Addendum to the lecture (04/30)

Here is the complete proof of the commutativity of the square

$$\begin{CD} H^k(U\cap V, U\cap V - K\cap L) @>f>> H^k(U, U-K) \\ @V{\mu_{K\cap L}}VV @VV{\mu_K \cap}V \\ H_{n-k}(U\cap V) @>i_*>> H_{n-k}(U) \end{CD}$$

Here $f := j^* \circ (i^*)^{-1}$ where $i : U\cap V \to U$ and $j : (U, U-K) \to (U, U-K\cap L)$ are inclusion maps. **The upshot** here is that f is **not** canonically induced by inclusion maps.

Proof. The commutativity of the diagram means the identity

$$i_*(\mu_{K\cap L} \cap \phi) = \mu_K \cap f\phi$$

For this proof, we use the excision isomorphism to expand the diagram into

$$\begin{array}{ccc} H^k(U, U-K\cap L) & & \\ \downarrow{\scriptstyle i^*} & \searrow^{j^*} & \\ H^k(U\cap V, U\cap V - K\cap L) & \xrightarrow{f} & H^k(U, U-K) \\ \downarrow{\scriptstyle \mu_{K\cap L}\cap} & & \downarrow{\scriptstyle \mu_K\cap} \\ H_{n-k}(U\cap V) & \xrightarrow{i_*} & H_{n-k}(U) \end{array}$$

where $j : (U, U-K) \to (U, U-K\cap L)$ is the inclusion map. Since $i^* : H^k(U, U-K\cap L) \to H^k(U\cap V, U\cap V - K\cap L)$ is an isomorphism, it is enough to prove

$$i_*(\mu_{K\cap L} \cap i^*a) = \mu_K \cap j^*a$$

for any $a \in H^k(U, U-K\cap L)$.

We represent a by a relative cocycle $\phi \in C^k(U, U-K\cap L) \subset C^k(U)$. Since $U-K \subset U-K\cap L$, it also defines a cocycle in $C^k(U, U-K) \subset C^k(U)$. Then $j^*\phi = \phi$ as a cochain in $C^k(U)$. Represent the class $\mu_{K\cap L}$ by a relative cycle $\alpha \in C_n(U\cap V, U\cap V - K\cap L)$ Then the naturality of cap product implies

$$i_*(\alpha \cap i^*\phi) = i_*\alpha \cap \phi.$$

But we also have $i_*\alpha \cap \phi = i_*\alpha \cap j^*\phi$ as $j^*\phi = \phi$ by the remark above. Hence we have

$$i_*(\alpha \cap i^*\phi) = i_*\alpha \cap j^*\phi$$

as a chain. By the uniqueness property of orientation class $\mu_{(.)}$, $i_*\alpha$ represents μ_K and hence this identity descends to

$$i_*(\mu_{K\cap L} \cap i^*a) = \mu_K \cap j^*a$$

which finishes the proof.

□