < \cdots < j_q} \sum_{j=1}^{n} Z_j \omega_{j_1,\ldots,j_q} \, d\bar{z}_j \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \right]
= \sum_{j_1 < \cdots < j_q} \sum_{j=1}^{n} \sum_{k=1}^{n} \bar{Z}_k \bar{Z}_j \omega_{j_1,\ldots,j_q} \, d\bar{z}_k \wedge d\bar{z}_j \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}.
\]

But since \([\bar{Z}_k, \bar{Z}_j] = 0\), we have
\[
\bar{Z}_k \bar{Z}_j \omega_{j_1,\ldots,j_q} \, d\bar{z}_k \wedge d\bar{z}_j = -\bar{Z}_j \bar{Z}_k [\omega_{j_1,\ldots,j_q}] \, d\bar{z}_j \wedge d\bar{z}_k
\]
Thus in the inner sum, each term appears twice, but with opposite signs. Thus the entire sum is zero. \(\square\)

Since \(X_j\) and \(Y_j\) are real differential operators, integration by parts shows that if \(f, g \in C^l(\mathbb{H}^n)\) and at least one of the two functions has compact support, then
\[
(X_j[f],g)_{L^2(\mathbb{H}^n)} = -(f,X_j[g])_{L^2(\mathbb{H}^n)} \quad \text{and} \quad (Y_j[f],g)_{L^2(\mathbb{H}^n)} = -(f,Y_j[g])_{L^2(\mathbb{H}^n)}.
\]
It follows that
\[
(Z_j[f],g)_{L^2(\mathbb{H}^n)} = -(f,Z_j[g])_{L^2(\mathbb{H}^n)}.
\]

We introduce the standard \(L^2\)-norm on the space of \((0,q)\)-forms on \(\mathbb{H}^n\). Thus if \(\omega = \sum_{j_1 < \cdots < j_q} \omega_{j_1,\ldots,j_q} \, d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}\) and \(\psi = \sum_{j_1 < \cdots < j_q} \psi_{j_1,\ldots,j_q} \, d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}\), we set
\[
(\omega,\psi)_{L^2} = \sum_{j_1 < \cdots < j_q} \int_{\mathbb{H}^n} \omega_{j_1,\ldots,j_q}(z,t) \psi_{j_1,\ldots,j_q}(z,t) \, dz \, dt.
\]
We want to compute the formal adjoint of \(\bar{\partial}_b\). Let \(\omega\) as above be a \((0,q)\)-form, and let \(\theta = \sum_{k_1 < \cdots < k_{q+1}} \theta_{k_1,\ldots,k_{q+1}} \, d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_{q+1}}\) be a \((0,q+1)\)-form. Was assume that the coefficients are of class \(C^1\) and either \(\omega\) or \(\theta\) has compact support. Then
\[
\bar{\partial}_b[\omega] = \sum_{j_1 < \cdots < j_q} \sum_{k=1}^{n} \bar{Z}_k [\omega_{j_1,\ldots,j_q}] \, d\bar{z}_k \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}
= \sum_{m_1 < \cdots < m_{q+1}} \sum_{j_1 < \cdots < j_q} \sum_{k=1}^{n} \bar{Z}_k [\omega_{j_1,\ldots,j_q}] \epsilon_{m_1,m_2,\ldots,m_{q+1}}^{j_1,j_2,\ldots,j_q} \, d\bar{z}_{m_1} \wedge \cdots \wedge d\bar{z}_{m_{q+1}}
\]

where
\[
\epsilon_{m_1,m_2,\ldots,m_{q+1}}^{j_1,j_2,\ldots,j_q} = \begin{cases} 0 & \text{if } \{m_1,m_2,\ldots,m_{q+1}\} \neq \{k,j_1,\ldots,j_q\}, \\ \text{sgn}(\sigma) & \text{if } \sigma(m_1,m_2,\ldots,m_{q+1}) = (k,j_1,\ldots,j_q). \end{cases}
\]
It now follows that
\[
(\bar{\partial}_b[\omega],\theta)_{L^2(\mathbb{H}^n)_{(0,q+1)}} = \sum_{m_1 < \cdots < m_{q+1}} \sum_{j_1 < \cdots < j_q} \sum_{k=1}^{n} \epsilon_{m_1,m_2,\ldots,m_{q+1}}^{j_1,j_2,\ldots,j_q} \left(\bar{Z}_k [\omega_{j_1,\ldots,j_q}],\theta_{m_1,\ldots,m_{q+1}}\right)_{L^2(\mathbb{H}^n)}
= -\sum_{j_1 < \cdots < j_q} \sum_{m_1 < \cdots < m_{q+1}} \sum_{k=1}^{n} \epsilon_{m_1,m_2,\ldots,m_{q+1}}^{j_1,j_2,\ldots,j_q} \left(\omega_{j_1,\ldots,j_q},\bar{Z}_k [\theta_{m_1,\ldots,m_{q+1}}]\right)_{L^2(\mathbb{H}^n)}.\]
It follows that
\[ \bar{\partial}_b^q[\theta] = - \sum_{j_1 < \cdots < j_q < m_1 < \cdots < m_{q+1}} \sum_{k=1}^n \epsilon_{k,j_1,\cdots,j_q} m_{j_1} m_{j_2} \cdots m_{j_q+1} Z_k[\theta_{m_1,\cdots,m_{q+1}}] d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_q. \]

To eliminate the need for so many subscripts, we introduce the following notation. \( I_q \) denotes the set of all strictly increasing \( q \)-tuples \( J = (j_1 < \cdots < j_q) \) of integers with \( 1 \leq j_1 < j_q \leq n \). If \( J_1 = (j_1^1, \ldots, j_q^1) \) and \( J_2 = (j_1^2, \ldots, j_q^2) \) are any ordered \( q \)-tuples of integers with \( 1 \leq j_1^1, j_2^1 \leq n \), we set
\[ \epsilon_{J_1}^1 = \begin{cases} 0 & \text{if either } J_1 \text{ or } J_2 \text{ has a repeated element}, \\ 0 & \text{if the set } J_1 \text{ is not equal to the set } J_2, \\ \text{sgn}(\sigma) & \text{if } \sigma : J_1 \to J_2 \text{ with } \sigma \in \mathfrak{S}_q. \end{cases} \]

Then with this notation, if \( \omega \) is a \((0,q)\)-form and if \( \theta \) is a \((0,q+1)\)-form,
\[ \bar{\partial}_b[\omega] = \sum_{K \in I_{q+1}} \left\{ \sum_{J \in I_q} \sum_{j=1}^n \epsilon_{J}^K Z_j[\omega_J] \right\} d\bar{z}_K, \]
\[ \bar{\partial}_b^q[\theta] = - \sum_{L \in I_q} \left\{ \sum_{K \in I_{q+1}} \sum_{\ell=1}^n \epsilon_{\ell,L}^K Z_{\ell}[\theta_K] \right\} d\bar{z}_L. \]

We now compute the operator \( \Box_b = \bar{\partial}_b \bar{\partial}_b^q + \bar{\partial}_b^q \bar{\partial}_b \).

**Proposition 2.6.** If \( \omega = \sum_{J \in I_q} \omega_{J} d\bar{z}_J \), then
\[ \Box_b[\omega] = \sum_{J \in I_q} \left\{ \sum_{j \in J} \bar{Z}_j Z_{\ell}[\omega_{J}] + \sum_{j \notin J} Z_j \bar{Z}_j[\omega_{J}] \right\} d\bar{z}_J. \]

**Proof.** We have
\[ \bar{\partial}_b \bar{\partial}_b^q[\omega] = \bar{\partial}_b^q \left[ \sum_{K \in I_{q+1}} \left\{ \sum_{J \in I_q} \sum_{j=1}^n \epsilon_{J}^K Z_j[\omega_J] \right\} d\bar{z}_K \right] \]
\[ = - \sum_{L \in I_q} \left\{ \sum_{K \in I_{q+1}} \sum_{\ell=1}^n \left( \sum_{J \in I_q} \sum_{j=1}^n \epsilon_{\ell,L}^K \epsilon_{J,j}^K Z_{\ell}[\omega_J] \right) \right\} d\bar{z}_L \]
\[ = - \sum_{L \in I_q} \left\{ \sum_{J \in I_q} \sum_{\ell=1}^n \left( \sum_{j=1}^n \epsilon_{j,L}^K Z_{\ell}[\omega_J] \right) \right\} d\bar{z}_L \]
\[ = - \sum_{L \in I_q} \left\{ \sum_{j=1}^n \left( \sum_{\ell=1}^n \epsilon_{j,L}^K Z_{\ell}[\omega_J] \right) \right\} d\bar{z}_L. \]

Now in the inner sum, we can have \( J = L \) and \( j = \ell \notin J = L \), so that \( \epsilon_{L,L}^J = 1 \), which gives rise to the sum
\[ I = - \sum_{L \in I_q} \left\{ \sum_{\ell \notin L} Z_{\ell} \bar{Z}_{\ell}[\omega_L] \right\} d\bar{z}_L. \]

The other possibility is that \( j \neq \ell, j \in L, \) and \( \ell \in J \), in which case there exists \( M \in I_{q-1} \) such that \( L = M \cup \{j\}, J = M \cup \{\ell\} \). In this case
\[ \epsilon_{L,L}^J = \epsilon_{L,J,M}^J \epsilon_{L,M}^J \epsilon_{M,L}^J = -\epsilon_{L,M}^J \epsilon_{M,L}^J. \]
and \( \omega_J = \epsilon_{J,M}^L \omega_{J,M} \). Thus we can write

\[
\sum_{J \in I_q} \left( \sum_{j=1}^n \sum_{\ell=1}^n \epsilon_{J,L}^j Z_{\ell j} [\omega_J] \right) = - \sum_{M \in I_{q-1}} \sum_{j: \ell \notin M \atop j \neq \ell} \epsilon_{L,M}^j \epsilon_{J,L}^j \epsilon_{L,M}^J \omega_{J,M} Z_{\ell j}[\omega_{J,M}]
\]

\[
= - \sum_{M \in I_{q-1}} \sum_{j: \ell \notin M \atop j \neq \ell} \epsilon_{L,M}^j Z_{\ell j}[\omega_{J,M}]
\]

This gives rise to the sum

\[
II = \sum_{L \in I_q} \left[ \sum_{M \in I_{q-1}} \sum_{j: \ell \notin M \atop j \neq \ell} \epsilon_{L,M}^j Z_{\ell j}[\omega_{J,M}] \right] d\bar{\varepsilon}_L.
\]

On the other hand

\[
\partial_b \partial_b^* [\omega] = \partial_b \left[ - \sum_{M \in I_{q-1}} \left[ \sum_{J \in I_q} \sum_{m=1}^n \epsilon_{m,M}^j Z_m [\omega_J] d\bar{\varepsilon}_M \right] \right]
\]

\[
= - \sum_{L \in I_q} \left[ \sum_{M \in I_{q-1}} \left[ \sum_{J \in I_q} \sum_{m=1}^n \epsilon_{L,M}^j \epsilon_{m,M}^j Z_{\ell j} Z_m [\omega_J] \right] \right] d\bar{\varepsilon}_L
\]

\[
= - \sum_{L \in I_q} \left[ \sum_{J \in I_q} \left[ \sum_{m=1}^n \sum_{M \in I_{q-1}} \epsilon_{L,M}^j \epsilon_{m,M}^j Z_{\ell j} Z_m [\omega_J] \right] \right] d\bar{\varepsilon}_L
\]

Now for a given \( L \in I_q, \epsilon_{L,M}^j = 0 \) unless \( \ell \in L \) and \( L = M \cup \{\ell\} \). Moreover, \( \epsilon_{m,M}^j = 0 \) unless \( m \notin M \) and \( J = M \cup \{m\} \). It can happen that \( \ell = m \), in which case \( J = L, \ell = m \in J = L \), and \( \epsilon_{L,M}^j \epsilon_{m,M}^j = 1 \). This gives rise to the sum

\[
III = - \sum_{L \in I_q} \left[ \sum_{\ell \in L} Z_{\ell j} \bar{Z}_L [\omega_L] \right] d\bar{\varepsilon}_L.
\]

The other possibility is that \( \ell \neq m \), in which case \( \ell, m \notin M \), and \( L = \{\ell\} \cup M \), \( J = \{m\} \cup M \), and \( \omega_J = \epsilon_{m,M}^j \omega_{m,M} \). Then

\[
\sum_{\ell=1}^n \sum_{m=1}^n \sum_{M \in I_{q-1}} \epsilon_{L,M}^j \epsilon_{m,M}^j \bar{Z}_L Z_m [\omega_J] = \sum_{M \in I_{q-1}} \sum_{\ell, m \notin M \atop \ell \neq m} \epsilon_{L,M}^j \epsilon_{m,M}^j \epsilon_{m,M}^j Z_{\ell j} Z_m [\omega_{J,M}]
\]

\[
= \sum_{M \in I_{q-1}} \sum_{\ell, m \notin M \atop \ell \neq m} \epsilon_{L,M}^j Z_{\ell j} Z_m [\omega_{J,M}].
\]

This gives rise to the term

\[
IV = - \sum_{L \in I_q} \left[ \sum_{M \in I_{q-1}} \sum_{\ell, m \notin M \atop \ell \neq m} \epsilon_{L,M}^j Z_{\ell j} Z_m [\omega_{J,M}] \right] d\bar{\varepsilon}_L.
\]
For \( \ell \neq m \), \([ \bar{Z}_\ell, Z_m ] = 0 \), so the terms \( II \) and \( IV \) cancel. Thus
\[
\Box_b[\omega] = - \sum_{L \in \mathcal{I}_q} \left[ \sum_{\ell \notin L} Z_\ell \bar{Z}_\ell[\omega_L] \right] d\bar{z}_L - \sum_{L \in \mathcal{I}_q} \left[ \sum_{\ell \in L} \bar{Z}_\ell Z_\ell[\omega_L] \right] d\bar{z}_L.
\]
This completes the proof. \(\Box\)

**Remark 2.7.** The operator \( \Box_b : C^\infty_{(0,q)} \to C^\infty_{(0,q)} \) is a system of second order partial differential operators, but in this case, the system is actually diagonal.

We now write
\[
\Box_j = \frac{1}{2} \left[ Z_j \bar{Z}_j + \bar{Z}_j Z_j \right]
\]
\[
= \frac{1}{2} \left( (X_j - iY_j)(X_j + iY_j) + (X_j + iY_j)(X_j - iY_j) \right)
\]
\[
= \frac{1}{2} \left[ X_j^2 + i[X_j, Y_j] + Y_j^2 \right]
\]
\[
= X_j^2 + Y_j^2.
\]

Now
\[
[Z_j, \bar{Z}_j] = [(X_j - iY_j), (X_j + iY_j)] = i[X_j, Y_j] = -4iT,
\]

Thus
\[
Z_j \bar{Z}_j = \frac{1}{2} \left[ Z_j \bar{Z}_j + \bar{Z}_j Z_j \right] + \frac{1}{2} \left[ Z_j \bar{Z}_j - \bar{Z}_j Z_j \right] = \Box_j + \frac{1}{2} \left[ Z_j, \bar{Z}_j \right] = \Box_j - 2iT
\]
\[
\bar{Z}_j Z_j = \frac{1}{2} \left[ Z_j \bar{Z}_j + \bar{Z}_j Z_j \right] - \frac{1}{2} \left[ Z_j \bar{Z}_j - \bar{Z}_j Z_j \right] = \Box_j - \frac{1}{2} \left[ Z_j, \bar{Z}_j \right] = \Box_j + 2iT.
\]

Finally, we let
\[
\mathcal{L} = - \sum_{j=1}^n \Box_j = - \sum_{j=1}^n (X_j^2 + Y_j^2).
\]

Then it follows that if \( L \in \mathcal{I}_q \),
\[
- \sum_{j \in L} \bar{Z}_j Z_j - \sum_{j \notin L} Z_j \bar{Z}_j = \mathcal{L} - 2qiT + 2(n - q)iT
\]
\[
= - \mathcal{L} + 2i(n - 2q)T.
\]

**Definition 2.8.** For \( \alpha \in \mathbb{R} \), put
\[
\mathcal{L}_\alpha = - \sum_{j=1}^n (X_j^2 + Y_j^2) + 2i\alpha T.
\]

**Theorem 2.9** (Folland and Stein, 1974). Let
\[
\phi_\alpha(z, t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}} \left( |z|^2 + it \right)^{-\frac{n-\alpha}{2}}.
\]
If \( \delta_0 \) denotes the “delta function” supported at the origin of \( \mathbb{H}^n \), then
\[
\mathcal{L}_\alpha[\phi_\alpha] = \gamma_\alpha \delta_0
\]
where
\[
\gamma_\alpha = \frac{2^{2-n} n^{n+1}}{\Gamma \left( \frac{n+\alpha}{2} \right) \Gamma \left( \frac{n-\alpha}{2} \right)}.
\]