

SOLUTIONS

Problem 1:

(a)

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 4 & 0 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

The associated equation is

$$\begin{cases} x_1 + 2x_2 + 2x_3 & = 0 \\ x_2 + x_3 + 2x_4 & = 0 \end{cases}$$

and hence x_3 and x_4 are free variables. Set $x_3 = t$, $x_4 = s$ and solve for x_1, x_2 to get

$$x_2 = -t - 2s, x_1 = -2(-t - 2s) - 2t = 4s.$$

Therefore the general solution is given by

$$x_1 = 4s, x_2 = -t - 2s, x_3 = t, x_4 = s, \quad t, s \text{ arbitrary.}$$

(b) Write the solution in a form of column vectors :

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 4s \\ -t - 2s \\ t \\ s \end{pmatrix} = s \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}.$$

This shows that any solution is a linear combination of the two vectors

$$\left\{ \begin{pmatrix} 4 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

This is a basis of the solution space.

(c) There are 2 elements in the above basis and so the dimension of the solution space is 2.

Problem 2:(a) A basis is $\{1, t, t^2\}$.

(b) Consider the defining equation of the linear independence

$$c_1(t - 1) + c_2(t^2 + 1) = \vec{0}$$

and see if this polynomial equation has only the trivial solution for c_1, c_2 . By turning the equation into the standard form,

$$c_2t^2 + c_1t + (-c_1 + c_2) = \vec{0}$$

we obtain $c_2 = 0, c_1 = 0, -c_1 + c_2 = 0$. Therefore $c_1 = c_2 = 0$ is the only solution and hence $\{t - 1, t^2 + 1\}$ is linearly independent.(c) Just add the constant polynomial 1 to the set $S : \{t - 1, t^2 + 1, 1\}$ is a basis extending S .**Problem 3:**(a) $\|u_1\| = \sqrt{1+1} = \sqrt{2}$.

- (b) We compute $\|u_2\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$ and $(u_1, u_2) = 1 \times 1 + 0 \times (-2) + 1 \times 2 = 3$. Therefore we have

$$\cos \theta = \frac{(u_1, u_2)}{\|u_1\| \|u_2\|} = \frac{3}{\sqrt{2} \times 3} = \frac{1}{\sqrt{2}},$$

and so $\theta = \frac{\pi}{4}$.

- (c) We start with the formula

$$v_1 = u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

and

$$v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1 = u_2 - \frac{3}{2} v_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -2 \\ \frac{1}{2} \end{pmatrix}$$

to obtain an **orthogonal** basis. Then we **normalize** them to obtain an **orthonormal** basis :

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{3\sqrt{2}} \\ -\frac{2}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix} \right\}.$$

Problem 4 :

- (a) See Problem 6 for the similar problem.
 (b) We can write

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and hence a basis is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Then we express

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Problem 5 :

We reduce $A = \begin{pmatrix} 5 & -3 & 1 \\ 1 & -2 & 1 \\ 3 & 1 & -1 \end{pmatrix}$ to a row echelon form :

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -\frac{4}{7} \\ 0 & 0 & 0 \end{pmatrix}$$

- (a) To find a basis of the row space of A , we just take the non-zero rows of the row echelon form

$$\left\{ (1, -2, 1), (0, 1, -\frac{4}{7}) \right\}.$$

To find a basis of the column space of A , we find the pivot columns of the row echelon form : In this example, the pivot columns are the 1-st and the 2-nd columns. Then we take the corresponding column vectors of the original matrix A , which are

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

(b) To find a basis of the null space, we solve the equation associated to the above row-echelon form :

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_2 - \frac{4}{7}x_3 = 0. \end{cases}$$

Setting the free variable $x_3 = t$, we solve

$$x_2 = \frac{4t}{7}, x_1 = \frac{t}{7}.$$

Writing this into the vector form, we obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{t}{7} \\ \frac{4t}{7} \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{7} \\ \frac{4}{7} \\ 1 \end{pmatrix}.$$

Therefore a basis of the null space is given by

$$\left\{ \begin{pmatrix} \frac{1}{7} \\ \frac{4}{7} \\ 1 \end{pmatrix} \right\}.$$

(c) rank $A = 2$, nullity $A = 1$.

(d) Knowing that

$$\left\{ \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

is a basis of the column space of A , we check if the equation

$$c_1 \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

has a non-trivial solution : The associated augmented matrix is

$$\left(\begin{array}{cc|c} 5 & -3 & 4 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{array} \right).$$

Its row echelon form is

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -\frac{1}{7} \\ 0 & 0 & -1 \end{array} \right).$$

Therefore the associated equation is **inconsistent** and so the vector is **not** in the column space.

Problem 6:

(a) We have only to prove :

1. If $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ are in W , then $A_1 + A_2$ is also in W .
2. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in W , then $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is also in W for all λ .

We start with (1). By the definition of W , which is *the sum of its (1,1)-entry and (2,2) being zero*, the matrices A_1, A_2 satisfy $a_1 + d_1 = 0, a_2 + d_2 = 0$. We have

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}.$$

Its (1, 1)-entry and (2, 2)-entry are $a_1 + a_2$ and $d_1 + d_2$ respectively. We write their sum

$$(a_1 + a_2) + (d_1 + d_2) = (a_1 + d_1) + (a_2 + d_2) = 0 + 0 = 0.$$

Hence $A_1 + A_2$ is in W .

For (2), we note that the corresponding entries of λA are λa , λd respectively. So their sum becomes

$$\lambda a + \lambda d = \lambda(a + d) = \lambda \cdot 0 = 0.$$

Therefore λA is in W for all λ .

This shows that W is a subspace of $M_{2 \times 2}(\mathbf{R})$.

- (b) From the equation $a + d = 0$, we have $d = -a$. Other than this, the choice of a , b , c are completely free. So W is the same as the space consisting of the matrices of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

which can be rewritten as

$$a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

So we obtain a basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Problem 7: We consider the defining equation

$$c_1(3 + x + x^2) + c_2(2 + 2x + 5x^2) + c_3(4 - 3x^2) = \vec{0}$$

for the linear independence. Writing it into the standard form

$$(3c_1 + 2c_2 + 4c_3) + (c_1 + 2c_2)x + (c_1 + 5c_2 - 3c_3)x^2 = \vec{0}$$

we obtain

$$\begin{cases} 3c_1 + 2c_2 + 4c_3 & = 0 \\ c_1 + 2c_2 & = 0 \\ c_1 + 5c_2 - 3c_3 & = 0 \end{cases}$$

By solving this equation, we find that it has non-trivial solution e.g., $c_1 = -2$, $c_2 = 1$, $c_3 = 1$ and so the set is **linearly dependent**.

Problem 8:

- (a) To find the transition matrix **from S to T** we need to express the elements of $S = \{v_1, v_2\}$ as linear combinations of $T = \{w_1, w_2\}$, which are already given as

$$v_1 = -w_1 + 4w_2, \quad v_2 = 5w_1 - 3w_2.$$

Therefore the matrix should be

$$P_{(T \leftarrow S)} = \begin{pmatrix} -1 & 5 \\ 4 & -3 \end{pmatrix}.$$

- (b) By the definition of $[x]_S$, we have

$$[x]_S = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

If you were asked to find $[x]_T$, we would apply the coordinate change formula

$$[x]_T = P_{(T \leftarrow S)}[x]_S = \begin{pmatrix} -1 & 5 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \end{pmatrix}.$$

Warning : You should note that I swapped the roles of letters S and T differently from the way how the textbook used. This, of course, is not important since they are just dummy variables. Any other letters as you want can replace the letters.

Problem 9 :

- (a) $\vec{0}$ = The zero polynomial
 (b) Look at the problem 5 (a) to see how to do it.