Addendum to the lecture (04/07)

Here is the proof of the statement "when M is connected orientable, then $\widetilde{M}$ is disconnected" :

Let the function $x \in M \mapsto \mu_x \in H_n(M, M - \{x\}; \mathbb{Z})$ be an orientation of M. Then $[x \mapsto -\mu_x]$ is also an orientation. Obviously $\mu_x \neq -\mu_x$ and so their images are disjoint in $\widetilde{M}$

Denote by $\mu : M \to \widetilde{M}$ be the associated function defined by $\mu(x) = \mu_x$, and denote by $-\mu$ its negative. Obviously we have $\pi \circ \mu = id_M$. Local consistency requirement of the orientation $x \mapsto \mu_x$ implies continuity of the function μ.

It follows that $\widetilde{M} = \text{Image}\, \mu \cup \text{Image}(-\mu)$ and the local consistency again implies that both $\text{Image}\, \mu$ and $\text{Image}(-\mu)$ are open. Therefore they are also closed and so they represent two disjoint components of $\widetilde{M}$ which finished the proof.

Incidentally, local consistency also implies that there are precisely two different orientations when M is connected orientable.