

## Addendum to the lecture (05/07)

**Lemma 1.** Consider the open neighborhoods  $V$  of  $M$  such that

$$V \cong \mathbb{R}_+^n \cap B^n(0)$$

near the boundary  $\partial M$ . Then there are unique homomorphisms

$$\partial_V : \Gamma_R(\dot{V}) \rightarrow \Gamma_R(\partial V)$$

which are compatible under restrictions to smaller neighborhoods.

*Proof.* Let  $\alpha \in \Gamma_R(\dot{V})$ . We will define a section  $\beta \in \Gamma_R(\partial V)$  out of it. By definition of section  $\beta$ , we need to assign an element

$$\beta(y) \in H_{n-1}(\partial V, \partial V - \{y\})$$

as its value at each point of  $y \in \partial M$ . We will do this a composition of various isomorphisms applied to the value

$$\alpha(x) \in H_n(\dot{V}, \dot{V} - \{x\})$$

for a fixed  $x \in \dot{V} = V \setminus \partial V$ .

**(1):** We note that  $M - \dot{V}$  is a strong deformation retract of  $M - \{x\}$ . Therefore we have

$$f_1 : H_n(M, M - \dot{V}) \rightarrow H_n(M, M - \{x\}) \cong H_n(\dot{M}, \dot{M} - \{x\}) \cong H_n(\dot{V}, \dot{V} - \{x\})$$

where the second isomorphism comes from the fact that  $\dot{M} = M \setminus \partial M$  and  $\partial M$  has a collar neighborhood.

**(2):** Now we consider the triple

$$(M, M - \dot{V}, M - V)$$

and the associated long exact sequence

$$\rightarrow H_n(M, M - V) \rightarrow H_n(M, M - \dot{V}) \xrightarrow{\partial} H_{n-1}(M - \dot{V}, M - V) \rightarrow H_{n-1}(M, M - V) \rightarrow$$

Since  $M - V$  is a strong deformation retract of  $M$ , we have  $H_i(M, M - V) = 0$  for all  $i$  and hence

$$\partial : H_n(M, M - \dot{V}) \rightarrow H_{n-1}(M - \dot{V}, M - V)$$

is an isomorphism.

**(3):** Now consider  $y \in \partial V \subset \partial M \cap V$ . We note that  $M - V$  is a strong deformation retract of  $M - \dot{V} - \{y\}$  which gives rise to an isomorphism

$$f_2 : H_{n-1}(M - \dot{V}, M - V) \rightarrow H_{n-1}(M - \dot{V}, M - \dot{V} - \{y\})$$

Similarly  $\partial M - \partial V$  is a strong deformation retract of  $\partial M - \{y\}$  and so an isomorphism

$$f_3 : H_{n-1}(\partial M, \partial M - \partial V) \rightarrow H_{n-1}(\partial M, \partial M - \{y\}) \cong H_{n-1}(\partial V, \partial V - \{y\})$$

(4): We note that  $M - \dot{V} - \{y\}$  is open in  $M - \dot{V}$ , and that the closure of  $\dot{M} - \dot{V}$  is  $M - V$  which is contained in the open subset  $M - \dot{V} - \{y\}$  of  $M - \dot{V}$ . Therefore we can excise  $\dot{M} - \dot{V}$  to obtain the excision isomorphism

$$f_4 : H_{n-1}(\partial M, \partial M - \partial V) \rightarrow H_{n-1}(M - \dot{V}, M - \dot{V} - \{y\})$$

(5): Now we define the section  $\beta \in H_{n-1}(\partial V, \partial V - \{y\})$  by

$$\beta(y) := f_3 \circ f_4^{-1} \circ f_2 \circ \partial(f_1(\alpha(x)))$$

and  $\partial_V(\alpha) := \beta$ .

(6): By construction, the compatibility under the restricting to smaller open subsets is evident. (Note that the above construction does not depend on the location of  $x \in \dot{V}$ , and all the maps  $f_i$  are induced by the inclusions.)  $\square$

We are ready to prove

**Proposition 1.** *Let  $M$  be  $R$ -orientable with boundary  $\partial M$ . Then the boundary map  $H_n(M, \partial M) \rightarrow H_{n-1}(\partial M)$  sends a fundamental class  $[M]$  of  $M$  to a fundamental class of  $\partial M$ . In particular,  $\partial M$  is  $R$ -orientable.*

*Proof.* When  $M$  is orientable, the fundamental class  $[M]$  defines a section  $\Gamma_R(\dot{V})$  on any coordinate neighborhood  $V$  which is either homeomorphic to  $\mathbb{R}^n$  or  $\mathbb{R}_+^n$ . The above lemma then implies that the restriction to the second type of neighborhood defines a section  $\beta$  on  $\partial V$  that satisfies the consistency condition and so defines a global section of  $(\partial M)_R \rightarrow \partial M$  whose value at each  $y \in \partial M$  is a generator of  $H_{n-1}(\partial M, \partial M - \{y\})$ .

By construction, in the proof of the above lemma, and using the excision isomorphism,  $j_* : H_n(V, V - \dot{V}) \cong H_n(M, M - \dot{V})$  where  $j : (V, V - \dot{V}) \rightarrow (M, M - \dot{V})$  is the inclusion, and chasing the diagram above, we derive

$$\beta(y) = (i_y^{\partial V})_*(\partial\alpha).$$

We also denote the inclusion by  $j : (\partial V, \partial V - \{y\}) \rightarrow (\partial M, \partial M - \{y\})$ . Then But we also have

$$\begin{aligned} (i_y^{\partial V})_*(\partial\alpha) &= (i_y^{\partial V})_*(j_*^{-1} \circ i_*[M]) \\ &= (i_y^{\partial V})_*(j_*^{-1} \circ i_*\partial[M]) = j_*^{-1} \left( (i_y^{\partial M})_*[\partial M] \right) \end{aligned}$$

where  $i : (M, \partial M) \rightarrow (M, M - \dot{V})$  is the inclusion. In other words, we have

$$(i_y^{\partial M})_*[\partial M] = j_*(i_y^{\partial V})_*(\partial\alpha) = (j \circ i_y^{\partial V})_*(\partial\alpha) = (j_{\partial V})_*(\partial\alpha)$$

where  $j_{\partial V} : (\partial V, \partial V - \{y\}) \rightarrow (\partial M, \partial M - \{y\})$  is the inclusion. Since the local sections  $(\partial\alpha, \partial V)$  were shown to be consistent over the choice of  $V$  and  $(i_y^{\partial V})_*(\partial\alpha)$  is a generator of  $H_n(\partial V, \partial V - \{y\})$ , this identity proves that  $\beta = \partial_V\alpha$  is locally the section associated to the homology class  $[\partial M]$ . Therefore by definition  $[\partial M]$  is the fundamental class of  $\partial M$ .  $\square$