HOMOTOPY INVARIANCE OF SPECTRAL INVARIANTS OF TOPOLOGICAL HAMILTONIAN FLOWS AND ITS LAGRANGIAN ANALOG

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ABSTRACT. In this paper, we prove that on any closed connected 2 dimensional symplectic manifold (M, ω) the spectral invariant $\rho(\lambda; a)$ of a topological Hamiltonian path is invariant under the hamiltonian homotopy for any quantum cohomology class $a \in QH^*(M)$, *provided* both λ and the homotopy are supported in $U = M \setminus B$ for a fixed ball $B \subset M$. The case of high dimensional rational symplectic manifolds will be treated in a sequel to this paper.

This homotopy invariance for $a = 1$ is a crucial ingredient of the author's extension of Calabi homomorphism of the disc to the group $Hameo(D^2, \partial D^2)$ consisting of compactly supported Hamiltonian homeomorphisms (also succinctly called *hameomorphisms*) and in turn the author's proof of nonsimpleness of the area preserving homeomorphism group of D^2 in a companion paper.

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CONTENTS

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1. Introduction and the main results

In [OM], Müller and the author introduced the group $Hameo(M,\omega)$ of *hameomorphisms* which is defined as the completion of $Ham(M,\omega)$ with respect to the Hofer distance and the C^0-distance on the space of Hamiltonian paths. For the sphere S^2, or the disc $(D^2, \partial D^2)$, they conjectured that this set of hameomorphisms is a proper subset of the area preserving homeomorphism group. In [Oh11], the author proves the properness for the case of $(D^2, \partial D^2)$ postponing a vanishing result of Calabi invariants of contractible *topological Hamiltonian loops* whose explanation is in order.

We always assume that the ambient manifolds M or N are connected throughout the entire paper.

1.1. Hamiltonian topology and hamiltonian homotopy.

In [OM], Müller and the author introduced the notion of Hamiltonian topology on the space

$$\mathcal{P}^{ham}(Symp(M,\omega), id)$$

of Hamiltonian paths $\lambda : [0,1] \to Symp(M,\omega)$ with $\lambda(t) = \phi_H^t$ for some time-dependent Hamiltonian H. We would like to emphasize that we do *not* assume that H is normalized *unless otherwise said explicitly.* This is because we need to consider both compactly supported and mean-normalized Hamiltonians and suitably transform one to the other in the course of the proof of the main theorem of this paper. One novelty of the present paper is an extensive and careful usage of the normalization constants of the Hamiltonian which naturally arise in various contexts in the course of the proof of the main theorems. It turns out that this analysis of the normalization constants is one of the crucial elements in the proofs of various results in the present paper.

In this subsection, we first recall the definition of this Hamiltonian topology.

We start with the case of closed (M, ω). For a given continuous function $h : M \to \mathbb{R}$, we denote

$$\mathrm{osc}(h) = \max h - \min h.$$

We define the C^0-distance $\overline{d}$ on $Homeo(M)$ by the symmetrized C^0-distance

$$\overline{d}(\phi, \psi) = \max\left\{ d_{C^0}(\phi, \psi), d_{C^0}(\phi^{-1}, \psi^{-1}) \right\}$$

and the C^0-distance, again denoted by $\overline{d}$, on

$$\mathcal{P}^{ham}(Symp(M, \omega), id) \subset \mathcal{P}(Homeo(M), id)$$

by

$$\overline{d}(\lambda, \mu) = \max_{t \in [0,1]} \overline{d}(\lambda(t), \mu(t)).$$

The Hofer length of Hamiltonian path $\lambda = \phi_H$ is defined by

$$\mathrm{leng}(\lambda) = \int_0^1 \mathrm{osc}(H_t)\, dt = \|H\|.$$

Following the notations of [OM], we denote by ϕ_H the Hamiltonian path

$$\phi_H : t \mapsto \phi_H^t;\ [0,1] \to Ham(M, \omega)$$

and by $\mathrm{Dev}(\lambda)$ the associated normalized Hamiltonian

$$\mathrm{Dev}(\lambda) := \underline{H}, \quad \lambda = \phi_H \tag{1.1}$$

where $\underline{H}$ is defined by

$$\underline{H}(t,x) = H(t,x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M H(t,x)\, \omega^n. \tag{1.2}$$

We normalize ω so that $\mathrm{vol}_\omega(M) = \int_M \omega^n = 1$ but do not remove the normalizing factor $\frac{1}{\mathrm{vol}_\omega(M)}$ to make the meaning of $\underline{H}$ more conspicuous.

Definition 1.1. Let (M, ω) be a closed symplectic manifold. Let λ, μ be smooth Hamiltonian paths. The *Hamiltonian topology* is the metric topology induced by the metric

$$d_{ham}(\lambda, \mu) := \overline{d}(\lambda, \mu) + \mathrm{leng}(\lambda^{-1}\mu). \tag{1.3}$$

Now we recall the notion of topological Hamiltonian flows and Hamiltonian homeomorphisms introduced in [OM].

Definition 1.2 ($L^{(1,\infty)}$ topological Hamiltonian flow)**.** A continuous map $\lambda : \mathbb{R} \to Homeo(M)$ is called a topological Hamiltonian flow if there exists a sequence of smooth Hamiltonians $H_i : \mathbb{R} \times M \to \mathbb{R}$ satisfying the following:

(1) $\phi_{H_i} \to \lambda$ locally uniformly on $\mathbb{R} \times M$.
(2) the sequence H_i is Cauchy in the $L^{(1,\infty)}$-topology locally in time and so has a limit H_∞ lying in $L^{(1,\infty)}$ on any compact interval $[a, b]$.

We call any such ϕ_{H_i} or H_i an *approximating sequence* of λ. We call a continuous path $\lambda : [a, b] \to Homeo(M)$ a *topological Hamiltonian path* if it satisfies the same conditions with $\mathbb{R}$ replaced by $[a, b]$, and the limit $L^{(1,\infty)}$-function H_∞ called a $L^{(1,\infty)}$ *topological Hamiltonian* or just a *topological Hamiltonian.*

Following the notations from [OM], we denote by $Sympeo(M,\omega)$ the closure of $Symp(M,\omega)$ in $Homeo(M)$ with respect to the C^0-metric $\overline{d}$, and by $\mathcal{H}_m([0,1]\times M,\mathbb{R})$ the set of mean-normalized topological Hamiltonians, and by

$$ev_1 : \mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id) \to Sympeo(M,\omega), id) \tag{1.4}$$

the evaluation map defined by $ev_1(\lambda) = \lambda(1)$. By the uniqueness theorem of Buhovsky-Seyfaddini [BS] (see also [V2] for the L^∞-context), we can extend the map Dev given in (1.1) to

$$\overline{\text{Dev}} : \mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id) \to \mathcal{H}_m([0,1]\times M,\mathbb{R})$$

in an obvious way. Following the notation of [OM, Oh10], we denote the topological Hamiltonian path $\lambda = \phi_H$ when $\overline{\text{Dev}}(\lambda) = \underline{H}$ in this general context.

Definition 1.3 (Hamiltonian homeomorphism group)**.** We define

$$Hameo(M,\omega) = ev_1\left(\mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id)\right)$$

and call any element therein a *Hamiltonian homeomorphisms*.

The group property and its normality in $Sympeo(M,\omega)$ are proved in [OM].

In [OM], only the (strong) Hamiltonian topology given in Definition 1.1 is studied except at Remark 3.27 [OM]. It appears that the weak Hamiltonian topology, which is induced by the metric on the path space $\mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega), id)$

$$d^{weak}_{ham}(\lambda,\mu) := d_{C^0}(\lambda(1),\mu(1)) + \text{leng}(\lambda^{-1}\mu), \tag{1.5}$$

will also play some significant role in the study of C^0 symplectic topology in relation to Lagrangian submanifolds especially *on the cotangent bundle*, as it will be clear in the statement of various theorems stated in the present paper. This prospect is worthwhile to pursue further which will be a subject of future research.

The following notion of hamiltonian homotopy of topological hamiltonian paths is introduced in [Oh11].

Definition 1.4 (Hamiltonian homotopy)**.** Let $\lambda_0,\ \lambda_1 \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$. A hamiltonian homotopy $\Lambda : [0,1]^2 \to Sympeo(M,\omega)$ between λ_0 and λ_1 based at the identity is the map such that

$$\Lambda(0,t) = \lambda_0(t),\ \Lambda(1,t) = \lambda_1(t), \tag{1.6}$$

and $\Lambda(0,s) \equiv id$ for all $s \in [0,1]$, and arises as follows: there is a sequence of smooth maps $\Lambda_j : [0,1]^2 \to Ham(M,\omega)$ that satisfy

1. (1) $\Lambda_j(s,0) = id$,
2. (2) $\Lambda_j \to \Lambda$ in C^0-topology,
3. (3) Any 'horizontal' section $\Lambda_{j,s} : \{s\}\times[0,1] \to Ham(M,\omega)$ converges in hamiltonian topology in the following sense: If we write

$$\text{Dev}\left(\Lambda_{j,s}\Lambda_{j,0}^{-1}\right) =: H_j(s),$$

then $H_j(s)$ converges in hamiltonian topology uniformly over $s \in [0,1]$. We call any such Λ_j an *approximating sequence* of Λ.

When $\lambda_0(1) = \lambda_1(1) = \psi$, a *hamiltonian homotopy relative to the ends* is one that satisfies $\Lambda(s,0) = id$, $\Lambda(s,1) = \psi$ for all $s \in [0,1]$ in addition.

We say that $\lambda_0,\ \lambda_1 \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ are *hamiltonian homotopic* (resp. relative to the ends), if there exists a hamiltonian homotopy (resp. a hamiltonian homotopy relative to the ends).

We emphasize that by the requirement (3),

$$H_j(0) \equiv 0 \tag{1.7}$$

in this definition.

All the above definitions can be modified to handle the case of open manifolds, either noncompact or compact with boundary, by considering compactly supported H's as done in section 6 [OM]. Our main interest of noncompact case is the cotangent bundle T^*N where N is a closed manifold. We recall the definitions of topological Hamiltonian paths and Hamiltonian homeomorphisms supported in an open subset $U \subset M$ from [OM].

We first define $\mathcal{P}^{ham}(Symp_U(M,\omega), id)$ to be the set of smooth Hamiltonian paths supported in U. The following definition is taken from Definition 6.2 [OM] to which we refer readers for more detailed discussions. First for any open subset $V \subset U$ with compact closure $\overline{V} \subset U$, we can define the completion of $\mathcal{P}^{ham}(Symp_{\overline{V}}(M,\omega), id)$ using the same metric above which we denote by

$$\mathcal{P}^{ham}(Symp_K(M,\omega), id), \quad K = \overline{V}.$$

Definition 1.5. Let $U \subset M$ be an open subset. Define $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$ to be the union

$$\mathcal{P}^{ham}(Sympeo_U(M,\omega), id) := \bigcup_{K\subset U} \mathcal{P}^{ham}(Sympeo_K(M,\omega), id)$$

with the direct limit topology, where $K \subset U$ is a compact subset. We define $Hameo_c(U,\omega)$ to be the image

$$Hameo_c(U,\omega) := ev_1(\mathcal{P}^{ham}(Sympeo_U(M,\omega), id).$$

We would like to emphasize that this set is not necessarily the same as the set of $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ with compact $\operatorname{supp}\lambda \subset U$. The same definition can be applied to general open manifolds or manifolds with boundary.

1.2. **Lagrangian spectral invariants.** Let N be a compact manifold without boundary and let T^*N be its cotangent bundle equipped with θ the Liouville one-form defined by

$$\theta_x(\xi_x) = p(d\pi(\xi_x)), \quad x = (q,p) \in T^*N.$$

The canonical symplectic form ω_0 on T^*N is defined by

$$\omega_0 = -d\theta = \sum_{k=1}^n dq^k \wedge dp_k \tag{1.8}$$

where $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ is the canonical coordinates of T^*N associated to the coordinates $(q^1,\ldots,q^n)$ of N. We put a density ρ_N on o_N (or a volume form when N is oriented), i.e., consider o_N as a *weighted Lagrangian submanifold* (o_N,ρ_N) in the sense of Weinstein [W2].

Consider Hamiltonian $H = H(t,x)$ such that H_t is asymptotically constant, i.e., the ones whose Hamiltonian vector field X_H is compactly supported. We define

$$\operatorname{supp}_{asc} H = \operatorname{supp} X_H := \bigcup_{t\in[0,1]} X_{H_t}.$$

For each given compact set $K \subset T^*N$ and $R \in \mathbb{R}_+$, we define

$$\mathcal{P}C^\infty_{R,K} = \{H \in C^\infty([0,1]\times T^*N,\mathbb{R}) \mid \operatorname{supp}_{asc} H \subset D^R(T^*N), \|H\| \le K\} \tag{1.9}$$

which provides a natural filtration of the space $C^\infty([0,1]\times T^*N,\mathbb{R})$. We also denote

$$\mathcal{P}C^\infty_R = \bigcup_{K\in\mathbb{R}_+} \mathcal{P}C^\infty_{K,R}, \quad \mathcal{P}C^\infty_{asc} = \bigcup_{R\geq 0} \mathcal{P}C^\infty_R. \tag{1.10}$$

By definition, each element H_t is independent of $x=(q,p)$ if $|p|$ is sufficiently large and so carries a smooth function $c_\infty:[0,1]\to\mathbb{R}$ defined by

$$c_\infty(t) = H(t,\infty).$$

Therefore we have the natural evaluation map

$$\pi_\infty : \mathcal{P}C^\infty_{asc} \to C^\infty([0,1],\mathbb{R}).$$

For each given smooth function $c:[0,1]\to\mathbb{R}$, we denote

$$\mathcal{P}C^\infty_{asc;c} := \pi_\infty^{-1}(c). \tag{1.11}$$

We then introduce the space of Hamiltonian deformations of the zero section and denote

$$\mathfrak{Iso}(o_N;T^*N) = \{\phi^1_H(o_N) \mid H\in \mathcal{P}C^\infty_{asc}\}$$

following the terminology of [W2], and

$$\mathfrak{Iso}(o_N;D^R(T^*N)) := \{\phi^1_H(o_N) \mid H\in \mathcal{P}C^\infty_R\}.$$

Definition 1.6. We define the *Hamiltonian topology* on $\mathfrak{Iso}(o_N;D^R(T^*N))$ as the quotient topology of the weak Hamiltonian topology of $\mathcal{P}^{ham}(Symp_{D^R}(T^*N,\omega),id)$ under the surjective map $\phi_H\mapsto \phi^1_H(L_0)$ where $D^R=D^R(T^*N)$. Then we equip

$$\mathfrak{Iso}(o_N;T^*N) = \lim_{R\to\infty} \mathfrak{Iso}(o_N;D^R(T^*N))$$

with the direct limit topology of the Hamiltonian topology of $\mathfrak{Iso}(o_N;D^R(T^*N))$.

For any given time-dependent Hamiltonian $H=H(t,x)$, the classical action functional on the space

$$\mathcal{P}(T^*N) := C^\infty([0,1],T^*N)$$

is defined by

$$\mathcal{A}^{cl}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(t,\gamma(t))\,dt.$$

We define the subset $\mathcal{P}(T^*N;o_N)$ by

$$\mathcal{P}(T^*N;o_N) = \{\gamma:[0,1]\to T^*N \mid \gamma(0)\in o_N\}.$$

The assignment $\gamma\mapsto \pi(\gamma(1))$ defines a fibration

$$\mathcal{P}(T^*N;o_N)\to o_N\cong N$$

with fiber at $q\in N$ given by

$$\mathcal{P}(T^*N;o_N,T^*_qN) := \{\gamma:[0,1]\to T^*N \mid \gamma(0)\in o_N,\ \gamma(1)\in T^*_qN\}.$$

For given $x\in L_H$, we denote the Hamiltonian trajectory

$$z^H_x(t) = \phi^t_H((\phi^1_H)^{-1}(x))$$

which is a Hamiltonian trajectory such that, by definition,

$$z^H_x(0)\in o_N, \quad z^H_x(1) = x. \tag{1.12}$$

We denote $L_H=\phi^1_H(o_N)$ and by $i_H:L_H\hookrightarrow T^*N$ the inclusion map.

Motivated by Weinstein's observation that the action functional

$$\mathcal{A}_H^{cl} : \mathcal{P}(T^*N; o_N) \to \mathbb{R}$$

can be interpreted as the canonical generating function of L_H, the present author constructed a family of spectral invariants of L_H by performing a mini-max theory via the chain level Floer homology theory in [Oh2, Oh3]. Indeed, the function defined by

$$h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \tag{1.13}$$

is a canonical generating function of L_H in that

$$i_H^*\theta = dh_H. \tag{1.14}$$

We call h_H the *basic generating function* of L_H. As a function on N, not on L_H, it is a multi-valued function.

One of the constructions in [Oh2, Oh3] considers the Lagrangian pair

$$(o_N, o_N)$$

and its associated Floer complex $CF(H; o_N, o_N)$ generated by the Hamiltonian trajectory $z : [0,1] \to T^*N$ satisfying

$$\dot z = X_H(t, z(t)), \quad z(0) \in o_N,\, z(1) \in o_N. \tag{1.15}$$

Denote by $\mathcal{C}hord(H; o_N, o_N)$ the set of solutions thereof. (In fact, the construction in [Oh2] is performed for arbitrary submanifolds $S \subset N$ by considering its conormal bundle N^*S. It becomes just the fiber T_q^*N when $S = \{q\}$ and the zero section o_N when $S = N$. In this paper, we will only consider the case $S = N$ or $S = \{pt\}$.)

The differential $\partial_{(H,J)}$ on $CF(H; o_N, o_N)$ is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(\tau, 0),\, u(\tau, 1) \in o_N, . \end{cases} \tag{1.16}$$

An element $\alpha \in CF(H; o_N, o_N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in \mathcal{C}hord(H; o_N, o_N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We denote the *level* of the chain α by

$$\lambda_H(\alpha) := \max_{z \in \operatorname{supp}\alpha} \{\mathcal{A}_H^{cl}(z)\}. \tag{1.17}$$

When a cohomology class $a \in H^*(N, \mathbb{Z})$ is given, using the canonical isomorphism

$$\Phi_H : H_*(N; \mathbb{Z}) \to HF_*(H; o_N, o_N),$$

we choose a Floer cycle α in class $[a]^\flat := PD(a)$ and take the mini-max value

$$\rho(H; a) = \inf_{\alpha \in [a]^\flat} \lambda_H(\alpha). \tag{1.18}$$

(A similar construction using the generating function method was earlier given by Viterbo [V1] and it is shown in [M, MO] that both invariants coincide *modulo a normalization constant.*) The number $\rho(H; a)$ depends on H, not just on $L_H = \phi_H^1(o_N)$

In this paper, both Lagrangian spectral invariants defined in [Oh2] and the Hamiltonian spectral invariants defined in [Oh6] (see also [Sc]) will be used. Because of this, we differentiate them by denoting the Lagrangian spectral invariants by ρ^{lag} and the Hamiltonian spectral invariants by ρ^{ham}.

We would like to emphasize that the above mentioned ambiguity of normalization constant in the equivalence statement of the two constructions is not a trivial matter to handle, especially when one attempts to relate the Lagrangian spectral invariants constructed in [Oh2] and the Hamiltonian spectral invariants constructed in [Oh6].

1.3. **Statement of main results.** Recall the definition

$$\operatorname{Dev}(\lambda) = \underline{F}$$

which is the normalized smooth Hamiltonian of F with $\lambda = \phi_F$, and

$$\rho^{ham}(\lambda; 1) := \rho^{ham}(\operatorname{Dev}(\lambda); 1) = \rho^{ham}(\underline{F}; 1) \tag{1.19}$$

by definition. We also denote by $\overline{\operatorname{Dev}}$ its extension to topological Hamiltonian paths.

Combining these theorems, we prove

Theorem 1.1. *Let (M, ω) be a closed connected 2 dimensional surface. Let $B \subset M$ be a closed ball. Let $\lambda \in \mathcal{P}^{ham}(Sympeo_U(M, \omega), id)$ with $U = M \setminus B$ be a topological Hamiltonian loop hamiltonian homotopic to the identity path via hamiltonian homotopy of loops in $\mathcal{P}^{ham}(Sympeo_U(M, \omega), id)$. Then $\rho^{ham}(\lambda; 1) = 0 = \rho^{ham}(\lambda^{-1}; 1)$.*

This is precisely the result whose proof was postponed from [Oh11].

Unraveling the definition of hamiltonian homotopy of topological Hamiltonian loops, this theorem is equivalent to the following

Theorem 1.2. *Let (M, ω) be closed connected 2 dimensional surface. Suppose the sequence $\lambda_i = \phi_{F_i}$ of smooth Hamiltonian paths where F_i is a sequence such that there exists a two parameter Hamiltonians $H_i = H_i(s, t, x)$ with $F_i = H_i(1)$ satisfying the following:*

1. *$H_i(0, t, x) = H_i(s, 0, x) = K_i(s, 0, x) \equiv 0$,*
2. *there exists a ball $B \subset M$ such that $\operatorname{supp} H_i \subset M \setminus B$ for all i,*
3. *$\max_{s \in [0,1]} \overline{d}(\phi^1_{H_i(s)}, id) \to 0$ as $i \to \infty$,*
4. *$H_i(s)$ converges in $L^{(1,\infty)}$ as $i \to \infty$ uniformly over $s \in [0, 1]$.*

Then

$$\lim_{i\to\infty} \rho^{ham}(\lambda_i; 1) = 0 = \lim_{i\to\infty} \rho^{ham}(\lambda_i^{-1}; 1). \tag{1.20}$$

We would like to emphasize that the uniform $L^{(1,\infty)}$-limit $H(1) := \lim_{i\to\infty} H_i(1)$ in this theorem may not be zero, i.e., $H(1)$ may generate a *non-constant* topological Hamiltonian loop. On the other hand, we recall Buhovsky-Seyfaddini's uniqueness theorem [BS] would imply the limit $H(1)$ must be zero, if we assumed the stronger condition on the *path* $t \mapsto \phi^t_{H_i(1)}$ converging to the identity path in C^0 instead of just the time-one map $\phi^1_{H_i}$ which corresponds to that *constant* topological Hamiltonian loop. The vanishing result stated in this theorem is an easy consequence of vanishing of the topological Hamiltonian in this case.

Corollary 1.3. *Assume (M,ω) is any closed connected 2 dimensional surface. Let λ_0, $\lambda_1 \in \mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$ with $U = M \setminus B$ for some $B \subset M$ with $\lambda_0(1) = \lambda_1(1)$, and assume that they are hamiltonian homotopic. Then*

$$\rho^{ham}(\lambda_0; a) = \rho^{ham}(\lambda_1; a)$$

for all $a \in QH^*(M)$ in $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$.

However the Hamiltonian naturally appearing in the proof of the main theorem is not mean-normalized one but has the support property

$$\operatorname{supp} H \subset U = M \setminus B$$

for a closed ball B instead. To exploit the results established on spectral invariants of such Hamiltonians for the *mean-normalized* Hamiltonians, we reduce our proof to the *engulfable* case by partitioning the given topological hamiltonian homotopy into to small pieces. The definition of topological hamiltonian homotopy enables us to make such a reduction.

We recall the definition of engulfable Hamiltonians following [Oh7, Sp]. For this, we first need to define the corresponding notion of Lagrangian submanifolds in general. Let $L \subset (M,\omega)$ be a compact Lagrangian submanifold and V_L its Darboux neighborhood.

Definition 1.7. A Hamiltonian $F = F(t,x)$ is called V_L-engulfable if $\phi_H^t(L) \subset V_L$ for all $t \in [0,1]$. When there exists such a Darboux neighborhood, without explicit mentioning thereof, we just call such F engulfable with respect to L.

Following the notations of [Oh13] we define

$$\mathcal{H}_\delta^{engulf}(L; V_L)$$

to be the set of V_L-engulfable Hamiltonian $F : [0,1] \times M \to \mathbb{R}$ that satisfies $\overline{d}(\phi_F^1, id) \leq \delta$. Then we define

$$\mathfrak{Iso}_\delta^{engulf}(L; V_L) = \{L' \in \mathfrak{Iso}(L) \mid L' = \phi_H^1(L),\ H \in \mathcal{H}_\delta^{engulf}(L; V_L)\}. \tag{1.21}$$

Going back to the Hamiltonian diffeomorphisms, we fix a Darboux neighborhood $(U_\Delta, -d\Theta) \subset (M,\omega) \times (M,-\omega)$ depending only on (M,ω) once and for all. We denote by $\mathcal{U} = \mathcal{U}(V_\Delta) \subset \mathcal{L}_0(M)$ the set of paths the graph of whose image is contained in V_Δ. We call a Hamiltonian F engulfable if there exists a Darboux neighborhood $V_\Delta \supset \Delta$ such that

$$\operatorname{Graph} \phi_F^t \subset V_\Delta \quad \text{for all } t \in [0,1].$$

for all t.

Now let $H_i = H_i(s,t,x)$ be the t-Hamiltonians on (M,ω) arising from the approximating sequence Λ_i of a engulfable topological hamiltonian homotopy Λ contracting to the identity. We apply the localization process for such engulfable topological hamiltonian loops developed in [Oh13] and define the local version of spectral invariant which we denote $\rho_{\mathcal{U}}^{ham}(H; 1_0)$ where $\mathcal{U} = \mathcal{U}(V_\Delta) \subset \mathcal{L}_0(M)$ is an open neighborhood of constant paths whose images are contained in a Darboux neighborhood V_Δ of the diagonal $\Delta \subset M \times M$. With this preparation, it turns out to be crucial to express the spectral invariant $\rho^{ham}(\underline{H_i}(1); 1)$ into the form

$$\rho^{ham}(\underline{H_i}(1); 1) = \rho_{\mathcal{U}}^{ham}(\underline{H_i}(1); 1_0) + \big(\rho^{ham}(\underline{H_i}(1); 1) - \rho_{\mathcal{U}}^{ham}(\underline{H_i}(1); 1_0)\big) \tag{1.22}$$

in the proof. Then we study the first summand and the second one of the right hand side of the equation separately.

For the first term, we first compare $\rho^{ham}_{\mathcal U}(\underline{H_i}(1);1_0)$ and its Lagrangian counterpart $\rho^{lag}_{V_\Delta}(\underline{H_i}(1)\oplus 0;1_0)$ in a Darboux chart of the diagonal $\Delta \subset M\times M$ and establish the equality

$$\lim_{i\to\infty}\rho^{ham}_{\mathcal U}(\underline{H_i}(1);1_0)=\lim_{i\to\infty}\rho^{lag}_{V_\Delta}(\underline{H_i}(1)\oplus 0;1_0) \tag{1.23}$$

which can be succinctly stated as

$$\rho^{ham}_{\mathcal U}(\underline{H}(1);1_0)=\rho^{lag}_{V_\Delta}(\underline{H}(1)\oplus 0;1_0)$$

in terms of the limit topological Hamiltonian H. On the other hand, the following coincidence theorem of the local Lagrangian spectral invariant and the global one

$$\rho^{lag}_{V_\Delta}(\underline{H_i}(1)\oplus 0;1_0)=\rho^{lag}(\underline{H_i}(1)\oplus 0;1_0)$$

follows from Theorem 1.5 of [Oh13]. Combining the two, we establish

$$\rho^{ham}_{\mathcal U}(\underline{H}(1);1_0)=\rho^{lag}(\underline{H}(1)\oplus 0;1_0) \tag{1.24}$$

for any engulfable topological Hamiltonian loop ϕ_H.

Once we convert the problem of hamiltonian spectral invariants to that of Lagrangian ones, there are two crucial ingredients that enter in the proof of the vanishing result

$$\rho^{lag}(\underline{H}(1)\oplus 0;1_0)=\lim_{i\to\infty}\rho^{lag}(\underline{H_i}(1)\oplus 0;1)=0. \tag{1.25}$$

The first one is the following hamiltonian continuity result, which is a Lagrangian analog to Corollary 1.2 of S. Seyfaddini's recent paper [Sey]. Denote the maximum C^0-oscillation of o_N under the Hamiltonian diffeomorphism ϕ by

$$\operatorname{osc}_{C^0}(\phi;o_N):=\max\left\{\max_{x\in o_N} d(\phi(x),x),\max_{x\in o_N} d(\phi^{-1}(x),x)\right\}. \tag{1.26}$$

Theorem 1.4. *Let (M,ω) be any closed connected symplectic manifold. Let $\lambda_i=\phi_{F_i}$ where $F_i=F_i(t,x)$ be a sequence of smooth Hamiltonians such that*

1. *there exists $R>0$ such that* $\operatorname{supp} X_{H_i}\subset D^R(T^*N)$ *for all i and $s\in[0,1]$.*
2. *There exists a closed ball $B\subset N$ such that* $\operatorname{supp}\phi_{F_i}\cap o_B=\emptyset$ *for all i's.*
3. *There exists a tubular neighborhood $T\supset o_B$ such that $\phi^1_{H_i}\equiv id$ on T for all i's.*
4. $\operatorname{osc}_{C^0}(\phi^1_{F_i};o_N)\to 0$ *as $i\to\infty$.*

Then $\lim_{i\to\infty}\big(\rho(F_i;1)-\rho(F_i;[pt]^\#)\big)=0$.

We would like to remark that the condition (3) above automatically satisfies for the Lagrangianization Graph $\phi^1_{F_i}$ of the sequence of Hamiltonians F_i given in Theorem 1.2 since Graph $\phi^1_{F_i}=\phi_{F_i\oplus 0}(\Delta)$ and $\phi_{F_i\oplus 0}\equiv id$ on $B\times M$ if $\phi_{F_i}\equiv id$ on B.

It turns out that the differences of two spectral invariants like $\rho^{lag}(F;1)-\rho^{lag}(F;[pt]^\#)$ do not depend on the choice of normalization. Therefore we can define

$$\gamma(L;o_N):=\rho^{lag}(F;1)-\rho(F;[pt]^\#)$$

unambiguously which we call the *spectral capacity* of L (relative to the zero section o_N). (See [V1], [Oh3].) We would like to emphasize that a priori it is possible that both $\rho^{lag}(F;1)$ and $\rho^{lag}(F;[pt]^\#)$ can have the same sign. This phenomenon is quite a nuisance when one handles the spectral numbers themselves. Because of this, this theorem itself does not tell much about the individual number $\rho^{lag}(F_i;1)$ e.g., it does not imply $\lim_{i\to\infty}\rho^{lag}(F_i;1)=0$.

To properly handle the individual number $\rho^{lag}(F;1)$ and relate it to the Lagrangian submanifold $L_F = \phi_F^1(o_N)$ itself, not just to F, we need to put an additional normalization condition relative to L_F. In this regard, it is useful to take the point of view of weighted Lagrangian submanifolds (L, ρ_N) introduced in [W2], where ρ_N is a probability density on N. Using this ρ_N, we can put the normalization condition (3.1), which is the Lagrangian analog to the mean-normalization of Hamiltonians

$$\int_M F(t,x)\,\omega^n = 0.$$

It is worthwhile to mention that the normalization (3.1) on the Hamiltonian F is canonically defined on the set of asymptotically constant Hamiltonians but not on the set of compactly supported ones. This is a reason why we allow more general class of asymptotically constant Hamiltonians.

We define

$$\operatorname{supp}_{asc} F := \operatorname{supp} X_F = \bigcup_{t\in[0,1]} \operatorname{supp} X_{F_t} \tag{1.27}$$

for a Hamiltonian defined on the cotangent bundle T^*N.

For a given two-parameter family $H = H(s,t,x)$ of t-Hamiltonians $H(s)$ defined by $H(s)(t,x) = H(s,t,x)$, we denote the associated s-Hamiltonian i.e., the Hamiltonian generating the vector field

$$\frac{\partial \phi^t_{H(s)}}{\partial s} \circ \left(\phi^t_{H(s)}\right)^{-1}$$

by $K = K(s,t,x)$ in general. We note the identity $\phi^s_{K^1} = \phi^1_{H(s)}$. We note that the above mentioned mean normalization on H_i will automatically hold for the measure induced by the pull-back form

$$\rho_\Delta = i_\Delta^*(\omega^n \oplus 0)$$

where $i_\Delta : \Delta \to M\times M$ is the diagonal embedding and the pull-back two-parameter family of engulfable Hamiltonians derived from that of the mean-normalized Hamiltonians $\underline{H_i}(s)$ on M.

Another crucial ingredients in our proof is the inequality

$$\lim_{i\to\infty} \rho^{lag}(\underline{H_i}(1)\oplus 0; 1) \geq 0 \geq \lim_{i\to\infty} \rho^{lag}(\underline{H_i}(1)\oplus 0; [pt]^\#) \tag{1.28}$$

which can be also written as

$$\rho^{lag}(\underline{H}(1)\oplus 0; 1) \geq 0 \geq \rho^{lag}(\underline{H}(1)\oplus 0; [pt]^\#)$$

for the limit topological Hamiltonian H.

Besides the extensive usage of Floer theory via the spectral invariants, in the course of proving the vanishing result (1.25), we introduce two crucial new additional ingredients of the more classical symplectic geometry of Lagrangian submanifolds and Hamiltonian flows on the cotangent bundle. One is our usages of the *basic phase functions* f_F and a *discontinuous, measurable but almost everywhere differentiable map* $\varphi^F : N \to N$. The latter map is defined to be the composition of the *basic Lagrangian selector* of L_F followed by the inverse of the flow map and is introduced in section 6 for the study of Lagrangian spectral invariants and for the proof of a crucial vanishing result, Lemma 12.2. This map provides a solution to the well-known difficulty of handling the multi-valuedness of the basic generating function $h_H : L_H \to \mathbb{R}$ as a function on N. The other is a new calculation carried out in section 11 which involves the basic generating function $\widetilde{h}_F$ and the weighted Lagrangian submanifold Δ equipped with density provided by $\rho_\Delta = i_\Delta^*(\omega^n \oplus 0)$. Especially Theorem 11.5 is a theorem of its own interest and somewhat unexpected convergence result in the spirit of topological Hamiltonian dynamics in that the theorem does not require anything about behavior of Hamiltonians but only require C^0-convergence of the flow. It appears to the author that both results seem to carry some significance in relation to C^0-symplectic topology and Hamiltonian dynamics, which may be worthwhile to pursue further in the future.

Finally the vanishing of the second term in (1.22) can be proved by a judicious combination of the Ostrover's trick from [Os] enhanced by Seyfaddini [Sey] and the Lagrangian version of the triangle inequality. This last step is the only place where rationality of (M, ω) is used in the entirety of the present paper. Recall that (M, ω) is called *rational* if the subgroup $\omega(\pi_2(M)) \subset \mathbb{R}$ is discrete. We denote by Σ_ω its positive generator, i.e.,

$$\omega(\pi_2(M)) = \mathbb{R} \cdot \Sigma_\omega. \tag{1.29}$$

(We set $\Sigma_\omega = \infty$ when $\omega(\pi_2(M)) = 0$.) We recall that any compact symplectic manifold of 2 dimension is rational. (However, although we do not pursue in this paper since it is not needed for the main purpose of the present paper, we suspect that the rationality hypothesis can be removed by a more sophisticated analysis of Floer complex. This will be a subject of future study.)

Organization of the contents of the paper is now in order. Section 2 performs the reduction to the engulfable case. After this reduction, sections 3 - 8 develop general theory of Lagrangian spectral invariants, basic phase functions and the relationship between them on the cotangent bundle. These sections have independent interest on their owns and can be read independently of the study of homotopy invariance of topological Hamiltonian paths leading to the main theorem. One highlight here among others is our introduction of discontinuous, almost everywhere differentiable but measurable map denoted by φ^H, which relates the basic generating function $\widetilde{h}_H$ and the basic phase function f_H. Section 9 introduces localization of Lagrangian Floer complex in general and specializes this localization to the cotangent bundle. After the classical procedure of Lagrangianization of Hamiltonian diffeomorphisms, combining all the materials on the Lagrangian spectral invariants developed in sections 3 - 8 and the localization results established in [Oh13], which are summarized in section 9, we complete the proof of homotopy invariance of topological Hamiltonian loops that satisfy the support hypothesis stated in Theorem 1.1. In regard to the Lagrangianization of Hamiltonian diffeomorphisms, section 11 contains crucial calculations involving weighted Lagrangian submanifolds and Weinstein's notion of Darboux family [W1]. We restrict our proofs to the case of two dimension only in the last part of section 11 and in the proof of Lemma 12.2. The arguments for the high dimensional cases are more involved than the two dimensional case which we will provide in a sequel [Oh14] to this paper.

We are extremely grateful to S. Seyfaddini for pointing out a crucial mistake in the very first attempt in our proof of nonsimpleness and also for recently sending us his very interesting preprint [Sey], which greatly helps us in proving the Hamiltonian continuity of Lagrangian spectral capacity stated in Theorem 1.4. We are also equally grateful to D. McDuff for her interest and suggestions on this work and for her careful reading of many previous versions of this paper. Without their kindness of patiently reading and pointing out numerous mistakes the author has made throughout this research, this work would not have been possible.

We also thank M. Usher for their useful comments on an early version of the present paper, and F. Zapolsky for attracting our attention to the preprint [MVZ] from which we have learned the Lagrangian version of the optimal triangle inequality which has been useful for our purpose.

Notations and Conventions

We follow the conventions of [Oh6, Oh9, Oh10] for the definition of Hamiltonian vector fields and action functional, and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants on general closed symplectic manifold. They are different from e.g., those used in [Po, EP] one way or the other, but coincide with those used in [Sey].

(1) We usually use the letter M to denote a symplectic manifold and N to denote a general smooth manifold.
(2) The Hamiltonian vector field X_H is defined by $dH = \omega(X_H, \cdot)$.
(3) The flow of X_H is denoted by $\phi_H : t \mapsto \phi_H^t$ and its time-one map by $\phi_H^1 \in Ham(M, \omega)$.
(4) We denote by $z_H^q(t) = \phi_H^t(q)$ the Hamiltonian trajectory associated to the initial point q.
(5) We denote by $z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$ the Hamiltonian trajectory associated to the final point x.
(6) $\overline{H}(t,x) = -H(t, \phi_H^t(x))$ is the Hamiltonian generating the inverse path $(\phi_H^t)^{-1}$.
(7) We denote by $H_1 \# H_2$ the Hamiltonian generating the product paths $\phi_{H_1}\phi_{H_2}$. More explicitly
$$H_1 \# H_2(t,x) = H_1(t,x) + H_2(t, (\phi_{H_2}^t)^{-1}(x)).$$
(8) When $H_1(1,x) \equiv H_2(0,x)$, we define the concatenation $H_1 * H_2$ by
$$(H_1 * H_2)(t,x) = \begin{cases} H_1(2t,x) & 0 \le t \le 1/2 \\ H_2(2t-1,x) & 1/2 \le t \le 1 \end{cases}$$
(9) For a two-parameter family $\phi(s,t) = \phi_{H(s)}^t$ of Hamiltonian diffeomorphisms, we call $H = H(s,t,x)$ the *t-Hamiltonian* and the Hamiltonian, denoted by $K = K(s,t,x)$, generating the vector field
$$\frac{\partial \phi}{\partial s} \circ \phi(s,t)^{-1}$$
the *s-Hamiltonian*. In this case, we denote by $H(s)$ the t-Hamiltonian $H(s)(t,x) = H(s,t,x)$ and K^t the s-Hamiltonian given by $K^t(s,x) = K(s,t,x)$.
(10) The canonical symplectic form on the cotangent bundle T^*N is denoted by $\omega_0 = -d\theta$ where θ is the Liouville one-form which is given by $\theta = \sum_i p_i \, dq^i$ in the canonical coordinates $(q^1, \cdots, q^n, p_1, \cdots, p_n)$.
(11) The classical Hamilton's action functional on the space of paths in T^*N is given by
$$\mathcal{A}_H^{cl}(\gamma) = \int \gamma^*\theta - \int_0^1 H(t, \gamma(t))\, dt.$$
(12) We denote by o_N the zero section of T^*N.

(13) We denote $\rho^{lag}(H;a)$ the Lagrangian spectral invariant on T^*N (relative to the zero section o_N) defined in [Oh2] for asymptotically constant Hamiltonian H on T^*N.
(14) We denote $\rho^{ham}(H;a)$ the spectral invariant on closed (M,ω) defined in [Oh6] but for any Hamiltonian which is not-necessarily mean-normalized.
(15) For a given smooth Hamiltonian path $\lambda : [0,1] \to Ham(M,\omega)$, M closed, we define $\mathrm{Dev}(\lambda)$ the mean-normalized Hamiltonian $\underline{H}$ given by

$$\mathrm{Dev}(\lambda)(t,x) = \underline{H}(t,x) := H(t,x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M H(t,x)\, \omega^n$$

where $\mathrm{vol}_\omega(M) = \int_M \omega^n$ is the Liouville volume.
(16) We define the spectral invariants of Hamiltonian path λ to be

$$\rho^{ham}(\lambda;a) := \rho^{ham}(\underline{H};a), \quad \text{when } \mathrm{Dev}(\lambda) = \underline{H}.$$

2. Reduction to the engulfable case

In this section, we reduce the proof of the main theorem, Theorem 1.1, to the case of engulfable topological Hamiltonian loop that is hamiltonian homotopic to the identity by engulfable hamiltonian homotopy.

Let λ be a topological Hamiltonian loop compactly supported in $U = M \setminus B \neq \emptyset$ for a closed ball $B \subset M$, and let F be the associated topological Hamiltonian with $F \equiv 0$ on B. Denote

$$c(t) = \int_M F(t,x)\, \omega^n = \int_U F(t,x)\, \omega^n.$$

Then we have

$$\overline{\mathrm{Cal}}_U^{path}(\lambda) = \int_0^1 c(t)\, dt.$$

Choose an approximating sequence $\lambda_i \in \mathcal{P}^{ham}(Symp_U(M,\omega), id)$.

By (the uniqueness and) the locality theorem of [BS] (see [V2], [Oh8] for the corresponding C^0-versions respectively), the hamiltonian convergence of

$$\lambda_i \in \mathcal{P}^{ham}(Symp_U(M,\omega), id)$$

to λ implies $\|F_i - F\| \to 0$ where F_i are supported in $M \setminus B$.

We denote $c_i(t) = \int_U F_i(t,x)\, \omega^n$. Since $F_i \equiv 0$ on $B = M \setminus U$,

$$\underline{F_{i,t}} \equiv -c_i(t) \quad \text{on } M \setminus U. \tag{2.1}$$

By the $L^{(1,\infty)}$-convergence of F_i to F,

$$e_i := \mathrm{Cal}_U^{path}(\lambda_i) = \int_0^1 c_i(t)\, dt \to \int_0^1 c(t)\, dt = \overline{\mathrm{Cal}}_U^{path}(\lambda)$$

as $i \to \infty$. This in turn implies $\rho^{ham}(\lambda_i;1) \to \rho(\lambda;1)$ by the identity

$$\rho^{ham}(F_i;a) = \rho^{ham}(\underline{F_i};a) - \int_0^1 c_i(t)\, dt$$

and the inequality

$$|\rho^{ham}(\lambda_i;1) - \rho^{ham}(\lambda;1)| \leq \|\overline{\mathrm{Dev}}(\lambda) - \mathrm{Dev}(\lambda_i)\| \to 0,$$

since we have

$$\mathrm{Dev}(\lambda_i) = \underline{F_i} = F_i - c_i(t), \quad c_i(t) = \int_M F_i(t,x)\, \omega^n$$

by definition. Applying this discussion to each $\lambda(s) := \Lambda(s, \cdot)$ for $s \in [0,1]$, we derive that $\rho^{ham}(\lambda_i(s); 1) \to \rho^{ham}(\lambda(s); 1)$ uniformly over s and so the function ρ_λ defined by $\rho_\lambda(s) := \rho^{ham}(\lambda(s); 1)$ is continuous.

We will prove Theorem 1.1 by contradiction. Denote

$$\rho^+_\lambda(s) := \max\{|\rho^{ham}(\lambda(s); 1)|, |\rho^{ham}(\lambda(s)^{-1}; 1)|\}$$

which is a continuous function with $\rho^+_\lambda(0) = 0$. Suppose to the contrary that $\rho^+_\lambda(1) \neq 0$. By changing the role of λ and λ^{-1} if necessary, we may assume

$$\rho^+_\lambda(1) = |\rho^{ham}(\lambda; 1)| \geq |\rho^{ham}(\lambda^{-1}; 1)|. \tag{2.2}$$

Remark 2.1. We would like to remark that if the signs of $\rho^{ham}(\lambda; 1)$ and $\rho^{ham}(\lambda^{-1}; 1)$ are different, the the triangle inequality $\rho^{ham}(\lambda; 1) + \rho^{ham}(\lambda^{-1}; 1) \geq 0$ implies that the maximum, $\max\{|\rho^{ham}(\lambda(s); 1)|, |\rho^{ham}(\lambda(s)^{-1}; 1)|\}$, is achieved by the positive one among the two. Therefore, under the assumption (2.2), we have

$$\rho^+_\lambda(1) = |\rho^{ham}(\lambda(1); 1)| = \rho^{ham}(\lambda(1); 1).$$

In particular, $\rho^{ham}(\lambda; 1) \geq 0$ under the assumption (2.2).

In particular, $|\rho_\lambda(1)| \neq 0$ but $|\rho_\lambda(0)| = 0$. Therefore the function ρ_λ is not locally constant.

Then it follows that for any given $\varepsilon_0 > 0$ there exists some $0 < s_0 < 1$ such that

$$c := |\rho_\lambda(s_0 + \varepsilon_0) - \rho_\lambda(s_0)| > 0.$$

Here we may assume $1 - s_0 > 0$ and $\varepsilon_0 < 1 - s_0$, recalling that $\lambda(s) \equiv \lambda(1)$ near $s = 1$.

Let $\eta > 0$ be given. We will fix a precise value $\eta > 0$ later. We consider the path

$$\lambda_{s_0,\epsilon_0} = (\lambda(s_0))^{-1}\lambda(s_0 + \epsilon_0).$$

By choosing ε_0 sufficiently small, we may also assume

$$\begin{aligned} \|\overline{\mathrm{Dev}}(\lambda_{s_0,\varepsilon_0})\| &\leq \eta \\ \overline{d}(\lambda_{s_0,\varepsilon_0}, id) &\leq \eta. \end{aligned}$$

This then also implies

$$0 < c \leq |\rho^{ham}(\lambda_{s_0,\varepsilon_0}; 1)| \leq \eta.$$

Here the inequality $c \leq |\rho^{ham}(\lambda_{s_0,\varepsilon_0}; 1)|$ follows from the triangle inequality

$$|\rho^{ham}(\lambda_{s_0,\varepsilon_0}; 1)| \geq |\rho^{ham}(\lambda(s_0 + \epsilon_0); 1) - \rho^{ham}(\lambda(s_0); 1)| = c$$

and the other inequality from the general equality

$$|\overline{\mathrm{Dev}}(\lambda_1^{-1}\lambda_2)| = \|\overline{\mathrm{Dev}}(\lambda_2) - \overline{\mathrm{Dev}}(\lambda_1)\|$$

for smooth Hamiltonian paths λ_1, λ_2. This equality is just the re-writing of the identity $\|\overline{F}_1 \# F_2\| = \|F_2 - F_1\|$ which in turn follows from the formula

$$\overline{F}_1 \# F_2(t, x) = -F_1(t, \phi^t_{F_1}(x)) + F_2(t, \phi^t_{F_1}(x)).$$

Therefore, by considering the path $\lambda_{s_0,\varepsilon_0}$ instead of λ, we may assume that λ itself satisfies

$$0 < c = |\rho^{ham}(\lambda; 1)| \leq \eta \tag{2.3}$$

$$\|\underline{H}(1)\| \leq \eta \tag{2.4}$$

$$\overline{d}(\lambda, id) \leq \eta \tag{2.5}$$

without loss of any generality. In particular $H(1)$ is *engulfable*.

We measure the size of V_Δ by the following constant

$$C(V_\Delta, \Theta) = \max_{x \in V_\Delta} |p(x)|, \quad x = (q(x), p(x)).$$

This constant is bounded away from 0 and so there exists some $\eta > 0$ depending only on $(U_\Delta, -d\Theta)$ (and so only on (M, ω)) such that whenever a smooth Hamiltonian F satisfies $\overline{d}(\phi_F, id) < 2\eta$, we have

$$\text{Graph}\, \phi_F^t \subset V_\Delta \quad \text{for all } t \in [0,1].$$

Now we fix any such constant $\eta > 0$ so that

$$2\eta < \min\left\{\frac{1}{2}C(U_\Delta, \Theta), \frac{\Sigma_\omega}{4}\right\}. \tag{2.6}$$

For the given hamiltonian homotopy Λ of a topological Hamiltonian loop λ contracting to the identity path in $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$, we consider an approximating sequence $\Lambda_i \in \mathcal{P}^{ham}(Symp_U(M,\omega), id)$ with $U = M \setminus B$ of Λ. We denote by $H_i = H_i(s,t,x)$ the t-Hamiltonian of Λ_i supported in U. *We note that these Hamiltonians are uniquely determined, without ambiguity of normalization constant,* since they are assumed to be compactly supported in U. Since $\Lambda_{i,0} \to id$ in hamiltonian topology $\Lambda_i \Lambda_{i,0}^{-1}$ itself is an approximating sequence of Λ. Therefore by replacing Λ_i by $\Lambda_i \Lambda_{i,0}^{-1}$, we may assume $\Lambda_i(s,0) \equiv id$. (See Definition 1.4.) So we will assume this in addition to the general properties of the approximating sequence in the discussion below. Then let $\underline{H_i} = \underline{H_i}(s,t,x)$ be the associated mean-normalized t-Hamiltonian of Λ_i on M.

By reparameterizing t, we may assume Λ_i are boundary flat both in t-direction. Using Lemma 13.1 in Appendix 13.3, for each given i, we can always reparameterize Λ_i in the form $\Lambda_i(s, \zeta(t))$ where $\zeta : [0,1] \to [0,1]$ are surjective monotonically increasing functions so that $\zeta'(t) \equiv 0$ for t near $\{0,1\}$ and $\|\zeta - id\|_{ham}$ become as small as we want, where $\|\cdot\|_{ham}$ is defined to be

$$\|\zeta\|_{ham} := \|\zeta\|_{C^0} + \|\zeta'\|_{L^1}.$$

We refer readers to Appendix for the details of this reparameterization process originally explained in [OM].

From now on, we assume that Λ_i are boundary flat in the above sense. Since λ_i is an approximating sequence of λ, the inequality (2.3) and (2.4) imply

$$0 < \frac{c}{2} \leq |\rho^{ham}(\lambda_i; 1)| \leq 2\eta, \quad \|H_i(1)\| \leq 2\eta \tag{2.7}$$

for all sufficiently large i's. It also follows

$$\text{supp}\, \phi_{H_i(1)}^s \subset U = M \setminus B \tag{2.8}$$

for all $s \in [0,1]$ and for all i.

We would now like to show

$$\rho^{ham}(\lambda; 1) = \lim_{i \to \infty} \rho^{ham}(\lambda_i; 1) = 0 \tag{2.9}$$

exploiting the convergence of $\overline{d}(\phi_{H_i(s)}^1, id) \to 0$ uniformly over $s \in [0,1]$ as $i \to \infty$. This latter is because $\Lambda_i(s,t) = \phi_{H_i(s)}^t$ is an approximating sequence of Λ, which is a hamiltonian homotopy of a topological Hamiltonian *loop* $\lambda = \phi_{H(1)}$ to the constant loop id.

For this purpose, we rewrite

$$\frac{c}{2} \leq \rho^{ham}(\underline{F_i}; 1) = \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0) + \left(\rho^{ham}(\underline{F_i}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0)\right). \tag{2.10}$$

We would like to emphasize that rewriting $\rho^{ham}(\underline{F_i}; 1)$ in this way is a crucial trick. For example, the first term $\rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0)$ can be studied entirely via Lagrangian spectral invariant using the equality (1.23) and (1.24) mentioned in the introduction. Sections 3-8 further develop the theory of Lagrangian spectral invariants needed for its proof beyond the one introduced in [Oh2, Oh3].

On the other hand, 'taking the difference' inside parenthesis of the second summand enables us to convert the spectral invariants of mean-normalized $\underline{F_i}$ into the ones of F_i with the support property $\operatorname{supp} F_i \subset U = M \setminus B$ by rewriting

$$\left(\rho^{ham}(\underline{F_i}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0)\right) = \left(\rho^{ham}(F_i; 1) - \rho^{ham}_{\mathcal{U}}(F_i; 1_0)\right).$$

This enables us to study this term using Ostrover's trick [Os], enhanced by Seyfaddini [Sey], together with a judicious usage of the Lagrangian triangle inequality given in Proposition 4.3.

3. Basic generating function h_H of Lagrangian submanifold

In this section, we recall the definition of *basic generating function*.

Let $H = H(t,x)$ be a Hamiltonian on T^*N which is *asymptotically constant* i.e., one whose Hamiltonian vector field X_H is compactly supported. Denote by $\mathcal{P}C^\infty_{asc}(T^*N, \mathbb{R})$ be the set of such a family of functions. We denote $L_H = \phi^1_H(o_N)$ and denote by $i_H : L_H \hookrightarrow T^*N$ the inclusion map.

Example 3.1. Consider a mean-normalized Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ on a closed symplectic manifold (M, ω). The manifold M carries a natural Liouville measure induced by ω^n. Consider the diagonal Lagrangian $\Delta \subset (M \times M, \omega \oplus -\omega)$ identified with the zero section $o_\Delta \subset T^*\Delta$ in a Darboux chart $(V_\Delta, -d\Theta)$ of Δ in $M \times M$. Then consider the Hamiltonian

$$\mathbb{H} : [0,1] \times T^*\Delta \to \mathbb{R}$$

defined by $\mathbb{H}(t,(x,y)) := \chi(d(x,y))H(t,x)$ where $\chi = \chi(r)$ is a cut-off function with $\operatorname{supp}\chi \subset [0,R)$ where we identify $V_\Delta \cong D^R(T^*\Delta)$. Then $\mathbb{H}$ is compactly supported and automatically satisfies the normalization condition.

$$\int_\Delta \mathbb{H}(t, \phi^t_{\mathbb{H}}(q))\, \rho_\Delta = 0 \tag{3.1}$$

for all $t \in [0,1]$ where ρ_Δ is the measure on Δ induced by the Liouville measure under the projection to the first factor.

Recall the classical action functional is defined as

$$\mathcal{A}^{cl}_H(\gamma) = \int \gamma^*\theta - \int_0^1 H(t,\gamma(t))\, dt$$

on the space $\mathcal{P}(T^*N)$ of paths $\gamma : [0,1] \to T^*N$, and its first variation formula is given by

$$d\mathcal{A}^{cl}_H(\gamma)(\xi) = \int_0^1 \omega(\dot\gamma - X_H(t,\gamma(t)), \xi(t))\, dt - \langle\theta(\gamma(0)), \xi(0)\rangle + \langle\theta(\gamma(1)), \xi(1)\rangle. \tag{3.2}$$

For given $q \in o_N \cong N$, we denote

$$z_H^q(t) = \phi_H^t(q)$$

which is a Hamiltonian trajectory such that

$$z_H^q(0) = q \in o_N, \tag{3.3}$$

which specifies the *initial point* $q \in o_N$. (We remark that the notation here is slightly different from that of [Oh2, Oh3] in that z_H^q therein denotes z_q^H in this paper. We adopt the current notation to be consistent with that of [Oh12] and other recent papers of the author.)

We define the function $\widetilde{h}_H : [0,1] \times N \to \mathbb{R}$ by

$$\widetilde{h}_H(t,q) = \int \left(z_H^q|_{[0,t]}\right)^* \theta - \int_0^t H(u, \phi_H^u(q))\, du. \tag{3.4}$$

The following basic lemma follows immediately from (3.2) whose proof we omit.

Lemma 3.1. *The function $\widetilde{h}_H$ satisfies*

$$\begin{aligned} d\widetilde{h}_H(t,q) &= ((z_H^q)^*\theta(t) - H(t, z_H^q(t))\, dt) + (\psi_H^t)^*\theta \\ &= \psi_H^*\theta - H(t, z_H^q(t))\, dt \end{aligned}$$

where $\psi_H : [0,1] \times N \to T^*N$ defined by $\psi_H(t,q) = \phi_H^t|_{o_N}$ and $\psi_H^t(q) = \psi_H(t,q)$.

It turns out that the following form of Hamiltonian trajectories

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x)) \tag{3.5}$$

are also useful, which specifies the *final point* of the trajectory instead of the initial point as specified in the trajectory z_H^q.

Denote $L_H = \phi_H^1(o_N)$. We would like to point out that the function

$$h_H : L_H \to \mathbb{R}\,;\, h_H(x) := \widetilde{h}_H(1, (\phi_H^1)^{-1}(x)) = \mathcal{A}_H^{cl}(z_x^H)$$

defines the natural generating function of L_H in that $dh_H = i_H^*\theta$ where $i_H : L_H \to T^*N$ is the canonical inclusion map. The image of the map

$$x \in L_H \mapsto (h_H(x), x)$$

defines a canonical Legendrian lift of L_H in the one-jet bundle $J^1(N) \cong \mathbb{R} \times T^*N$. (See [Oh2] for the relevant discussion.) We denote the corresponding Legendrian submanifold by R_H. However, as a function on N, h_H is multi-valued, while $\widetilde{h}_H$ is a well-defined single-valued function.

In general, the projection $R \to \mathbb{R} \times N$ of any Legendrian submanifold $R \subset J^1(N, \mathbb{R}) = \mathbb{R} \times T^*N$ is called the *wave front* [El] of the Legendrian submanifold R. We denote by $W_R \subset \mathbb{R} \times N$ by the front of R. We also define the (Lagrangian) action spectrum of H on T^*N by

$$\mathrm{Spec}(H; N) = \mathrm{Crit}(h_H) = \{\mathcal{A}_H^{cl}(z_x^H) \mid x \in L_H \cap o_N\}. \tag{3.6}$$

It follows that $\mathrm{Spec}(H;N)$ is a compact subset of $\mathbb{R}$ of measure zero.

Remark 3.2. We would like to note that we have no a priori control of C^0 bound for the functions h_H (or equivalently $\widetilde{h}_H$), even when H is bounded in $L^{(1,\infty)}$ norm. Getting this C^0-bound is equivalent to getting the bound for the actions of the relevant Hamiltonian chords. Indeed understanding the precise relationship between the action bound, the norm $\|H\|$ and the C^0-distance of the time-one map ϕ_H^1 is a heart of the matter in C^0 symplectic topology.

In section 6, we recall construction of *basic phase function* f_H from [Oh2] which is a particular single valued selection of the multivalued function h_H on N. This function was constructed via the Floer mini-max arguments similarly as the spectral invariants $\rho^{ham}(H;a)$ is defined, and its C_0-norm is bounded by $\|H\|$. It turns out that there is a *measurable, discontinuous but differentiable almost everywhere* map, denoted by $\varphi^H : N \to N$, which relates $\widetilde{h}_H$ and f_H via the identity

$$f_H = \widetilde{h}_H \circ \varphi^H \quad \text{almost everywhere.}$$

Furthermore $\varphi^H \to id_N$ almost everywhere in the L^∞-sense because $\phi_H^1 \to id$ in the C^0 topology. In particular $(\varphi^H)_*\rho \to \rho$ in measure for any given density ρ on N. This convergence property of φ^H plays a fundamental role in our proof of the main theorem in section 12.

4. Lagrangian spectral invariants

In this section, we first briefly recall the construction of Lagrangian spectral invariants $\rho^{lag}(H;a)$ for $L_H = \phi_H^1(o_N)$ performed by the author in [Oh3]. A priori, this invariant may depend on H, not just on L_H itself. In [Oh3], we prove that

$$\rho^{lag}(H;a) = \rho^{lag}(F;a) \tag{4.1}$$

for all $a \in H^*(N;\mathbb{Z})$ if $L_H = L_F$, *but modulo the addition of a constant* and then somewhat ad-hoc normalization to remove this ambiguity of a constant.

Consider the zero section o_N and the space

$$\mathcal{P}(o_N, o_N) = \{\gamma : [0,1] \to T^*N \mid \gamma(0),\, \gamma(1) \in o_N\}.$$

The set of generators of $CF(H; o_N, o_N)$ is that of solutions

$$\dot z = X_H(t, z(t)),\; z(0),\, z(1) \in o_N$$

and its Floer differential is defined by counting the number of solutions of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(\tau,0),\, u(\tau,1) \in o_N. \end{cases} \tag{4.2}$$

An element $\alpha \in CF(H; o_N, o_N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in Chord(H; o_N, o_N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We denote the *level* of the chain α by

$$\lambda_H(\alpha) := \max_{z \in \operatorname{supp} \alpha} \{\mathcal{A}_H^{cl}(z)\}. \tag{4.3}$$

For given non-zero cohomology class $a \in H^*(N, \mathbb{Z})$, we consider its Poincaré dual $[a]^\flat := PD(a) \in H_*(N, \mathbb{Z})$ and its image under the canonical isomorphism

$$\Phi : H_*(N, \mathbb{Z}) \to HF_*(H, J; o_N, o_N).$$

Definition 4.1. Let (H, J) be a Floer regular pair relative to (o_N, o_N) and let $(CF(H), \partial_{(H,J)})$ be its associated Floer complex. For any $0 \neq a \in H^*(N, \mathbb{Z})$, we define

$$\rho^{lag}(H;a) = \inf_{\alpha \in \Phi(a^\flat)} \{\lambda_H(\alpha)\}. \tag{4.4}$$

One important result is the following basic property, called *spectrality* in [Oh6], which is not explicitly stated in [Oh2] but can be easily derived by a compactness argument. (See the proof in [Oh6] given in the Hamiltonian context.)

Proposition 4.1. *Let $H = H(t,x)$ be any,* not necessarily nondegenerate*, smooth Hamiltonian. Then for any $0 \neq a \in H^*(N,\mathbb{Z})$, there exists a point $x \in L_H \cap o_N$ such that*

$$\mathcal{A}^{cl}_H(z^H_x) = \rho^{lag}(H;a).$$

In particular, $\rho^{lag}(H;a) \in \mathrm{Spec}(H;N)$.

The following notion of tightness is a useful notion introduced in [Oh9].

Definition 4.2. Let $a \in H^*(N,\mathbb{Z})$. Assume H is a generic Hamiltonian so that $\phi^1_H(o_N)$ intersects $T^*_q N$ transversely. A cycle α with $[\alpha] = PD[a]$ is called *tight* if it satisfies $\lambda_H(\alpha) = \rho^{lag}(H;a)$.

4.1. **Triangle inequality for Lagrangian spectral invariants.** We recall from, [Sc], [Oh6] that the triangle inequality of the Hamiltonian spectral invariants

$$\rho^{ham}(H\#F; a\cdot b) \leq \rho^{ham}(H;a) + \rho^{ham}(F;b)$$

for the product Hamiltonian $H\#F$ relies on the homotopy invariance property of spectral invariants which in turn relies on the existence of canonical normalization procedure of Hamiltonians on closed (M,ω) which is nothing but the *mean normalization*. On the other hand, one can directly prove

$$\rho^{ham}(H * F; a\cdot b) \leq \rho^{ham}(H;a) + \rho^{ham}(F;b)$$

more easily for the concatenated Hamiltonian. (See e.g., [FOOO3] for the proof.) Once we have the latter inequality, we can derive the former from the latter again by the homotopy invariance property of $\rho^{ham}(\cdot;a)$ *for the mean-normalized Hamiltonians.*

When one attempts to assign an invariant of Lagrangian submanifold $\phi^1_H(o_N)$ itself out of the spectral invariant $\rho^{lag}(H;a)$, one has to choose a normalization of the Hamiltonian *relative to* the Lagrangian submanifold. Since there is no canonical normalization unlike the Hamiltonian case, the invariance property of Lagrangian spectral invariants and so the triangle inequality is somewhat more nontrivial than the case of Hamiltonian spectral invariants. In this subsection, we clarify these issues of invariance property and of the triangle inequality.

The following parametrization independence follows immediately from the construction of Lagrangian spectral invariants and $L^{(1,\infty)}$-continuity of $H \mapsto \rho^{lag}(H;a)$.

Lemma 4.2. *Let $H = H(t,x)$ be any,* not necessarily nondegenerate*, smooth Hamiltonian and let $\chi : [0,1] \to [0,1]$ a reparameterization function with $\chi(0) = 0$ and $\chi(1) = 1$. Then*

$$\rho^{lag}(H;a) = \rho^{lag}(H^\chi;a)$$

where $H^\chi(t,x) = \chi'(t)H(\chi(t),x)$.

We first recall the following triangle inequality which was essentially proved in [Oh3]. (See Theorem 6.4 and Lemma 6.5 [Oh3]. In [Oh3], the cohomological version of the Floer complex was considered and hence the opposite inequality is stated. Other than this, the same proof can be applied here.)

Proposition 4.3. *Let H, $F \in \mathcal{PC}^\infty_{asc}(T^*N;\mathbb{R})$, and assume F is autonomous. Then we have*

$$\rho^{lag}(H\#F;ab) \leq \rho^{lag}(H;a) + \rho^{lag}(F;b). \tag{4.5}$$

Recently, Monzner, Vichery, and Zapolsky [MVZ] proved the following form of the triangle inequality which uses the concatenated Hamiltonian $H * F$ instead of the product Hamiltonian $H\#F$.

Proposition 4.4 (Proposition 2.4 [MVZ])**.** *Suppose $H(1,x) \equiv F(0,x)$ and $H * F$ be the concatenated Hamiltonian. Then*

$$\rho^{lag}(H * F;ab) \leq \rho^{lag}(H;a) + \rho^{lag}(F;b) \tag{4.6}$$

for all a, $b \in H^*(N)$.

In particular, this proposition applies to all pairs H, F which are boundary flat.

Remark 4.3. We suspect that (4.5) holds even for the non-autonomous F as in the Hamiltonian case but we did not check this, since it is not needed in the present paper.

4.2. **Assigning spectral invariants to Lagrangian submanifolds.** In this subsection, we identify a class, denoted by $\mathcal{PC}^\infty_{(B;e)}$, of Hamiltonians H among those satisfying $\phi^1_H(o_N) = \phi^1_F(o_N)$, such that the equality

$$\rho^{lag}(H;a) = \rho^{lag}(F;a)$$

holds for all H, $F \in \mathcal{PC}^\infty_{(B;e)}$. As the notation suggests, the class depends on the subset $B \subset N$ and the real number $e \in \mathbb{R}$.

We start with the following proposition. The proof closely follows that of Lemma 2.6 [MVZ] which uses Proposition 4.4 in a significant way. We need to modify their proof to obtain a somewhat stronger statement, which replaces the condition "$\phi^1_H = \phi^1_F$" used in [MVZ] by the conditions put in this proposition. Identifying the optimal condition as stated in this proposition turns out to be an essential element of our proof in section 12.

Proposition 4.5 (Compare with Lemma 2.6 [MVZ])**.** *Let H, $F \in \mathcal{PC}^\infty_{asc}(T^*N;\mathbb{R})$ be boundary-flat. Suppose in addition H, F satisfy the following:*

1. *$\phi^1_H(o_N) = \phi^1_F(o_N)$,*
2. *$H \equiv c(t)$, $F \equiv d(t)$ on a tubular neighborhood $T \supset B$ in T^*N of a closed ball $B \subset o_N$ where $c(t)$, $d(t)$ are independent of $x \in T$, and*
3. *they satisfy*

$$\int_0^1 c(t)\,dt = \int_0^1 d(t)\,dt.$$

Then $\rho^{lag}(H;a) = \rho^{lag}(F;a)$ holds for all $a \in H^*(N,\mathbb{Z})$ without ambiguity of constant.

Proof. We consider the Hamiltonian path $\phi_G : t \mapsto \phi^t_G$ with $G = \widetilde{F} * H$ with $\widetilde{F}(t,x) = -F(1-t,x)$. This defines a loop of Lagrangian submanifold

$$t \mapsto \phi^t_G(o_N), \quad \phi^1_G(o_N) = o_N$$

and satisfies $\phi^t_G|_B \equiv id$ and

$$G(t,q) = \begin{cases} -c(1-2t) & 0 \leq t \leq 1/2 \\ d(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

for all $q \in B \subset T$ by definition $G = \widetilde{F} * H$.

We claim $\rho^{lag}(G; a) = 0$ for all $0 \neq a \in H^*(N)$. This will be an immediate consequence of the following lemma and the spectrality of numbers $\rho^{lag}(G; a)$.

Lemma 4.6. *The value $\mathcal{A}^{cl}_G(z)$ does not depend on the Hamiltonian chord $z \in \mathcal{C}hord(G; o_N, o_N)$. In particular, $\mathcal{A}^{cl}_G(z) = 0$.*

Proof. Recall that any Hamiltonian chord in $\mathcal{C}hord(G; o_N, o_N)$ has the form

$$z(t) = z^q_G(t)$$

for some $q \in o_N$. Here we use the hypothesis $\phi^1_G(o_N) = o_N$. Consider any smooth path $\alpha : [0,1] \to o_N$ with $\alpha(0) = q$, $\alpha(1) = q'$. Then

$$\mathcal{A}^{cl}_G\left(z^{q'}_G\right) - \mathcal{A}^{cl}_G\left(z^q_G\right) = \int_0^1 \frac{d}{du}\mathcal{A}^{cl}_G\left(z^{\alpha(u)}_G\right)\, du.$$

But a straightforward computation using the first variation formula (3.2) implies

$$\frac{d}{du}\mathcal{A}^{cl}_G\left(z^{\alpha(u)}_G\right) = \left\langle \theta, \frac{\partial}{\partial u}(\phi_G(\alpha(u)))\right\rangle - \left\langle \theta, \frac{\partial}{\partial u}(\alpha(u))\right\rangle = 0 - 0 = 0$$

since $\phi_G(\alpha(u))$, $\alpha(u) \in o_N$.

For the second statement, we have only to consider the constant path $z \equiv c_q \in B$ for which

$$\begin{aligned}\mathcal{A}^{cl}_G(c_q) &= -\int_0^1 G(t,q)\, dt = \int_0^{1/2} c(1-2t)\, dt - \int_{1/2}^1 d(2t-1)\, dt \\ &= \int_0^1 c(t)\, dt - \int_0^1 d(t)\, dt = 0.\end{aligned}$$

This proves the lemma. ☐

Once we have the lemma, we can apply the triangle inequality (4.6)

$$\rho^{lag}(H; a) \leq \rho^{lag}(F; a) + \rho^{lag}(G; 1) = \rho^{lag}(F; a)$$

for any given $a \in H^*(N)$. By changing the role of H and F in the proof of the above lemma, we also obtain $\rho^{lag}(\widetilde{G}; 1) = 0$ and then obtain $\rho^{lag}(F; a) \leq \rho^{lag}(H; a)$ by triangle inequality. This finishes the proof of the proposition. ☐

This proposition motivates us to introduce the following definitions

Definition 4.4. For each given $B \subset N$, we define

$$\mathfrak{Iso}_B(o_N; T^*N) = \{L \in \mathfrak{Iso}(o_N; T^*N) \mid o_N \cap L \supset B\}.$$

When a function $c : [0,1] \to \mathbb{R}$ is given in addition, we define

$$\begin{aligned}\mathcal{P}C^\infty_{(B;e)} &= \{H \in \mathcal{P}C^\infty_{asc} \mid H_t \equiv c(t) \text{ on a neighborhood of } B \text{ in } T^*N \\ &\qquad \text{and } \int_0^1 c(t)\, dt = e\}.\end{aligned}$$

With these definitions, the proposition enables us to unambiguously define the following spectral invariant attached to L.

Definition 4.5. Suppose $L \in \mathfrak{Iso}_B(o_N;T^*N)$ and let $e \in \mathbb{R}$ be given. For each given such e, we define a spectral invariant of $L \in \mathfrak{Iso}_B(o_N;T^*N)$ by

$$\rho^{(B;e)}(L;a) := \rho^{lag}(H;a), \quad L = \phi_H^1(o_N)$$

for a (and so any) $H \in \mathcal{P}C^\infty_{(B;e)}$.

With this definition, we have the following obvious lemma

Lemma 4.7. *Let $H \in \mathcal{P}C^\infty_{(B;e)}$, then $\widetilde{H}, \overline{H} \in \mathcal{P}C^\infty_{(B;-e)}$.*

Then we prove the following duality statement of $\rho^{(B;e)}$.

Proposition 4.8. *Let $H \in \mathcal{P}C^\infty_{(B;e)}$ and $L = \phi_H^1(o_N)$. We denote $\widetilde{L} = \phi^1_{\widetilde{H}}(o_N) = \phi^1_{\overline{H}}(o_N)$. Then*

$$\rho^{(B;-e)}(\widetilde{L};1) = -\rho^{(B;e)}(L;[pt]^\#). \tag{4.7}$$

Proof. By the above lemma, $\widetilde{H} \in \mathcal{P}C^\infty_{(B;-e)}$ and so $\rho^{(B;-e)}(\widetilde{L};1)$ is given by

$$\rho^{(B;-e)}(\widetilde{L};1) = \rho^{lag}(\widetilde{H};1)$$

by definition. But it was proven in [V1, Oh2, Oh3] that

$$\rho^{lag}(\widetilde{H};1) = -\rho^{lag}(H;[pt]^\#) \tag{4.8}$$

which follows from the Poincaré duality argument, by studying the time-reversal flow of the Floer equation (1.10) $\widetilde{u}$ defined by $\widetilde{u}(\tau,t) = u(-\tau, 1-t)$. The map $\widetilde{u}$ satisfies the equation

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \tau} + \widetilde{J}\left(\frac{\partial \widetilde{u}}{\partial t} - X_{\widetilde{H}}(\widetilde{u})\right) = 0 \\ \widetilde{u}(\tau,0),\ \widetilde{u}(\tau,1) \in o_N. \end{cases}$$

Furthermore this equation is compatible with the involution of the path space

$$\iota : \Omega(o_N,o_N) \to \Omega(o_N,o_N)$$

defined by $\iota(\gamma)(t) = \widetilde{\gamma}(t)$ with $\widetilde{\gamma}(t) = \gamma(1-t)$ and the action functional identity

$$\mathcal{A}^{cl}_{\widetilde{H}}(\widetilde{\gamma}) = -\mathcal{A}^{cl}_H(\gamma).$$

We refer to [Oh3] for the details of the duality argument in the Floer theory used in the derivation of (4.8).

On the other hand, by definition,

$$\rho^{lag}(H;[pt]^\#) = \rho^{(B;e)}(L;[pt]^\#)$$

since $H \in \mathcal{P}C^\infty_{(B;e)}$. This finishes the proof. □

5. Comparison of two Cauchy-Riemann equations

So far we have looked at the Hamiltonian-perturbed Cauchy-Riemann equation (4.2), which we call the *dynamical version* as in [Oh2].

On the other hand, one can also consider the *genuine* Cauchy-Riemann equation

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^H \frac{\partial v}{\partial t} = 0 \\ v(\tau,0) \in \phi^1_H(o_N),\ v(\tau,1) \in o_N \end{cases} \tag{5.1}$$

for the path $u : \mathbb{R} \to \mathcal{P}(o_N, L)$ where $L = \phi^1_H(o_N)$ and

$$\mathcal{P}(o_N,L) = \{\gamma : [0,1] \to T^*N \mid \gamma(0) \in L,\ \gamma(1) \in o_N\}$$

and $J_t^H = (\phi_H^t \phi_H^{-1})_* J_t$. We call this version the *geometric version*.

We now describe the geometric version of the Floer homology in some more details. We refer readers to [Oh2] for the discussion on the further comparison of the two versions in the point of moduli spaces and others. The upshot is that there is a filtration preserving isomorphisms between the dynamical version and the geometric version of the Lagrangian Floer theories.

We denote by $\widetilde{\mathcal{M}}(L_H, o_N; J^H)$ the set of finite energy solutions and $\mathcal{M}(L_H, o_N; J^H)$ to be its quotient by $\mathbb{R}$-translations. This gives rise to the geometric version of the Floer homology $HF_*(o_N, \phi_H(o_N), \widetilde{J})$ of the type [Fl1, Oh3] whose generators are the intersection points of $o_N \cap \phi_H(o_N)$. An advantage of this version is that it depends only on the Lagrangian submanifold $L = \phi_H(o_N)$, only loosely on H. (The author proved in [Oh3] that $\rho(H; a)$ is the invariant of $L_H = \phi_H(o_N)$ up to this normalization by comparing these two versions of the Floer theory in [Oh2, Oh3].)

The following is a straightforward to check but is a crucial lemma.

Lemma 5.1. *Let $L = \phi_H^1(o_S)$.*

(1) *The map $\Phi_H : o_N \cap L \to \mathcal{C}hord(H; o_N, o_N)$ defined by*

$$x \mapsto z_x^H(t) = \phi_H^t\left((\phi_H^1)^{-1}(x)\right)$$

gives rise to the one-one correspondence between the set $o_N \cap L \subset \mathcal{P}(o_N, L)$ as constant paths and the set of solutions of Hamilton's equation of H.

(2) *The map $a \mapsto \Phi_H(a)$ also defines a one-one correspondence from the set of solutions of* (4.2) *and that of*

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^H \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H(o_N),\ v(\tau, 1) \in o_N \end{cases} \tag{5.2}$$

where $J^H = \{J_t^H\}$, $J_t^H := (\phi_H^t(\phi_H^1)^{-1})^* J_t$. Furthermore, (5.2) *is regular if and only if* (4.2) *is regular.*

Once we have transformed (4.2) to (5.2), we can further deform J^H to the constant family J_0 and consider

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H(o_N),\ v(\tau, 1) \in o_N. \end{cases} \tag{5.3}$$

This latter deformation preserves the filtration of the associated Floer complexes [Oh2]. A big advantage of considering this equation is that it enables us to study the behavior of spectral invariants for a sequence of L_i converging to o_N *in Hamiltonian distance*.

The following proposition provides the action functional associated to the equation (5.2), (5.3), which will give a natural filtration associated Floer homology $HF(L, o_N)$.

Proposition 5.2. *Let L and h_L be as in Lemma 3.1. Let $\Omega(L, o_N; T^*N)$ be the space of paths $\gamma : [0,1] \to \mathbb{R}$ satisfying $\gamma(0) \in L, o_N$, $\gamma(1) \in o_N$. Consider the effective action functional*

$$\mathcal{A}^{\text{eff}}(\gamma) = \int \gamma^*\theta + h_H(\gamma(0)).$$

Then $d\mathcal{A}^{\text{eff}}(\gamma)(\xi) = \int_0^1 \omega(\xi(t), \dot\gamma(t))\, dt$. In particular,

$$\mathcal{A}^{\text{eff}}(c_x) = h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \tag{5.4}$$

for the constant path $c_x \equiv x \in L \cap o_N$ *i.e., for any critical path* c_x *of* $\mathcal{A}^{\text{eff}}$.

We would like to highlight the presence of the 'boundary contribution' $h_H(\gamma(0))$ in the definition of the effective action functional above: This addition is needed to make the Cauchy-Riemann equation (5.1) or (5.3) into a *gradient trajectory equation* of the relevant action functional. We refer readers to section 2.4 [Oh2] and Definition 3.1 [KO] and the discussion around it for the upshot of considering the effective action functional and its role in the study of Cauchy-Riemann equation.

6. Basic phase function and measurable map φ^H

In this section, we first recall the definition of *basic phase function* constructed in [Oh2]. Then we introduce a crucial measurable map $\varphi^H : N \to N$, which is defined by a selection of of a single valued branch of the multivalued section

$$N \to L_H \subset T^*M$$

followed by $(\phi_H^1)^{-1}$. It is interesting to note that such a selection process was studied e.g., in the theory of multi-valued functions, or Q-valued functions, in the sense of Almgren [Al] in geometric measure theory. In particular, in [DGT], existence of such a single valued branch is studied in the general abstract setting of metric spaces and a finite group action of isometries. It would be interesting to see whether there would be any other significant intrusion of the theory of multivalued functions into the study of symplectic topology.

6.1. Graph selector of wave fronts.

The following theorem was proved in [Cha] and in [Oh2] by the generating function method and by the Floer theory respectively. (According to [PPS], the proof of this theorem was first outlined by Sikorav in Chaperon's seminar.)

Theorem 6.1 (Sikorav, Chaperon [Cha], Oh [Oh2])**.** *Let* $L \subset T^*N$ *be a Hamiltonian deformation of the zero section* o_N*. Then there exists a Lipschitz continuous function* $f : N \to \mathbb{R}$*, which is smooth on an open subset* $N_0 \subset N$ *of full measure, such that*

$$(q, df(q)) \in L$$

for every $q \in N_0$*. Moreover if* $df(q) = 0$ *for all* $q \in N_0$*, then* L *coincides with the zero section* o_N*. The choice of* f *is unique modulo the shift by a constant.*

The details of the proof of Lipschitz continuity of f is given in [PPS]. We denote by $\operatorname{Sing} f$ the set of non-differentiable points of f. Then by definition

$$N_0 = \operatorname{Reg} f := N \setminus \operatorname{Sing} f$$

is a subset of full measure and f is differentiable thereon.

We call such a function f a *graph selector* in general following the terminology of [PPS] and denote the corresponding graph part of the front of the Legendrian submanifold R by

$$G_f := \{(h_L(q, df(q)), q, df(q)) \mid q \in N\} \subset R.$$

By construction, the projection $\pi_R : G_f \to N$ restricts to a one-one correspondence and the function $f : \operatorname{Reg} f \to \mathbb{R}$ continuously extends to $\overline{\operatorname{Reg} f} = N$.

By definition,

$$|df(q)| \leq \max_{x \in L} |p(x)| \tag{6.1}$$

for any $q \in N_0$, where $x = (q(x), p(x))$ and the norm $|p(x)|$ is measured by any given Riemannian metric on N.

Proposition 6.2. *As* $d_{\rm H}(L, o_N) \to 0$, $|df(q)| \to 0$ *uniformly over* $q \in N_0$.

In [Oh2], a canonical choice of f is constructed via the chain level Floer theory, *provided* the generating Hamiltonian H of L is given. The author called the corresponding graph selector f the *basic phase function* of $L = \phi_H^1(o_N)$ and denoted it by f_H. We give a quick outline of the construction referring the readers to [Oh2] for the full details of the construction.

6.2. **The basic phase function f_H and its Lagrangian selector.** Another construction in [Oh2] is given by considering the Lagrangian pair

$$(o_N, T_q^*N), \quad q \in N$$

and its associated Floer complex $CF(H; o_N, T_q^*N)$ generated by the Hamiltonian trajectory $z : [0,1] \to T^*N$ satisfying

$$\dot z = X_H(t, z(t)), \quad z(0) \in o_N,\ z(1) \in T_q^*N. \tag{6.2}$$

Denote by $\mathcal{C}hord(H; o_N, T_q^*N)$ the set of solutions. The differential $\partial_{(H,J)}$ on $CF(H; o_N, T_q^*N)$ is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \\ u(\tau, 0) \in o_N,\ u(\tau,1) \in T_q^*N. \end{cases} \tag{6.3}$$

An element $\alpha \in CF(H; o_N, T_q^*N)$ is expressed as a finite sum

$$\alpha = \sum_{z \in \mathcal{C}hord(H; o_N, T_q^*N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We denote the level of the chain α by

$$\lambda_H(\alpha) := \max_{z \in \operatorname{supp} \alpha} \{\mathcal{A}_H^{cl}(z)\}.$$

The resulting invariant $\rho^{lag}(H; \{q\})$ is to be defined by the mini-max value

$$\rho^{lag}(H; \{q\}) = \inf_{\alpha \in [q]} \lambda_H(\alpha)$$

where $[q] \in H_0(\{q\}; \mathbb{Z})$ is a generator of the homology group $H_0(\{q\}; \mathbb{Z})$.

A priori, $\rho^{lag}(H; \{q\})$ is defined when $\phi_H^1(o_N)$ intersects T_qN^* transversely but can be extended to non-transversal q's by continuity. By varying $q \in N$, this defines a function $f_H : N \to \mathbb{R}$ which is precisely the one called the basic phase function in [Oh2]. (A similar construction of such a function using the generating function method was earlier given by Sikorav and Chaperon [Cha].) We call the associated graph part G_{f_H} the *basic branch* of the front W_{R_H} of R_H.

Theorem 6.3 ([Oh2, Oh6])**.** *There exists a solution* $z : [0,1] \to T^*N$ *of* $\dot z = X(t,z)$ *such that* $z(0) = q$, $z(1) \in o_N$ *and* $\mathcal{A}_H^{cl}(z) = \rho^{lag}(H; \{q\})$ whether or not $\phi_H^1(o_N)$ intersects T_q^*N transversely.

We summarize the main properties of f_H established in [Oh2].

Theorem 6.4 ([Oh2]). *When the Hamiltonian $H = H(t,x)$ such that $L = \phi_H^1(o_N)$ is given, there is a canonical lift f_H defined by $f_H(q) := \rho^{lag}(H;\{q\})$ that satisfies*

$$f_H \circ \pi(x) = h_H(x) = \mathcal{A}^{cl}_H(z^H_x) \tag{6.4}$$

for some Hamiltonian chord z^H_x ending at $x \in T^*_q N$. This f_H satisfies the following property in addition

$$\|f_H - f_{H'}\|_\infty \leq \|H - H'\|. \tag{6.5}$$

An immediate corollary of this theorem is

Corollary 6.5. *If H_i converges in $L^{(1,\infty)}$, then f_{H_i} converges uniformly.*

Based on this corollary, we will just denote the limit continuous function by

$$f_H := \lim_{i\to\infty} f_{H_i} \tag{6.6}$$

when $H_i \to H$ in $L^{(1,\infty)}$-topology, and call it the basic phase function of the topological Hamiltonian H or of the C^0-Lagrangian submanifold $L_H = \phi_H^1(o_N)$.

Note that $\pi_H = \pi|_{L_H} : L_H = \phi_H^1(o_N) \to N$ is surjective for all H (see [LS] for its proof) and so $\pi_H^{-1}(\pi_H^{-1}(q)) \subset o_N$ is a non-empty compact subset of $o_N \cong N$. Therefore we can regard the 'inverse' $\pi_H^{-1} : N \to L_H \subset T^*N$ as a everywhere defined multivalued section of $\pi : T^*N \to N$.

We introduce the following general definition

Definition 6.1. Let $L \subset T^*N$ be a Lagrangian submanifold projecting surjectively to N. We call a single valued section σ of T^*N with values lying in L a *Lagrangian selector* of L.

For any given Lagrangian selector σ of $L = L_H = \phi_H^1(o_N)$, we define the map $\varphi^\sigma : N \to N$ to be

$$\varphi^\sigma(q) = (\phi_H^1)^{-1}(\sigma(q)).$$

Recall that the graph G_{f_H} is a subset of the front W_{R_H} of R_H and for a generic choice of H the set $\operatorname{Sing} f_H \subset N$ consists of the crossing points of the two different branches and the cusp points of the front of W_{R_H}. Therefore it is a set of measure zero in N. (See [El], [PPS], for example.) Once the graph selector f_H of L_H is picked out, it provides a natural Lagrangian selector defined by

$$\sigma_H(q) := \text{Choice}\{x \in L_H \mid \pi(x) = q,\ \mathcal{A}^{cl}_H(z^H_x) = f_H(q)\}$$

via the axiom of choice where Choice is a choice function. We call this particular Lagrangian selector of L_H the *basic Lagrangian selector.*

The general structure theorem of the wave front (see [El], [PPS] for example) proves that the section σ_H is a differentiable map on a set of full measure for a generic choice of H which is, however, *not necessarily continuous*: This is because as long as $q \in N\backslash \operatorname{Sing} f_H$, we can choose a small open neighborhood of $U \subset N\backslash\operatorname{Sing} f_H$ of q and $V \subset L_H = \phi_H^1(o_N)$ of $x \in V$ with $\pi(x) = q$ so that the projection $\pi|_V : V \to U$ is a diffeomorphism.

Then we define a measurable map $\varphi^H : N \to N$ by

$$\varphi^H(q) = (\phi_H^1)^{-1}(\sigma_H(q)). \tag{6.7}$$

The map φ^H is a *measurable, but not necessarily continuous*, map which is however differentiable on a set of full measure for a generic choice of H. On the other hand, the map φ^H may not be continuous along the subset $\mathrm{Sing}\, f_H \subset N$ which is a set of measure zero. By definition, we have

$$f_H(q) = \mathcal{A}_H^{cl}\left(z_H^{\varphi^H(q)}\right) = \widetilde{h}_H(\varphi^H(q)). \tag{6.8}$$

This relationship between f_H and $\widetilde{h}_H$ is the reason why we introduce the map φ^H which will play a crucial role in the proof of main theorem in section 12.

The following lemma is obvious from the definition of φ^H which will be used later in section 12. We note

$$d_{\mathrm{H}}(\phi_H^1(o_N), o_N) \leq \mathrm{osc}_{C^0}(\phi_H^1; o_N)$$

where $d_{\mathrm{H}}(\phi_H^1(o_N), o_N)$ is the Hausdorff distance.

Lemma 6.6. *We have*

$$d(\varphi^H(x), x) \leq d_{\mathrm{H}}(\phi_H^1(o_N), o_N) + \mathrm{osc}_{C^0}(\phi_H^1; o_N) \leq 2\mathrm{osc}_{C^0}(\phi_H^1; o_N)$$

for all $x \in N_0$. In particular, if $\mathrm{osc}_{C^0}(\phi_H^1; o_N) \to 0$, then $\max_{x\in N_0} d(\varphi^H(x), x) \to 0$ uniformly over $x \in N_0$.

7. Triangle product in Floer homology

We first remark that both $\rho^{lag}(H;1)$ and f_H remain unchanged under the change of H outside a neighborhood of $\bigcup_{t\in[0,1]} \phi_H^t(o_N)$.

The main theorem we prove in this section is the following whose proof occupies the entirety of this section.

Theorem 7.1. *For any Hamiltonian $H \in \mathcal{P}C^\infty_{asc}$,*

$$\max f_H \leq \rho^{lag}(H;1).$$

Remark 7.1. One might recall the general inequality $\rho^{lag}(H;[pt]^\#) \leq \rho^{lag}(H;1)$ and so wonder whether the inequality $\rho^{lag}(H;[pt]^\#) \leq \min f_H$ from below would hold or not. However this inequality fails to hold in general. See Example 9.4 [Oh2] which studies an example of Lagrangians on T^*S^1. In that example, one can check that $\rho^{lag}(H;[pt]^\#) = 0$ which is realized by the level of the Floer cycle $z_1 + z_3$ in the example. But the minimum of f_H is realized by a negative number at a non-smooth point of the function f_H.

We first recall the definition of the triangle product described in [Oh3], [FO] and put it into a more modern context in the general Lagrangian Floer theory such as in [FOOO1] and in other more recent literatures.

Let $q \in N$ be given. Consider the Hamiltonians $H : [0,1] \times T^*N \to \mathbb{R}$ such that L_H intersects transversely both o_N and T_q^*N. We consider the Floer complexes

$$CF(L_H, o_N), \quad CF(o_N, T_q^*N), \quad CF(L_H, T_q^*N)$$

each of which carries filtration induced from the effective action function given in Proposition 5.2. We denote by $\mathfrak{v}(\alpha)$ the level of the chain α in any of these complexes.

More precisely, $CF(L_H, o_N)$ is filtered by the effective functional

$$\mathcal{A}^{(1)}(\gamma) := \int \gamma^*\theta + h_H(\gamma(0)),$$

$CF^\mu(o_N, T_q^*N)$ by

$$\mathcal{A}^{(2)}(\gamma) := \int \gamma^*\theta,$$

and $CF(L_H, T_q^*N)$ by

$$\mathcal{A}^{(0)}(\gamma) := \int \gamma^*\theta + h_H(\gamma(0))$$

respectively. We recall the readers that h_H is the potential of L_H and the zero function the potentials of o_N, T_q^*N.

We now consider the triangle product in the chain level, which we denote by

$$\mathfrak{m}_2 : CF(L_H, o_N) \otimes CF(o_N, T_q^*N) \to CF(L_H, T_q^*N) \tag{7.1}$$

following the general notation from [FOOO1], [Se]. This product is defined by considering all triples

$$x_1 \in L_H \cap o_N,\ x_2 \in o_N \cap T_q^*N,\ x_0 \in L_H \cap T_q^*N$$

with the polygonal Maslov index $\mu(x_1, x_2; x_0)$ whose associated analytical index, or the virtual dimension of the moduli space

$$\mathcal{M}_3(D^2; x_1, x_2; x_0) := \widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0)/PSL(2, \mathbb{R})$$

of J-holomorphic triangles, becomes zero and counting the number of elements thereof. The precise formula of the index is irrelevant to our discussion which, however, can be found in [Se], [FOOO2].

Definition 7.2. Let $J = J(z)$ be a domain-dependent family of compatible almost complex structures with $z \in D^2$. We define the space $\widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0)$ by the pairs $(w, (z_0, z_1, z_2))$ that satisfy the following:

(1) $w : D^2 \to T^*N$ is a continuous map satisfying $\overline{\partial}_J w = 0$ $D^2 \setminus \{z_0, z_1, z_2\}$,
(2) the marked points $\{z_0, z_1, z_2\} \subset \partial D^2$ with counter-clockwise cyclic order,
(3) $w(z_1) = x_1$, $w(z_2) = x_2$ and $w(z_0) = x_0$,
(4) the map w satisfies the Lagrangian boundary condition

$$w(\partial_1 D^2) \subset L_H,\ w(\partial_2 D^2) \subset o_N,\ w(\partial_3 D^2) \subset T_q^*N$$

where $\partial_i D^2 \subset \partial D^2$ is the are segment in between x_i and x_{i+1} ($i \mod 3$).

The general construction is by now well-known and e.g., given in [FOOO1]. In the current context of exact Lagrangian submanifolds, the detailed construction is also given in [Oh3] and [Se]. One important ingredient in relation to the study of the effect on the level of Floer chains under the product is the following (topological) energy identity where the choice of the *effective* action functional plays a crucial role. For readers' convenience, we give its proof here.

Proposition 7.2. *Suppose $w : D^2 \to T^*N$ be any smooth map with finite energy that satisfy all the conditions given in 7.2, but not necessarily J-holomorphic. We denote by $c_x : [0, 1] \to T^*N$ the constant path with its value $x \in T^*N$. Then we have*

$$\int w^*\omega_0 = \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}). \tag{7.2}$$

Proof. Recall $\omega_0 = -d\theta$ and $i^*\theta = dh_H$ on L_H and $i^*\theta = 0$ on o_N and T_q^*N where i's are the associated inclusion maps of L_H, o_N, $T_q^*N \subset T^*N$ respectively. Therefore

$$\begin{aligned} \int_{D^2} w^*\omega_0 &= -\int_{\partial D^2} w^*\theta = -\int_{\partial_1 D^2} w^*\theta - \int_{\partial_2 D^2} w^*\theta - \int_{\partial D_3^2} w^*\theta \\ &= -\int_{\partial_1 D^2} w^* dh_H - 0 - 0 = h_H(w(z_1)) - h_H(w(z_2)) \\ &= h_H(x_1) - h_H(x_0) = \mathcal{A}^{(1)}(c_{x_1}) - \mathcal{A}^{(0)}(c_{x_0}) \\ &= \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}). \end{aligned}$$

Here the last equality comes since $\mathcal{A}^{(2)}(c_{x_2}) = \int c_{x_2}^*\theta = 0$. This finishes the proof. □

An immediate corollary of this proposition from the definition of $\mathfrak{m}_2$ is that the map (7.1) restricts to

$$\mathfrak{m}_2 : CF^\lambda(L_H, o_N) \otimes CF^\mu(o_N, T_q^*N) \to CF^{\lambda+\mu}(L_H, T_q^*N).$$

It is straightforward to check that this map satisfies

$$\partial(\mathfrak{m}_2(x,y)) = \mathfrak{m}_2(\partial(x), y) \pm \mathfrak{m}_2(x, \partial(y))$$

and in turn induces the product map

$$*_F : HF^\lambda(L_H, o_N) \otimes HF^\mu(o_N, T_q^*N) \to HF^{\lambda+\mu}(L_H, T_q^*N) \tag{7.3}$$

in homology. This is because if w is J-holomorphic $\int w^*\omega \geq 0$. (We refer to [Oh3] and [FO] for the general construction of product map $\mathfrak{m}_2$ and to [Oh3], [MVZ] for the study of filtration. Similar study of filtration is also performed in [Sc], [Oh6] in the Hamiltonian Floer homology setting.)

With these preparations, we are ready to wrap-up the proof of Theorem 7.1:

Proof of Theorem 7.1. We consider a Floer cycle α representing the fundamental class $1^\flat = [M] \in HF(L_H, o_N)$ and $\beta = \{q\}$ representing the unique generator of $HF(o_N, T_q^*N) \cong \mathbb{Z}$. Then by definition

$$\mathfrak{v}(\alpha) \geq \rho^{lag}(H;1), \quad \mathfrak{v}(\beta) = \rho^{lag}(0;[q]) = 0.$$

Then its product cycle $\mathfrak{m}_2(\alpha,\beta) \in CF(L_H, T_q^*N)$ represents the homology class $[q] \in CF(L_H, T_q^*N) \cong \mathbb{Z}$ and so $\mathfrak{v}(\mathfrak{m}_2(\alpha,\beta)) \geq \rho^{lag}(H;\{q\}) = f_H(q)$ by definition of the latter. Applying the triangle inequality, we obtain

$$\mathfrak{v}(\alpha) + 0 = \mathfrak{v}(\alpha) + \mathfrak{v}(\beta) \geq \mathfrak{v}(\mathfrak{m}_2(\alpha,\beta)) \geq \rho^{lag}(H;\{q\}) = f_H(q).$$

Therefore we have derived

$$\mathfrak{v}(\alpha) \geq f_H(q)$$

for all cycle $\alpha \in CF(L_H, o_N)$ representing $[M]$. By definition of $\rho^{lag}(H;1)$, this proves

$$\rho^{lag}(H;1) \geq f_H(q).$$

Since this holds for any point $q \in N$, we have proved $\rho^{lag}(H;1) \geq \max f_H$. □

8. A Hamiltonian continuity theorem of spectral capacity

In this section, we prove the following Hamiltonian continuity of spectral capacity. The proof of this theorem is an adaptation to the Lagrangian context of the one used by Seyfaddini in his proof of Theorem 1 (or rather Corollary 1.2) [Sey]. The proof is also a variation of Ostrover's scheme used in [Os] and is an adaptation thereof. In our proof, we however use the Lagrangian analog to the notion of 'ε-shiftability' introduced by Seyfaddini [Sey], instead of 'displaceability' used in [Os] and in other literature such as [EP], [U]. In the Lagrangian context here, the ε-shiftable domain is realized as the graph of df of a function f having no critical points on the corresponding domain. In this regard, it appears to the author that the notion of ε-shiftability becomes more geometric and intuitive in the Lagrangian context than in the Hamiltonian context.

Consider the subset

$$C^\infty_{crit}(N;B) = \{f \in C^\infty(N) \mid \mathrm{Crit}\, f \subset \mathrm{Int}\, B\}.$$

We recall the notation

$$\mathrm{osc}_{C^0}(\phi_H^1; o_N) := \max\left\{\max_{x\in o_N} d\left(\phi_H^1(x), x\right),\ \max_{x\in o_N} d\left((\phi_H^1)^{-1}(x), x\right)\right\}.$$

from (1.26).

Theorem 8.1. *Let $\lambda_i = \phi_{H_i}$ where $H_i \in \mathcal{P}C^\infty_{asc}$ is a sequence such that*

1. *there exists $R > 0$ such that $\mathrm{supp}\, X_{H_i} \subset D^R(T^*N)$ for all i and $s \in [0,1]$,*
2. *There exists a closed ball $B \subset N$ such that $\mathrm{supp}\,\phi_{H_i} \cap o_B = \emptyset$ for all i where we recall*
$$\mathrm{supp}\,\phi_{H_i} = \bigcup_{t\in[0,1]} \mathrm{supp}\,\phi_{H_i}^t.$$
3. *There exists a uniform neighborhood $T \supset o_B$ in T^*N such that $\phi^1_{H_i} \equiv id$ on T for all i's.*

Then if $\lim_{i\to\infty} \mathrm{osc}_{C^0}(\phi^1_{H_i}; o_N) = 0$,

$$\lim_{i\to\infty}\left(\rho^{lag}(H_i;1) - \rho^{lag}(L_{H_i};[pt]^\#)\right) = 0.$$

The rest of the section is occupied by the proof of this theorem.

We fix a Riemannian metric g and the Levi-Civita connection on N. They naturally induces a metric on T^*N. Denote the latter metric on T^*N by $\widetilde{g}$ and the corresponding distance function by $d(x,y)$ for x, $y \in T^*N$. We denote by $D^r(T^*N)$ the disc bundle of T^*N of radius r.

The following is the well-known fact on this metric $\widetilde{g}$, which can be easily checked.

Lemma 8.2. *The metric $\widetilde{g}$ carries following properties:*

1. *$\widetilde{g}$ is invariant under the reflection $(q,p) \mapsto (q,-p)$ and in particular o_N is totally geodesic.*
2. *There exists a sufficiently small $r = r(N,g) > 0$ depending only on (N,g) such that the following triangle inequality holds: Let $x \in T^*N$ and denote $x = (q(x), p(x))$. Then*
$$d(o_{q(x)}, x) \geq \max\{|p(x)|, d(q, q(x))\} \geq |p(x)| \tag{8.1}$$
for all $x \in D^r(T^*N)$ where $|p(x)|$ is the norm on $T^*_{q(x)}N$.

We introduce a collection of the pairs (T, f) of a tubular neighborhood $T \supset o_B$ in T^*N and a Morse function $f \in C^\infty_{crit}(N; B, T)$ such that

(1) all of its critical points contained in $\operatorname{Int} B$,
(2) $\operatorname{Graph} df \subset D^r(T^*N)$ for $r = r(N, g)$ given in Lemma 8.2,
(3) $\operatorname{Graph}(df|_B) \subset T$.

Denote by $\mathcal{T}_B$ the set of all such pairs. We start with the following lemma

Lemma 8.3. *Let $H \in \mathcal{P}C^\infty_{asc}$ in T^*N such that*

$$\operatorname{supp} \phi_H \cap o_B = \emptyset, \tag{8.2}$$

and $\phi_H^1 \equiv id$ on a neighborhood $T \supset o_B$ in T^*N. Let $(T, f) \in \mathcal{T}_B$ be given such that H satisfies $\phi_H^1 \equiv id$, and

$$\operatorname{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N \setminus B, T)$$

where the constant $C_1^-(f; N \setminus B, T)$ is defined below in (8.3)*. Then we have*

$$L_f \cap o_N = \phi_H^1(L_f) \cap o_N$$

In particular all the Hamiltonian trajectories of $H\#(f \circ \pi)$, which have the form $z_p^{H\#(f\circ\pi)}$ for some $p \in L_f \cap o_N = \phi_H^1(L_f) \cap o_N$, are constant equal to p.

Proof. In the proof, we will denote $p \in N$ and the corresponding point in the zero section of T^*N by o_p for the notational consistency.

By the choice of the pair $(T, f) \in \mathcal{T}_B$, we have

$$\min\left\{\min_{p \in N \setminus B} |df(p)|,\ d_{\mathrm{H}}(N \setminus B, \operatorname{Crit} f)\right\} > 0.$$

where $d_{\mathrm{H}}(N \setminus B, \operatorname{Crit} f)$ is the Hausdorff distance. We define a positive constant

$$C_1^-(f; N \setminus B) := \min\left\{\min_{p \in N \setminus B} |df(p)|, d_{\mathrm{H}}(N \setminus B, \operatorname{Crit} f)\right\} \tag{8.3}$$

By definition of $C_1^-(f; N \setminus B, T)$, if $q \in N \setminus B$, we have

$$|df(q)|,\ d(q, \operatorname{Crit} f) \geq C_1^-(f; N \setminus B, T) > 0. \tag{8.4}$$

Obviously we have $\operatorname{Crit} f = L_f \cap o_B \subset \phi_H^1(L_f) \cap o_N$ since we assume $\phi_H^1 \equiv id$ on a neighborhood, T, of $o_B \supset \operatorname{Crit} f$.

We will now prove the opposite inclusion $\phi_H^1(L_f) \cap o_N \subset L_f \cap o_B$. Suppose $o_p \in \phi_H^1(L_f) \cap o_N$. Then we have $(\phi_H^1)^{-1}(o_p) \in L_f$.

Consider first the case $p \in B$. In this case since we assume $\phi_H^1 = id$ on a neighborhood of o_B, it in particular implies $o_p = (\phi_H^1)^{-1}(o_p)$ for all i and hence $o_p \in o_B \cap L_f \cong \operatorname{Crit} f$.

Now we will show that p cannot lie in $N \setminus B$. Suppose $p \in N \setminus B$ to the contrary and write

$$(\phi_H^1)^{-1}(o_p) = df(p')$$

for some $p' \in N$. Therefore

$$d(o_p, df(p')) = d(o_p, (\phi_H^1)^{-1}(o_p)) \leq \operatorname{osc}_{C^0}(\phi_H^1; o_N).$$

Furthermore we also have $|df(p')| \leq d(o_p, df(p'))$ by Lemma 8.2 since $\operatorname{Graph} df \subset D^r(T^*N)$. Therefore we have shown

$$|df(p')| \leq \operatorname{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N \setminus B, T). \tag{8.5}$$

This in particular implies $(\phi_H^1)^{-1}(o_p) = df(p')$ must lie in $\mathrm{Graph}\, df|_B \subset T$ for otherwise $|df(p')| \geq C_1^-(f; N \setminus B, T)$ by definition of $C_1^-(f; N \setminus B, T)$ which would contradict to (8.5).

This in turn implies $(\phi_H^1)^{-1}(o_p) \in T$. But ϕ_H^1 is assumed to be the identity map on T and hence follows

$$o_p = (\phi_H^1)^{-1}(o_p) = df(p').$$

In particular $df(p') \in o_N$ and so $p' \in \mathrm{Crit}\, f$ and hence $o_{p'} = df(p')$. This implies $p = p'$ and so $d(p, \mathrm{Crit}\, f) = 0$, i.e., $p \in \mathrm{Crit}\, f \subset B$, a contradiction to the hypothesis $p \in N \setminus B$.

Therefore p cannot lie in $N \setminus B$ and hence proves $o_p \in o_B \cap L_f \cong \mathrm{Crit}\, f$ for any $o_p \in \phi_H^1(L_f) \cap o_N$. This then finishes the proof of the first statement

$$L_f \cap o_N = \phi_H^1(L_f) \cap o_N. \tag{8.6}$$

To prove the second statement, we recall the definition

$$z_p^{H\#f\circ\pi}(t) = \phi_{H\#f\circ\pi}^t((\phi_{H\#f\circ\pi}^1)^{-1}(p))$$

and so $z_p^{H\#f\circ\pi}(1) = p$. But we have $df(p) = 0$ and $(\phi_H^1)^{-1}(o_p) = o_p$ since

$$p \in \phi_H^1(L_f) \cap o_N = L_f \cap o_N \subset o_B \cap \mathrm{Crit}\, f$$

and $\phi_H^1 \equiv id$ near p. Therefore

$$(\phi_{H\#f\circ\pi}^1)^{-1}(o_p) = (\phi_{f\circ\pi}^1)^{-1}(\phi_H^1)^{-1}(o_p) = o_p.$$

On the other hand $\phi_H^t \equiv id$ on a neighborhood $T_i' \supset o_B$ in T^*N since we assume $\mathrm{supp}\, \phi_H \cap o_B = \emptyset$. Therefore

$$\begin{aligned} z_p^{H\#f\circ\pi}(t) &= \phi_{H\#f\circ\pi}^t((\phi_{H\#f\circ\pi}^1)^{-1}(o_p)) = \phi_{H\#f\circ\pi}^t(o_p) \\ &= \phi_H^t(\phi_{f\circ\pi}^t(o_p)) = \phi_H^t(o_p) = o_p \end{aligned}$$

since $df(p) = 0$ and $\phi_H^t(o_p) = o_p$ for all $t \in [0,1]$. This finishes the proof. □

Remark 8.1. We would like to mention that in the above proof, the choice of the neighborhood T_i' may depend on i's and so may not be able to choose a uniform neighborhood T' independent of i's.

Motivated by the proof of this proposition, we introduce a collection, denoted by $C^\infty_{crit}(N; B, T) \subset C^\infty(N)$, of Morse functions f satisfying the condition in this lemma. We define the subset $C^\infty_{crit}(N; B) \subset C^\infty(N)$ to be the union

$$C^\infty_{crit}(N; B) = \bigcup_T C^\infty_{crit}(N; B, T).$$

It is easy to check that $C^\infty_{crit}(N; B, T) \neq \emptyset$ for any such $T \supset o_B$ by considering the λf for a sufficiently small $\lambda > 0$ for any given Morse function f with $\mathrm{Crit}\, f \subset \mathrm{Int}\, B$.

Lemma 8.4. *For any $f \in C^\infty_{crit}(N; B, T)$, the constant $C_1^-(f; N \setminus B, T)$ satisfies*

$$C_1^-(\lambda f; N \setminus B, T) = \min_{p \in N \setminus B} |d(\lambda f)(p)| \tag{8.7}$$

whenever λ is so small that

$$\min_{p \in N \setminus B} |d(\lambda f)(p)| < d_{\mathrm{H}}(N \setminus T, \mathrm{Crit}\, f).$$

In particular, we have

$$\lambda C_1^-(f; N \setminus B, T) = C_1^-(\lambda f; N \setminus B, T)$$

for such λ's.

Proof. First note that

$$\min_{p\in N\setminus B}|\lambda df(p)| = \lambda \min_{p\in N\setminus B}|df(p)| \to 0$$

as $\lambda \to 0$ but $d_{\rm H}(N\setminus B, \mathrm{Crit}(\lambda f))$ is independent of λ. Therefore the minimum in the definition

$$C_1^-(\lambda f; N\setminus B, T) = \min\left\{\min_{p\in N\setminus B}|d(\lambda f)(p)|, d_{\rm H}(N\setminus B, \mathrm{Crit}(\lambda f))\right\}$$

is realized by $\min_{p\in N\setminus B}|d(\lambda f)(p)|$ for all sufficiently small λ. Then the lemma follows. □

The following proposition is a crucial ingredient of the proof, which is a variation of Proposition 2.6 [Os], Proposition 3.3 [EP], Proposition 3.1 [U] and Proposition 2.3 [Sey].

Proposition 8.5. *Let $H \in \mathcal{PC}^\infty_{asc}$ in T^*N such that*

$$\mathrm{supp}\,\phi_H \cap o_B = \emptyset. \tag{8.8}$$

Take any $f \in C^\infty_{crit}(N;B)$ such that

$$\mathrm{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N\setminus B, T). \tag{8.9}$$

Then

$$\rho^{lag}(H;1) - \rho^{lag}(H;[pt]^\#) \le 2\,\mathrm{osc} f. \tag{8.10}$$

Proof. Denote $L_f := \mathrm{Graph}\, df$, $\quad L_t = \phi_H^t(L_f) = \phi_H^t(\mathrm{Graph}\, df)$. Note that the condition (8.8) implies

$$H_t|_B \equiv c_B(t) \tag{8.11}$$

for a function $c_B = c_B(t)$ depending only on t but not on $x \in B$.

The following lemma is the analogue of Lemma 5.1 [Os].

Lemma 8.6.

$$\rho^{lag}(H\#f;1) - \rho^{lag}(H\#f;[pt]^\#) \le \mathrm{osc} f. \tag{8.12}$$

Proof. By the spectrality of $\rho^{lag}(\cdot,1)$ in general, we have

$$\begin{aligned}\rho^{lag}(H\#f\circ\pi;1) &= \mathcal{A}^{cl}_{(H\#f\circ\pi)}\left(z^{H\#f\circ\pi}_{p_-}\right),\\ \rho^{lag}(H\#f\circ\pi;[pt]^\#) &= \mathcal{A}^{cl}_{(H\#f\circ\pi)}\left(z^{H\#f\circ\pi}_{p_+}\right)\end{aligned}$$

for some $p_\pm \in L_f \cap o_N$. Using the second statement of Lemma 8.3, we compute

$$\begin{aligned}&\mathcal{A}^{cl}_{(H\#f\circ\pi)}\left(z^{H\#f\circ\pi}_{p_+}\right) - \mathcal{A}^{cl}_{(H\#f\circ\pi)}\left(z^{H\#f\circ\pi}_{p_-}\right)\\ = &-\int_0^1 (H\#f\circ\pi)(t,p_+)\,dt + \int_0^1 (H\#f\circ\pi)(t,p_-)\,dt\\ = &-\int_0^1 c_B(t)\,dt - f(p_+) + \int_0^1 c_B(t)\,dt + f(p_-)\\ = &-f(p_+) + f(p_-) \le \max f - \min f = \mathrm{osc} f.\end{aligned}$$

Here for the equality in the line next to the last, we use the identity

$$(H\#f\circ\pi)(t,p_\pm) = H(t,p_\pm) + f(\phi_H^t(p_\pm)) = c_B(t) + f(p_\pm).$$

This finishes the proof. □

On the other hand, we have

$$\phi_H^1(L_f) = \phi_H^1(\phi_{f\circ\pi}^1(o_N)) = \phi_{H\# f\circ\pi}^1(o_N)$$

and so by the triangle inequality, Proposition 4.3,

$$\begin{aligned}\rho^{lag}(H\#(f\circ\pi);1) &\geq \rho^{lag}(H;1) - \rho^{lag}(-f\circ\pi;1)\\ \rho^{lag}(H\#(f\circ\pi);[pt]^\#) &\leq \rho^{lag}(H;[pt]^\#) + \rho^{lag}(f\circ\pi;1).\end{aligned}$$

(One can also use Proposition 4.4 using the concatenation $H * (f\circ\pi)$ instead. Here $f\circ\pi$ is not boundary flat, which is required in Proposition 4.4, but one can always reparameterize the flow $t\mapsto \phi_{f\circ\pi}^t$ by multiplying $\chi'(t)$ to $f\circ\pi$ so that the perturbation is as small as we want in $L^{(1,\infty)}$-topology which in turn perturbs ρ slightly. See Lemma 5.2 [Oh4], Remark 2.5 [MVZ] for the precise statement on this approximation procedure, or Appendix of the present paper. This enables us to apply the triangle inequality in Proposition 4.4 in the current context.)

Therefore subtracting the second inequality from the first and using the identity

$$\rho^{lag}(-f\circ\pi;1) = \max f, \quad \rho^{lag}(f\circ\pi;1) = -\min f$$

(see [Fl3], [Oh3] for its proof), we obtain

$$\begin{aligned}&\rho^{lag}(H\#(f\circ\pi);1) - \rho^{lag}(H\#(f\circ\pi);[pt]^\#)\\ \geq\ &\rho^{lag}(H;1) - \rho^{lag}(H;[pt]^\#) - (\max f - \min f)\end{aligned}$$

which in turn gives rise to

$$\begin{aligned}\rho^{lag}(H;1) - \rho^{lag}(H;[pt]^\#) &\leq \rho^{lag}(H\#(f\circ\pi);1) - \rho^{lag}(H\#(f\circ\pi);[pt]^\#)\\ &\quad +(\max f - \min f)\\ &\leq 2\,\mathrm{osc} f.\end{aligned}$$

We have finished the proof of the proposition. □

We now go back to the proof of Theorem 8.1.

Consider the elements H_i in the given sequence that satisfy (8.8). $\phi_{H_i}^1 \equiv id$ on a uniform $T \supset o_B$, and the oscillation $\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N)$ can be made arbitrarily small by letting $i\to\infty$.

If $\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) = 0$ for all sufficiently large i's, we have $\phi_{H_i}^1(o_N) = o_N$ and so $\rho^{lag}(H_i;1) - \rho^{lag}(H_i;[pt]^\#) = 0$ for which (8.10) obviously holds. Therefore we assume that there exists a subsequence, again denoted by H_i, such that $\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) \neq 0$.

Since $\mathrm{supp}\,\phi_{H_i} \cap o_B = \emptyset$ and $\phi_{H_i}^1 \equiv 0$ on T for all i, and $\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) \to 0$ as $i\to\infty$, we have

$$\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) < C_1^-(f; N\setminus B, T)$$

eventually. Recall from Lemma 8.3 that the choice of f depends only on the ball B and the neighborhood $T\subset B$ in T^*N. Then we choose $\lambda_i > 0$ such that

$$\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) = \lambda_i C_1^-(f; N\setminus B, T)$$

i.e.,

$$\lambda_i = \frac{\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N)}{C_1^-(f; N\setminus B, T)}.$$

Since $\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) \to 0$, $\lambda_i \to 0$. Obviously we have

$$\mathrm{osc}_{C^0}(\phi_{H_i}^1; o_N) < (\lambda_i + \varepsilon) C_1^-(f; N\setminus B, T)$$

for all $\varepsilon > 0$. Consider sufficiently large i's so that

$$\min_{p\in N\setminus B} |d(\lambda_i f)(p)| < d_{\rm H}(N\setminus B, \operatorname{Crit} f)$$

and hence

$$\lambda_i C_1^-(f; N\setminus B, T) = C_1^-(\lambda_i f; N\setminus B, T)$$

by Lemma 8.4.

Now we *fix* any such i. Lemma 8.4 also implies

$$(\lambda_i+\varepsilon) C_1^-(f; N\setminus B, T) = C_1^-((\lambda_i+\varepsilon) f; N\setminus B, T)$$

for all small $\varepsilon > 0$ such that

$$\min_{p\in N\setminus B} |(\lambda_i+\varepsilon) df(p)| < d(N\setminus B, \operatorname{Crit} f).$$

For example, we can choose any $\varepsilon > 0$ so that

$$0 < \varepsilon < \frac{d(N\setminus B, \operatorname{Crit} f)}{\min_{p\in N\setminus B}|df(p)|}. \tag{8.13}$$

Note that the upper bound does not depend on i's at all.

Since (8.10) holds for any pair H, f that satisfy (8.8) and (8.9), applying it to the pair $(H_i, (\lambda_i+\varepsilon)f)$ for $T\supset B$ chosen above independently of i's, we derive

$$\begin{aligned}\rho^{lag}(H_i;1)-\rho^{lag}(H_i;[pt]^\#) &\leq 2\operatorname{osc}((\lambda_i+\varepsilon)f) = 2(\lambda_i+\varepsilon)\operatorname{osc} f\\ &= 2\left(\frac{\operatorname{osc}_{C^0}(\phi^1_{H_i};o_N)}{C_1^-(f;N\setminus B,T)}+\varepsilon\right)\operatorname{osc} f.\end{aligned}$$

Since this holds for all $\varepsilon > 0$ satisfying (8.13), it follows

$$0\leq \rho^{lag}(H_i;1)-\rho^{lag}(H_i;[pt]^\#) \leq 2\left(\frac{\operatorname{osc} f}{C_1^-(f;N\setminus B,T)}\right)\operatorname{osc}_{C^0}(\phi^1_{H_i};o_N) \tag{8.14}$$

letting $\varepsilon\to 0$.

This inequality in particular finishes the proof of Theorem 8.1. □

The following upper bound of the spectral capacity involving the C^0-metric $\operatorname{osc}_{C^0}(\phi^1_H;o_N)$ has been obtained in the course of the above proof, which has some independent interest in its own right.

Theorem 8.7. *Let $B\subset N$ be a closed ball and $(T,f)\in\mathcal{T}_B$. Consider the set of Hamiltonians $H\in\mathcal{P}C^\infty_{asc,0}$ satisfying* $\operatorname{supp}\phi_H\cap o_B=\emptyset$ *and assume*

$$\operatorname{osc}_{C^0}(\phi^1_H;o_N) < C_1^-(f;N\setminus B,T).$$

Then we have

$$\frac{\rho^{lag}(H;1)-\rho^{lag}(H;[pt]^\#)}{\operatorname{osc}_{C^0}(\phi^1_H;o_N)}\leq\frac{2\operatorname{osc} f}{C_1^-(f;N\setminus B,T)}. \tag{8.15}$$

The following question seems to be an interesting question to ask in regard to the precise estimate of the upper bound in this theorem.

Question 8.2. For given H satisfying the condition in Theorem 8.7, what is an optimal estimate of the constant $\frac{2\operatorname{osc} f}{C_1^-(f;N\setminus B,T)}$ in terms of B, T and H? For example, can we obtain an upper bound independent of B or T?

9. Local Floer complex of engulfable topological Hamiltonian loop

We first recall the Lagrangian analogue of the Novikov ring $\Gamma_\omega = \Gamma(M, \omega)$ from [FOOO1]. Denote by $I_\omega : \pi_2(M, L) \to \mathbb{R}$ the evaluations of symplectic area. We also define another integer-valued homomorphism $I_\mu : \pi_2(M, L) \to \mathbb{Z}$ by

$$I_\mu(\beta) = \mu\left(w^*TM, (\partial w)^*TL\right)$$

which is the Maslov index of the bundle pair $(w^*TM, (\partial w)^*TL)$ for a (and so any) representative $w : (D^2, \partial D^2) \to (M, L)$ of β.

Definition 9.1. We define

$$\Gamma_{(\omega,L)} = \frac{\pi_2(M, L)}{\ker I_\omega \cap \ker I_\mu}.$$

and $\Lambda(L; \omega)$ to be the associated Novikov ring.

We briefly recall the basic properties on the Novikov ring $\Lambda_{(\omega,L)}(R)$ and its subring $\Lambda_{0,(\omega,L)}(R)$ where R is a commutative ring where R could be $\mathbb{Z}_2$, $\mathbb{Z}$ or $\mathbb{Q}$ for example. We put

$$q^\beta = T^{\omega(\beta)} e^{\mu_L(\beta)},$$

and

$$\deg(q^\beta) = \mu_L(\beta), \quad E(q^\beta) = \omega(\beta)$$

which makes $\Lambda_{(\omega,L)}$ and $\Lambda_{0,(\omega,L)}$ become a graded ring in general. We have the canonical valuation $\nu : \Lambda_{(\omega,L)} \to \mathbb{R}$ defined by

$$\nu\left(\sum_\beta a_\beta T^{\omega(\beta)} e^{\mu_L(\beta)}\right) = \min\{\omega(\beta) \mid a_\beta \neq 0\}$$

It induces a valuation on the subring $\Lambda_{0,(\omega,L)} \subset \Lambda_{(\omega,L)}$ which induces a natural filtration on it. This makes $\Lambda_{(\omega,L)}$ a filtered graded ring. For a general Lagrangian submanifold, this ring may not even be Noetherian but it is so if L is rational, i.e., $\Gamma(L; \omega)$ is discrete.

Next we recall the construction from [Oh1] of the local version of the Floer cohomology $HF(H; L, L)$ which singles out the contribution from the Floer trajectories whose images are contained in a given Darboux neighborhood U_L of L in M, provided

$$\phi_{H^t}(L) \subset V_L \subset \overline{V}_L \subset U_L \tag{9.1}$$

for all $t \in [0, 1]$. We will also show that $HF(H; L, L) \cong HF(F; L, L)$ provided there exists a family $\mathcal{H} = \{H(s)\}_{s\in[0,1]}$ such that

1. $H(0) = H, \quad H(1) = F$
2. The inclusions (9.1) hold for all $s \in [0, 1]$ for a family of neighborhoods $V_L(s) \subset \overline{V}_L(s) \subset U_L$.

This construction was introduced by Floer in [Fl2] in the Hamiltonian context which was further amplified in [Oh1] in the context of Lagrangian Floer homology. It is also proved in [Oh1] that this local contribution depends only on the pair (L, U_L) and so we can carry out its computation for the pair (o_L, V_L) where $V_L \subset T^*L$ is a neighborhood of the zero section $o_L \cong L$, *provided H is C^2-small (or ϕ_H^1 is C^1-small.* We refer to [Oh1] for the full details of construction thereof.

In this section, we recall the localization result from [Oh13] for *engulfable C^0-approximate loop* ϕ_H which replaces the C^1-smallness of ϕ_H in the construction of local Floer complex. Following the notations of [Oh13] we define

$$\mathcal{H}_\delta^{engulf}(L; V_L)$$

to be the set of $F : [0,1] \times M \to \mathbb{R}$ that satisfies

(1) it satisfies (9.1) for some pair of Darboux neighborhoods $V_L \subset U_L$ and
(2) $\overline{d}(\phi_F^1, id) \leq \delta$.

Then we define

$$\mathfrak{Iso}_\delta^{engulf}(L; V_L) = \{L' \in \mathfrak{Iso}(L) \mid L' = \phi_H^1(L),\, H \in \mathcal{H}_\delta^{engulf}(L; V_L)\}.$$

We consider

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H^1(L),\, v(\tau, 1) \in L. \end{cases} \tag{9.2}$$

Let $v : \mathbb{R} \times [0,1] \to M$ be a solution of (9.2) associated to H and J_0. The following theorem is proved in [Oh13]

Theorem 9.1 (Theorem 1.1 [Oh13])**.** *Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold and let $V_L \subset \overline{V}_L \subset U_L$ be a pair of Darboux neighborhoods of L. Consider a V_L-engulfable Hamiltonian path ϕ_H. Then whenever $\overline{d}(\phi_H^1, id) \leq \delta$ for any $\delta < d(V_L, \Theta)$, any solution of v of* (5.3) *satisfies one of the following alternatives:*

(1) *Either*
$$\operatorname{Image} v \subset D_\delta(L) \subset V_L \tag{9.3}$$
where $D_\delta(L)$ is the δ-neighborhood of L.
(2) *or* $\operatorname{Image} v \not\subset V_L$*. In this case, we also have $\int v^*\omega \geq C(J_0, V_L)$ where $C(J_0, V_L) > 0$ is a constant depending only on δ and V_L.*

Now consider a nondegenerate Hamiltonian H among those given in Theorem 9.1. Following [Che] we say that two elements of $\operatorname{Crit} \mathcal{A}_H^{cl}$ are said to be equivalent if they belong to the same connected component of the set

$$\pi^{-1}\left(\{\gamma \in \Omega(L, L) \mid \gamma([0,1]) \subset U_L\}\right) \subset \widetilde{\Omega}(L, L).$$

Then the projection $\pi : \widetilde{\Omega}(L, L) \to \Omega(L, L)$ bijectively maps each equivalence class of $\operatorname{Crit} \mathcal{A}_H^{cl}$ to $\mathcal{C}hord(L, L; H)$. There is a 'canonical equivalence class' represented by the pairs

$$[z, w_z]$$

where $z \in \mathcal{C}hord(L, L; H)$ and w_z is the (homotopically) unique cone-contraction of z to a point in L.

We denote this equivalence class by $\operatorname{Crit}^{can} \mathcal{A}_H^{cl} \subset \operatorname{Crit} \mathcal{A}_H^{cl}$. This induces the natural $\Gamma_{(\omega,L)}$-action on $\operatorname{Crit} \mathcal{A}_H^{cl}$ which induces the bijection

$$\operatorname{Crit}^{can} \mathcal{A}_H \times \Gamma_{(\omega,L)} \to \operatorname{Crit} \mathcal{A}_H.$$

We denote

$$\operatorname{Crit}^{[g]} \mathcal{A}_H = g \cdot \operatorname{Crit}^{can} \mathcal{A}_H, \quad g \in \Gamma_{(\omega,L)}.$$

With this notation, we have $\operatorname{Crit}^{[id]} \mathcal{A}_H = \operatorname{Crit}^{can} \mathcal{A}_H$. Then we denote their associated $\mathbb{Q}$ vector spaces by

$$CF_*^{[g]}((L, L), H; U_L), \quad CF_*^{[id]}((L, L), H; U_L) = CF_*^{can}((L, L), H; U_L).$$

We want to remark that $CF_*^{can}((L,L),H;U_L)$ is the one that was used in [Oh1] for the case of C^2-small cases.

The above discussion in turn gives rise to the isomorphism $CF_{[g]}((L,L),H;U_L))\otimes_R \Lambda_{(\omega,L)} \cong CF_*((L,L);H)$ as $\Lambda_{(\omega,L)}$-module for each $g \in \Lambda_{(\omega,L)}$.

Definition 9.2. The local Floer complex, denoted by $(CF_*^{[g]}((L,L),H;U_L),\partial_{(0)})$, of H in U_L associated to $g \in \Lambda_{(\omega,L)}$ is defined to be

$$CF_*^{[g]}((L,L),H;U_L) = \mathbb{Q}\cdot\{\mathrm{Crit}^{[g]}\,\mathcal{A}_H^{cl}\}, \quad \partial_{(0)}^{[g]} = \partial_{(0)}|_{CF_*^{[g]}((L,L),H;U_L)}$$

where $\partial_{(0)}$ is the contribution to ∂ arising from the thick-thin decomposition given below in Theorem 9.1.

Here we note that the Floer boundary map ∂ is $\Lambda_{(\omega,L)}$-equivariant and has the decomposition $\partial = \partial_{(0)} + \partial'$ so that

$$\widehat{g}\circ \partial_{(0)}|_{CF_*^{can}((L,L),H;U_L)} = \partial_{(0)}|_{CF_*^{[g]}((L,L),H;U_L)} \circ \widehat{g}$$

and $\widehat{g}$ carries a natural real grading given by

$$\mathcal{A}_F(g\cdot[z,w]) - \mathcal{A}_F(\cdot[z,w]),\ [z,w]\in \mathrm{Crit}\,\mathcal{A}_F$$

which does not depend on the choice of $[z,w]\in \mathrm{Crit}\,\mathcal{A}_F$. In fact this real grading is nothing but the value $\omega([g])$.

Then by definition the thin part of Floer moduli spaces for the pair $(\phi_H^1(L),L)$ does not bubble-off which then immediately proves the following [Fl3, Oh1, Che, Oh13]

Proposition 9.2. *Let $H \in \mathcal{H}_\delta^{engulf}(M;V_L)$ with $V_L \subset \overline{V}_L \subset U_L$. Then the local Floer homology*

$$HF_*^{[g]}((L,L),H;V_L) = \ker \partial_{(0)}^{[g]}/\operatorname{im}\partial_{(0)}^{[g]}$$

is well-defined.

In [Oh13], we prove the following theorem.

Theorem 9.3 (Theorem 7.2 [Oh13])**.** *Let F be an V_L-engulfable Hamiltonian with $F = H(1)$ for a family $\mathcal{H} = \{H(s)\}_{0\le s\le 1} \subset \mathcal{H}_\delta^{engulf}(L;V_L)$ with $H(0)=0$. Then*

$$HF^{can}(F,L;J';U_L) \cong H_*(L;\mathbb{Z})$$

for any J' sufficiently close to J_0 in C^∞-topology.

10. Lagrangianization of Hamiltonian Floer complex

For each given generic one-periodic $J=\{J_t\}$, the Hamiltonian Floer complex $(CF_*(F),\partial_{(F,J)})$ is defined by considering the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_F(u)\right) = 0 \tag{10.1}$$

and define a boundary map $\partial_{(F,J)} : CF_*(F) \to CF_{*-1}(F)$ by studying the moduli space of solutions of (10.1).

We will assume that ϕ_F is sufficiently close to the identity path in hamiltonian topology, i.e., we assume

$$d_{ham}(\phi_F, id) < \delta$$

for some small constant $\delta > 0$ depending only on (M,ω) in the next section. The precise size of $\delta > 0$ will be determined later. *In particular the Hamiltonian F will be engulfable.*

In this section, we consider Example 3.1 in the Darboux chart $(\mathcal{U}, -d\Theta)$ for a sufficiently C^0-small Hamiltonian paths ϕ_F for a mean-normalized engulfable. Hamiltonian $F : [0,1] \times M \to \mathbb{R}$. Put a density ρ_Δ on $\Delta \subset M \times M$ induced by ω^n by the diffeomorphism of the first projection $\Delta \to M$.

We fix a Darboux neighborhoods

$$V_\Delta \subset \overline{V}_\Delta \subset U_\Delta$$

and let $\omega \oplus -\omega = -d\Theta$ on U_Δ regarded as a neighborhood of the zero section of $T^*\Delta$ once and for all. We measure the size of U_Δ by the following constant

$$C(U_\Delta, \Theta) = \max_{x \in \mathcal{U}} |p(x)|. \tag{10.2}$$

Then if we choose $\delta > 0$ sufficiently small depending only on (M,ω) and $(U_\Delta, -d\Theta)$, then

$$\operatorname{Graph} \phi_F^t \subset V_\Delta \quad \text{for all } t \in [0,1].$$

We define a Hamiltonian $\mathbb{F}$ by

$$\mathbb{F}(t,(x,y)) = F(t,x)$$

on $T^*\Delta$. This itself is not supported in U_Δ but we can multiply a cut-off function χ of U_Δ so that

$$\chi \equiv 1 \quad \text{on } V_\Delta, \quad \operatorname{supp} \chi \subset U_\Delta$$

and consider $\chi(x,y)\mathbb{F}(t,(x,y))$ so that the associated Hamiltonian deformations of $\psi^t(o_N)$ are unchanged. We note that $\mathbb{F}$ *is compactly supported in $T^*\Delta$.*

We now construct a canonical filtration preserving one-one correspondence between the local Hamiltonian Floer complex of $\lambda = \phi_F$ and that of the local Lagrangian Floer complex pair $(o_\Delta, \operatorname{Graph}(\lambda))$, *provided* $\operatorname{Graph}(\phi_F^t)$ *are all supported in* V_Δ and so the local Lagrangian Floer complexes $CF(\operatorname{Graph}(\lambda), o_\Delta; U_\Delta)$ are defined. A complete discussion on such correspondences are given in section 4.2 [Oh4], section 8.4 [Oh13] for the case *whenever the graph of the image of Floer trajectory is contained in* V_Δ, to which we refer readers for the detailed explanations.

Therefore we will focus on the discussion on the relationship between the associated action functionals for the two cases.

When δ is sufficiently small, any 1-periodic trajectory z of F carries a canonical bounding disc obtained by taking the cone of the loop from its center of mass whose graph in $M \times M$ is contained in V_Δ. We denote by $\mathcal{L}_0(M)$ the set of contractible loops and by $\widetilde{\mathcal{L}}_0(M)$ the associated Novikov covering space consisting of the pairs $[\gamma, w_\gamma^{can}]$ with $\gamma \in \mathcal{L}_0(M)$ and $w : D^2 \to M$ satisfying $w|_{\partial D^2} = \gamma$.

Then for each given element $[\gamma, w] \in \widetilde{\mathcal{L}}_0(M)$, we consider the pair

$$\Gamma(t) = (z(t), z(0)), \quad W(z) = (w(z), w(0,0))$$

where we identity $z = se^{2\pi\sqrt{-1}t}$. We recall $w(0,t) \equiv x_0$ is a point in M and so $W : (D^2, \partial D^2) \to M \times M$ defines a well-defined map satisfying the boundary condition

$$W(0,t) \equiv (w(0,0), w(0,0)),\ W(s,0),\ W(s,1) \in \Delta,\ W(1,t) = \Gamma(t).$$

This map extends to a C^0-neighborhood $\mathcal{U} := \mathcal{U}(V_\Delta)$ of constant paths in M to a C^0-neighborhood, which we denote by $\Omega_0(\Delta, \Delta; V_\Delta)$ of the constant paths $t \mapsto (x, x)$ which in fact defines a one-one correspondence. We denote this map by

$$\Phi : \mathcal{U}(V_\Delta) \to \Omega_0(\Delta, \Delta; V_\Delta).$$

Now we consider the action functionals $\mathcal{A}_{F\oplus 0} : \Omega_0(F \oplus 0, \Delta; V_\Delta) \to \mathbb{R}$ defined by

$$\mathcal{A}_{F\oplus 0}([\Gamma, W]) = -\int W^*(\omega \oplus (-\omega)) - \int_0^1 (F \oplus 0)(\Gamma(t))\, dt. \tag{10.3}$$

By definition, one can check the identity

$$\mathcal{A}_F([\gamma, w]) = (\mathcal{A}_{F\oplus 0} \circ \Phi)([\gamma, w]). \tag{10.4}$$

We use this functional $\mathcal{A}_{F\oplus 0}$ for the construction of the Lagrangian spectral invariants

$$\rho^{lag}_{V_\Delta}(F \oplus 0; 1_0)$$

in the local Floer complex on $V_\Delta \supset \Delta$. We denote by $\rho^{ham}_{\mathcal{U}}(F; 1_0)$ the associated local spectral invariant of F on $\mathcal{U} \subset \widetilde{\mathcal{L}}_0(M)$ as in [Oh13], where $\mathcal{U}$ is the set of short loops such that their graphs are contained in the Darboux neighborhood $V_\Delta \subset M \times M$.

Once we establish this correspondence of the action functionals precisely, the discussion on the local Floer homology carried out in the previous section and (10.4) immediately give rise to

Proposition 10.1. *Consider Example 3.1 in the Darboux chart $(V_\Delta, -d\Theta)$. Denote by $\rho^{ham}_{\mathcal{U}}(F; 1_0)$ the spectral invariant corresponding to $1 \in H^*(M)$ in the local Floer complex. Let $\rho^{lag}_{V_\delta}(F \oplus 0; 1)$ be the (global) Lagrangian spectral invariant on $T^*\Delta$ defined in section 4. Then we have*

$$\rho^{ham}_{\mathcal{U}}(F; 1_0) = \rho^{lag}_{\Delta_V}(F \oplus 0; 1).$$

Proof. It remains to prove the second equality. By Stokes' formula, we obtain

$$\begin{aligned} \mathcal{A}_{F\oplus 0}([\Gamma, W]) &= -\int W^*(\omega \oplus (-\omega)) - \int_0^1 (F \oplus 0)\, dt \\ &= \int \Gamma^*\Theta - \int_0^1 (F \oplus 0)(t, \Gamma(t))\, dt \end{aligned}$$

where the right hand side is nothing but the classical action functional

$$\mathcal{A}^{cl}_{F\oplus 0}(\Gamma)$$

on $\Omega_0(o_\Delta, o_\Delta)$, which was used to define the (global) Lagrangian spectral invariant $\rho^{lag}(F \oplus 0; 1)$ on the cotangent bundle $T^*\Delta$ in section 4. This finishes the proof. □

The following result is also proved in [Oh13]

Theorem 10.2 (Theorem 1.5 [Oh13])**.** *Fix an open neighborhood $V \subset T^*L$ of $o_L \subset T^*L$ that is J_0-convex. Let $\mathcal{H} = \{H(s)\}$ be an engulfable isotopy with $H(0) = 0$ and $H(1) = F$. Then for any $F \in \mathcal{H}^{engulf}_\delta(M; V)$,*

$$\rho^{lag}_V(F; 1_0) = \rho^{lag}(F; 1).$$

An immediate corollary of Proposition 10.1 and Theorem 10.2 is the equality

Corollary 10.3. *Let F be as in Proposition 10.1. Then*

$$\rho^{ham}_{\mathcal{U}}(F; 1_0) = \rho^{lag}(F \oplus 0; 1).$$

11. Lagrangianization of smooth Hamiltonian homotopy

The following is the main theorem of this section.

Theorem 11.1. *Assume* $\dim M = 2$. *Let H be a engulfable hamiltonian homotopy of contractible topological Hamiltonian loop ϕ_F contained in $\mathcal{P}^{ham}(Sympeo_U(M,\omega))$ with $U = M \setminus B$ that satisfies*

$$H(0) = H(0,t,x) \equiv 0, \quad H(1) = F.$$

Consider an approximating sequence H_i in $\mathcal{P}^{ham}(Symp_U(M,\omega))$ such that

$$H_i(0,t,x) \equiv H_i(s,0,x) \equiv 0. \tag{11.1}$$

Put a density ρ_Δ on $\Delta \subset M \times M$ induced by the $2n$-form $i_\Delta^*(\omega^n \oplus 0)$ where $i_\Delta : \Delta \to M \times M$ is the diagonal embedding. Then

$$\lim_{i\to\infty} \int_\Delta \widetilde{h}_{(\underline{\mathbb{H}_i}(1))}(\mathbf{q})\, \rho_\Delta = 0, \quad \mathbf{q} = (q,q) \tag{11.2}$$

where $\underline{\mathbb{H}_i} = \underline{H_i} \oplus 0$, i.e., $\underline{\mathbb{H}_i}(s,\mathbf{x}) = \underline{H_i}(s,x)$ for $\mathbf{x} = (x,y)$.

Proof. In the beginning of the proof, we analyze the integral $\int_\Delta \widetilde{h}_{\underline{\mathbb{H}_i}}(1)\, \rho_\Delta$ in general and so drop the subindex i from $\mathbb{H}_i$ and do computations for arbitrary smooth $\mathbb{H}$ associated to engulfable Hamiltonian $H = H(s,t,\mathbf{x})$, which enables us to do computations on a Darboux-Weinstein neighborhood V_Δ of the diagonal $\Delta \subset M \times M$. *We do not assume the two dimensionality until at the last moment of the proof, which we will mention when we need it.* At the end, we will apply the computations to the given approximating sequence of topological hamiltonian homotopy of contractible topological Hamiltonian loop.

Recall $H \equiv 0$ on a neighborhood of B by definition of $\mathcal{P}^{ham}(Symp_U(M,\omega))$ with $U = M \setminus B$. We denote by $K = K(s,t,x)$ a s-Hamiltonian of the 2-parameter family $\Lambda = \left\{\phi^t_{H(s)}\right\}$ with $K(s,0,\cdot) \equiv 0$: The latter choice is possible we have the s-Hamiltonian flow $s \mapsto \phi^0_{H(s)} \equiv id$ and so we can set $K(s,0,\cdot) \equiv 0$.

We first prove a few lemmata.

The following lemma immediately follows from the same calculation done in [Oh5]. For readers' convenience, we give its complete proof.

Lemma 11.2. *$K \equiv 0$ on a neighborhood of $B \subset M$.*

Proof. We recall the identity

$$\frac{\partial K}{\partial t} = \frac{\partial H}{\partial s} - \{K, H\}.$$

Recall $H(s,t,\mathbf{x}) \equiv 0$ on a neighborhood of B because we assume that H is compactly supported in $U = M \setminus B$ by definition. From this, it follows $\frac{\partial K}{\partial t} \equiv 0$ thereon. Together with the initial condition $K(s,0,\cdot) \equiv 0$, this proves $K(s,0,x) \equiv 0$ for all x in a neighborhood of B. □

This in particular implies $\phi_{K^1} \in \mathcal{P}^{ham}(Symp_U(M,\omega), id)$. Next we have the following coincidence of the Calabi invariant.

Lemma 11.3.
$$\mathrm{Cal}_U(K^1) = \mathrm{Cal}_U(H(1))$$

Proof. First note $\phi^1_{K^1} = \phi^1_{H(1)}$. Denote by $\Lambda(s,t) = \phi^t_{H(s)}$ the two-parameter family associated to H. Then
$$\Lambda(0,t) \equiv id \equiv \Lambda(s,0)$$
by (11.1). Therefore the Hamiltonian path $t \mapsto \phi^t_{H(1)} := \Lambda(1,t)$ is smoothly homotopic to the path $s \mapsto \phi^s_{K^1} := \Lambda(s,1)$ relative to the ends and hence we have the lemma by the smooth homotopy invariance of Cal_U: In fact, an explicit homotopy $\Upsilon : [0,1]^2 \to Symp_U(M,\omega)$ between them is given by the formula
$$\Upsilon(s,t) = \begin{cases} \Lambda(t, 1+2s(t-1)) & \text{for } 0 \le s \le \frac{1}{2} \\ \Lambda\left(2(s-1/2)+2t(1-s), t\right) & \text{for } \frac{1}{2} \le s \le 1. \end{cases}$$
The map Υ satisfies
$$\begin{aligned} \Upsilon(0,t) &= \Lambda(t,1) = \phi^t_{K^1}, \quad \Upsilon(1,t) = \phi^t_{H(1)}, \\ \Upsilon(s,0) &= id, \quad \Upsilon(s,1) = \Lambda(1,1) = \phi^1_{H(1)} = \phi^1_F \end{aligned}$$
and hence is the required homotopy relative to the ends. □

Lemma 11.4. $\widetilde{h}_{\mathbb{K}^1} = \widetilde{h}_{\mathbb{H}(1)}$,

Proof. We apply the first variation formula (3.2) to $z^{\mathbf{q}}_{\mathbb{K}^1}(s)$ and $z^{\mathbf{q}}_{\mathbb{H}(1)}(t)$ respectively, and obtain
$$\begin{aligned} d\widetilde{h}_{\mathbb{K}^1}(v) = \langle \Theta(\phi^1_{\mathbb{K}^1}(\mathbf{q})), T\phi^1_{K^1}(v)\rangle \\ d\widetilde{h}_{\mathbb{H}^1}(v) = \langle \Theta(\phi^1_{\mathbb{H}(1)}(\mathbf{q})), T\phi^1_{\mathbb{H}(1)}(v)\rangle \end{aligned}$$
for any given $v \in T_{\mathbf{q}}\Delta$. Since $\phi^1_{\mathbb{K}^1} = \phi^1_{\mathbb{H}(1)}$, we have proved $d\widetilde{h}_{\mathbb{K}^1} = d\widetilde{h}_{\mathbb{H}(1)}$. On the other hand, for any point $\mathbf{q} \in \Delta_B$, $\mathbb{H} \equiv 0 \equiv \mathbb{K}^1$ on a neighborhood of $\mathbf{q}$ in $T^*\Delta$ and so both $z^{\mathbf{q}}_{\mathbb{K}^1}$ and $z^{\mathbf{q}}_{\mathbb{H}(1)}$ are constant. Therefore the values of both $\widetilde{h}_{\mathbb{K}^1}$ and $\widetilde{h}_{\mathbb{H}(1)}$ are zero at such a point $\mathbf{q} \in \Delta_B$. This finishes the proof. □

Then we also have
$$\widetilde{h}_{\underline{\mathbb{K}}^1} = \widetilde{h}_{\mathbb{K}^1} + \mathrm{Cal}_U(K^1) = \widetilde{h}_{\mathbb{H}(1)} + \mathrm{Cal}_U(H(1)) = \widetilde{h}_{\underline{\mathbb{H}}(1)}$$
With this preparation, in our proof, we will use $\mathbb{K}^1$ instead of $\mathbb{H}(1)$ in our proof. This is because we will exploit the fact that the s-Hamiltonian flow of K^1 is C^0-small for the approximating sequence of hamiltonian homotopy of (contractible) topological Hamiltonian loop. Note that the t-Hamiltonian flow of $\mathbb{H}(1)$ will not be small in general.

From now on we suppress the underline from $\underline{\mathbb{K}}^1 = \underline{K}^1 \oplus 0$ and just denote $\mathbb{K}^1$ or K^1 assuming K^1 is already normalized. We first recall the definition
$$\widetilde{h}_{\mathbb{K}^1}(\mathbf{q}) = \mathcal{A}^{cl}_{\mathbb{K}^1}\left(z^{\mathbf{q}}_{\mathbb{K}^1}\right), \mathbf{q} \in o_\Delta$$
and $z^{\mathbf{q}}_{\mathbb{K}^1}(s) = \phi^s_{\mathbb{K}^1}(\mathbf{q}) = \phi^1_{\mathbb{H}(s)}(\mathbf{q})$, which we remind the readers will let as C^0-close to the identity as we want later. Recalling the definitions $\widetilde{h}_{\mathbb{K}^1}$, we carefully analyze

the integral

$$\begin{aligned}\int_\Delta \widetilde{h}_{\mathbb{K}^1}(\mathbf{q})\,\rho_\Delta &= \int_\Delta \mathcal{A}^{cl}_{\mathbb{K}^1}\left(z^{\mathbf{q}}_{\mathbb{K}^1}\right)\,\rho_\Delta \\ &= \int_\Delta \left(\int (z^{\mathbf{q}}_{\mathbb{K}^1})^*\Theta - \int_0^1 \mathbb{K}^1(s, z^{\mathbf{q}}_{\mathbb{K}^1}(s))\,ds\right)\rho_\Delta. \qquad (11.3)\end{aligned}$$

Using the normalization, the definition $z^{\mathbf{q}}_{\mathbb{K}^1}(s) = \phi^s_{\mathbb{K}^1}(\mathbf{q})$, and the symplectic property of $\phi^s_{K^1}$, the integral of the Hamiltonian term $\mathbb{K}^1$ vanishes for each $s \in [0,1]$ after interchanging the order of integration, and so does the whole double integral.

Now we apply the above discussion to the sequence of s-Hamiltonians K_i^1 of the 2-parameter families $H_i = H_i(s,t,x)$ associated to the given hamiltonian homotopy H and have reduced the proof of the theorem to the proof of

$$\int_\Delta \left(\int (z^{\mathbf{q}}_{\mathbb{K}^1_i})^*\Theta\right)\rho_\Delta \to 0$$

as $i \to \infty$. This then is a special case of the following general theorem below since $\overline{d}(\phi_{K_i^1}, id) \to 0$, for which we assume two dimensionality in this paper.

This concludes the proof of Theorem 11.1. □

The remaining section will be occupied by the proof of the following theorem.

Remark 11.1. We would like to emphasize that this vanishing result is a consequence of the interplay on the diagonal $\Delta = \Delta_M$ between the particular choice of density ρ_Δ on o_Δ induced by $i^*_\Delta(\omega \oplus 0)$ and the particular form of the Hamiltonian vector field $X_{\mathbb{G}} = X_G \oplus \vec{0}$ on $T^*\Delta$. We do not expect this kind of vanishing result to hold on a general cotangent bundle T^*N for a general density ρ on the zero section o_N and for a general Hamiltonian vector field on T^*N. Furthermore, because this theorem also involves the Liouville one-form Θ on $T^*\Delta$ which does not reside in the original symplectic manifold (M,ω), it is not clear, at least at the moment of writing this paper, whether there is an interpretation of this theorem purely in terms of the original Hamiltonian vector field X_G on M itself.

Theorem 11.5. *Assume* $\dim M = 2$. *Suppose* $G_i : [0,1]\times M \to \mathbb{R}$ *be a sequence of any engulfable Hamiltonian on* M *such that* $\overline{d}(\phi_{G_i}, id) \to 0$. *Let* V_Δ *be an associated Darboux-Weinstein neighborhood of the diagonal* $\Delta \subset M \times M$ *with* $-d\Theta = \omega \oplus -\omega$. *Let* ρ_Δ *be a density given as in Theorem 11.1. We define the function* $\mathbb{G} : [0,1] \times M \times M \to \mathbb{R}$ *by* $\mathbb{G}(s,\mathbf{x}) = G(s,x)$ *for* $\mathbf{x} = (x,y)$. *Then*

$$\lim_{i\to\infty}\int_\Delta \int (z^{\mathbf{q}}_{\mathbb{G}_i})^*\Theta\,\rho_\Delta = \lim_{i\to\infty}\int_\Delta\left(\int_0^1 \left\langle \Theta(\phi^s_{\mathbb{G}}(\mathbf{q}))), X_{\mathbb{G}_i}(s, \phi^s_{\mathbb{G}_i}(\mathbf{q}))\right\rangle ds\right)\rho_\Delta = 0.$$

Proof. In the beginning of the proof, we analyze the integral $\int_\Delta \int (z^{\mathbf{q}}_{\mathbb{G}_i})^*\Theta\,\rho_\Delta$ in general and so drop the subindex i from G_i and do calculations for arbitrary smooth engulfable Hamiltonian $G = G(s,x)$ and its associated $\mathbb{G}$. The main consequences of these calculations are Proposition 11.6, Lemma 11.7 and Proposition 11.10. At the end, we will apply the calculations to the given sequence G_i satisfying $\overline{d}(\phi_{G_i}, id) \to 0$.

For the proof, we first consider the standard Lagrangian suspension construction associated to the s-Hamiltonian G, which is given by the map

$$\Psi : [0,1] \times M \to T^*\Delta \times T^*[0,1]$$

defined by

$$\Psi(s,x) = ((\phi_G^s(x), x), s, -G(t, \phi_G^s(x))) = (\phi_{\mathbb{G}}^s(\mathbf{x}), s, -\mathbb{G}(t, \phi_{\mathbb{G}}^s(\mathbf{x}))) . \tag{11.4}$$

We denote by $\mathrm{pr} : V_\Delta \times T^*[0,1] \to V_\Delta$ the projection to V_Δ and $\pi_i : M \times M \to M$ the i-th projection for $i = 1, 2$. Then we denote

$$\mathrm{pr}_i = \pi_i \circ \mathrm{pr}.$$

It is well-known that the map Ψ defines a Lagrangian embedding of $[0,1] \times M$ into $T^*\Delta \times T^*[0,1]$ with respect to the symplectic form

$$\mathrm{pr}^*(-d\Theta) + ds \wedge db$$

on $V_\Delta \times T^*[0,1] \subset T^*\Delta \times T^*[0,1]$.

Remark 11.2. Strictly speaking, we should also denote the second sum as the pull-back of the form by the corresponding projection to $T^*[0,1]$. This should not cause any confusion, while we feel necessary to make everything clear for the first summand $\mathrm{pr}^*(-d\Theta)$. We hope that this somewhat excessive cautious measure really helps readers following the computations henceforth without confusion, which we ourselves have experienced quite a bit. However this being already alerted, we will omit the projections from notations, unless we feel absolutely necessary to avoid possible confusion.

We note that the symplectic form can be also written as

$$\mathrm{pr}^*(\omega \oplus -\omega) + ds \wedge db$$

on $V_\Delta \times T^*[0,1]$ regarding it as a subset of $M \times M \times T^*[0,1]$. We denote

$$\Omega_1 = \mathrm{pr}_1^*\omega, \quad \Omega_2 = \mathrm{pr}_2^*\omega$$

as a two-form on $V_\Delta \times T^*[0,1]$, and then we have

$$\mathrm{pr}^*(-d\Theta) = \Omega_1 - \Omega_2$$

on $V_\Delta \times T^*[0,1]$.

We also note that

$$\Psi^*(\mathrm{pr}^*(\omega \oplus -\omega) + ds \wedge db) = \Psi^*(\mathrm{pr}^*(-d\Theta) + ds \wedge db) = 0$$

since Ψ is a Lagrangian embedding. In other words,

$$\Psi^*(db \wedge ds) = \Psi^*\mathrm{pr}^*(-d\Theta) = \Psi^*(\Omega_1 - \Omega_2) \tag{11.5}$$

on $[0,1] \times M$ and each restriction at each fixed time s

$$\Psi_s = \Psi \circ i_s : M \to V_\Delta \subset M \times M; \quad \Psi_s(x) = (\psi_G^s(x), x)$$

is a Lagrangian embedding for each $s \in [0,1]$, where

$$i_s : M \times M \to \{s\} \times M \times M \hookrightarrow [0,1] \times M \times M,$$

the obvious embedding. We denote its associated Lagrangian submanifold by

$$R_\Psi = \mathrm{Im}\,\Psi \subset V_\Delta \times T^*[0,1].$$

We consider the integral

$$\int_\Delta \left(\int_0^1 \langle \Theta(\phi_{\mathbb{G}}^s(\mathbf{q})), X_{\mathbb{G}}(s, \phi_{\mathbb{G}}^s(\mathbf{q})) \rangle \, dt \right) \rho_\Delta \tag{11.6}$$

and note ρ_Δ is the measure induced by the form $i_\Delta^*(\omega^n \oplus 0) = i_\Delta^*(\Omega_1^n)$. Recalling the definition of ρ_Δ, a straightforward computation shows

$$\begin{aligned} & \int_\Delta \left(\int_0^1 \langle \Theta(\phi_{\mathbb{G}}^s(\mathbf{q}))), X_{\mathbb{G}}(s, \phi_{\mathbb{G}}^s(\mathbf{q}))\rangle \; ds \right) \rho_\Delta \\ = & \int_\Delta \left(\int_0^1 \langle \Theta(\phi_{\mathbb{G}}^s(\mathbf{q}), X_{\mathbb{G}}(s, \phi_{\mathbb{G}}^s(\mathbf{q}))\rangle \; ds \right) i_\Delta^*(\Omega_1^n) \\ = & \int_{R_\Psi} \langle \Theta(\mathbf{x}), X_{\mathbb{G}}(s, \mathbf{x})\rangle \, ds \wedge \Omega_1^n \end{aligned}$$

where $\mathbf{x} = (x, y)$ and the last integral is the integration of $2n+1$-form

$$\langle \Theta(\mathbf{x}), X_{\mathbb{G}}(s, \mathbf{x})\rangle \; ds \wedge \Omega_1^n$$

(defined on $V_\Delta \times T^*[0,1] \subset (M \times M) \times T^*[0,1]$) against the $(2n+1)$-dimensional submanifold $R_\Psi \subset V_\Delta \times T^*[0,1]$.

We decompose $\Theta = \eta' + \eta$ where η' is a section of $\Lambda^1(\pi_1^* TM)$ and η is that of $\Lambda^1(\pi_2^* TM)$ for the projections $\pi_i : M \times M \to M$ for $i = 1, 2$. To make the type of η, η' look conspicuous, we also denote $\eta = 0 \oplus \eta$ and $\eta' = \eta' \oplus 0$ in calculations below, when we feel convenient. Then

$$\begin{aligned} \langle \Theta(\mathbf{x}), X_{\mathbb{G}}(s, \mathbf{x})\rangle \; ds \wedge \Omega_1^n &= \langle \Theta(\mathbf{x}), X_{\mathbb{G}}(s, \mathbf{x})\rangle \; ds \wedge (\omega^n \oplus 0) \\ &= (X_G \rfloor \eta') \, ds \wedge (\omega^n \oplus 0) \\ &= \eta' \wedge ((ndG \wedge ds \wedge (\omega^{n-1} \oplus 0)) \\ &= n \, \eta' \wedge dG \wedge ds \wedge \Omega_1^{n-1}. \end{aligned}$$

Here we use the vanishing $\eta' \wedge \Omega_1^n = (\eta' \oplus 0) \wedge (\omega^n \oplus 0) = 0$ for the second equality. We would like to remark that we have been doing our calculations on $M \times M \times T^*[0,1]$, not restricted to $\mathrm{Graph}\, \Psi$ so far.

Therefore

$$\begin{aligned} & \langle \Theta(\phi_{\mathbb{G}}^s(\mathbf{q}), X_{\mathbb{G}}(s, \phi_{\mathbb{G}}^s(\mathbf{q}))\rangle \; ds \wedge \Omega_1^n \\ = \;& \Psi^* \left(\langle \Theta, X_{\mathbb{G}} \rangle \, ds \wedge \Omega_1^n \right) = \Psi^*(\eta' \wedge db \wedge ds \wedge \Omega_1^{n-1}). \end{aligned} \tag{11.7}$$

Then, exploiting the Lagrangian property (11.5), we derive

$$\begin{aligned} \Psi^*(\eta' \wedge db \wedge ds \wedge \Omega_1^{n-1}) &= \Psi^*(\eta' \wedge (\Omega_1 - \Omega_2) \wedge \Omega_1^{n-1}) \\ &= \Psi^*((\eta' \wedge \Omega_1^n) - \Psi^* \left(\eta' \wedge \Omega_1^{n-1} \wedge \Omega_2 \right) \\ &= -\Psi^* \left(\eta' \wedge \Omega_1^{n-1} \wedge \Omega_2 \right). \end{aligned} \tag{11.8}$$

Here we use the vanishing $\eta' \wedge \Omega_1^n = 0$ again.

We summarize the above calculations into the following crucial proposition, whose validity strongly depends on our particular choice of density ρ_Δ on Δ and on the particular form of Hamiltonian vector field $X_{\mathbb{G}} = X_G \oplus 0$.

Proposition 11.6. *Let $G = G(t, x)$ be any engulfable Hamiltonian and $\mathbb{G} = G \oplus 0$. Denote by Ψ the Lagrangian suspension of the Lagrangian isotopy $\Psi : [0,1] \times M \to V_\Delta \times T^*[0,1]$ defined as above, and let $R_\Psi = \mathrm{Im}\, \Psi$ be the associated Lagrangian* *submanifold. Then*

$$\begin{aligned}\int_\Delta \left(\int (z^{\mathbf{q}}_{\mathbb{G}})^*\Theta\right) \rho_\Delta &= \int_{[0,1]\times M} \Psi^*\left(-\eta' \wedge \Omega_1^{n-1}\wedge \Omega_2\right)\\ &= \int_{R_\Psi} -\eta' \wedge \Omega_1^{n-1}\wedge\Omega_2.\end{aligned}$$

It remains to examine the contribution of this last integral (or equivalently the next to the last). For this purpose, we need to study the structure of the Liouville one-form Θ of the $V_\Delta \subset T^*\Delta$ more closely in a Darboux neighborhood of each point of (M,ω). By choosing Darboux coordinates $(q_1,\cdots,q_n,p_1,\cdots,p_n)$ of (M,ω) at each given point $x \in M$, we first examine the case of $\mathbb{R}^{2n}$ with $\omega_0 = \sum_{i=1}^n dq_i\wedge dp_i$.

Remark 11.3. Weinstein [W1] previously used such a uniform choice of Darboux charts varying smoothly on the point of M and called such a family a *Darboux family.* This is precisely what we use in the following discussion. In regard to our considering engulfable Hamiltonians G, the pair of points $(\phi_G^s(x),x)$ for all $s\in[0,1]$ will be assumed to be contained in V_Δ and the points $\phi_G^s(x)$ contained in the given Darboux chart of x. This choice of family enable us to perform the following calculations on $\mathbb{R}^{2n}$, allowing a universal constant depending only on the family. For readers' convenience, we provide some systematic discussion on the framework of Darboux family and its associated exponential map.

Now we fix a Darboux family $\Phi = \{(U_x,\Phi_x)\}_{x\in M}$ such that each chart (U_x,Φ_x) is centered at x, i.e., $\Phi_x(x) = \vec{0}\in\mathbb{R}^{2n}$. Then we fix the Darboux-Weinstein neighborhood V_Δ of Δ in $M\times M$ that we have been using so that

$$V_\Delta \subset \bigcup_{y\in M} U_y\times\{y\}\subset M\times M \tag{11.9}$$

and there exists some $\delta_\Phi>0$ independent of $y\in M$ such that

$$V_\Delta \supset U_y\times\{y\}\supset \Phi_y^{-1}(B^{2n}(\delta_\Phi))\times\{y\} \tag{11.10}$$

for all $y\in M$ for the given Darboux family Φ. (See Appendix 13.1 for more discussion on the Darboux family.) This is possible because Φ is smooth and M is compact. By construction, the diameter of $\Phi_y(U_y)$ is not be bigger than

$$\mathrm{Lip}(\Phi_y)\cdot \mathrm{diam}\, U_y$$

where $\mathrm{Lip}(\Phi_y)$ is the Lipschitz constant of the map Φ_y with respect to any given compatible metric on (M,ω) and the standard metric on $\mathbb{R}^{2n}$. We then define

$$C_\Phi = \sup_{y\in M}\mathrm{Lip}(\Phi_y) < \infty \tag{11.11}$$

whose finiteness follows from the smoothness of the family Φ and compactness of M. From now on, we will consider only the V_Δ-engulfable family of Hamiltonians for such V_Δ and

$$\overline{d}(\phi_G, id) < \frac{\delta_\Phi}{C_\Phi}$$

so that $d(\Phi_y(\phi_G(y)),\Phi_y(y)) = |\Phi_y(\phi_G(y))|\le\delta_\Phi$ for all y. Here $d(\Phi_y(\phi_G(y)),\Phi_y(y))$ is the distance and $|\Phi_y(\phi_G(y))|$ is the norm with respect to the standard metric $\mathbb{R}^{2n}$. (Recall $\Phi_y(y)=0$.)

We denote by $(Q_1, \cdots, Q_n, P_1, \cdots, P_n)$ the canonical coordinates of the first factor and by $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ the one for the second of $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ so that

$$\omega \oplus (-\omega) = \sum_{i=1}^n dQ_i \wedge dP_i - \sum_{i=1}^n dq_i \wedge dp_i.$$

In this coordinates, the diagonal is given by the condition

$$q_i = Q_i,\, p_i = P_i,\, i = 1, \cdots, n.$$

Therefore a canonical one form Θ of $T^*\Delta$ satisfying $-d\Theta = \omega \oplus (-\omega)$ and $\Theta \equiv 0$ on Δ is given by

$$\Theta = \sum_{i=1}^n (q_i - Q_i) d\left(\frac{P_i + p_i}{2}\right) + (P_i - p_i) d\left(\frac{Q_i + q_i}{2}\right).$$

Then we obtain

$$\begin{aligned} \eta &= \sum_{i=1}^n \frac{1}{2}(q_i - Q_i)\, dp_i + \frac{1}{2}(P_i - p_i) dq_i \\ \eta' &= \sum_{i=1}^n \frac{1}{2}(q_i - Q_i)\, dP_i + \frac{1}{2}(P_i - p_i) dQ_i. \end{aligned}$$

And in this coordinates, we have $\mathbb{G}(s, \mathbf{x}) = G(s, (\vec{Q}, \vec{P}))$ with $\vec{Q} = (Q_1, \cdots, Q_n)$, $\vec{P} = (P_1, \cdots, P_n)$. In the presentation following, we will do not distinguish the point $x \in M$ and its coordinates as in this expression.

11.1. **Two dimensional case.** To make the following calculations more transparent, let us consider the case $n = 1$, i.e., the case of two dimensional surface in this paper. (The case of general dimension will be treated in [Oh14] in preparation.)

In this case, there is no summation and hence (11.8) becomes

$$\Psi^*(-\eta' \wedge \Omega_2)$$

Therefore when $n = 1$, we have converted the integral (11.6) into

$$\int_{[0,1]\times M} \Psi^*(-\eta' \wedge \Omega_2).$$

Recall $\Omega_2 = \mathrm{pr}_2^*\omega$ and

$$\eta' = \frac{1}{2}(q - Q)\, dP + \frac{1}{2}(P - p) dQ.$$

Now we consider the twisted projection $\mathrm{pr}_G = \mathrm{pr}_1 \circ \Psi : [0,1] \times M \to M$ defined by

$$\mathrm{pr}_G(s, (Q, P)) = \phi_G^s(Q, P). \tag{11.12}$$

We note that the M-coordinate travels along the Hamiltonian trajectory $z_G^{(Q,P)}$ in our notation, i.e., the one with its initial point given by (Q, P). We also remark that the point in $R_\Psi = \mathrm{Im}\,\Psi$ for its (Q, P)-coordinate given by $z_G^{(Q,P)}(s)$ has its (q, p)-coordinate fixed at

$$(q, p) = (Q, P)$$

which is nothing but the initial point of the trajectory $z_G^{(Q,P)}$.

Now we foliate $[0,1] \times M$ into the union of space-time trajectories

$$[0,1] \times M = \bigcup_{(Q,P) \in M} Z(Q,P)$$

where the subset $Z(Q,P)$ is defined by

$$Z(Q,P) = \bigcup_{s \in [0,1]} \left\{ \left(s, z_G^{(Q,P)}(s) \right) \right\}.$$

We remark that this is the projection of the curve $\Psi_{(Q,P)} : [0,1] \to R_\Psi$ of R_Ψ given by

$$\Psi_{(Q,P)}(s) = \Psi \circ i_{(Q,P)}(s) = \left(z_G^{(Q,P)}(s), (Q,P), s, -G(s, z_G^{(Q,P)}(s)) \right) \tag{11.13}$$

where $i_{(Q,P)} : [0,1] \to [0,1] \times M$ is the inclusion map $i_{(Q,P)}(s) = (s, (Q,P))$. We call the curve $\Psi_{(Q,P)}$ the *characteristic curve* of R_Ψ issued at (Q,P). Then we also have the foliation of R_Ψ into

$$R_\Psi = \bigcup_{(Q,P) \in M} \operatorname{Im} \Psi_{(Q,P)}.$$

Denote by $\pi_M : [0,1] \times M \to M$ the projection. Then we transform the above integral using the integration over the fiber, which is

$$\int_{R_\Psi} -\eta' \wedge \Omega_2 = \int_M (\pi_M)_! \Psi^*(-\eta' \wedge \Omega_2).$$

We note that the fiber of π_M is nothing but the curve $Z(Q,P)$ whose initial point is given by the second projection $(q,p) = (Q,P)$, which is fixed. The following push-forward formula is standard. For reader's convenience, we include its proof in Appendix.

Lemma 11.7.

$$(\pi_M)_! \Psi^*(\eta' \wedge \Omega_2) = \left(\int_{\Psi_{(Q,P)}} \eta' \right) \omega.$$

The relevant push-forward formula for the high dimensional case and its application to the proof below are more subtle and requires more sophisticated treatment whose discussion we postpone in [Oh14].

This lemma now reduces the integral to

$$\int_{R_\Psi} -\eta' \wedge \Omega_2 = \int_M (\pi_M)_! \Psi^*(-\eta' \wedge \Omega_2) = \int_M \left(\int_{\Psi_{(Q,P)}} -\eta' \right) \omega. \tag{11.14}$$

It remains to evaluate the line integral $\int_{\Psi_{(Q,P)}} \eta'$.

To properly relate our study of the above integral to the C^0-closeness of G or ϕ_G *without involving derivatives in the final estimates*, we proceed as follows. Recall that the image (q,p) of the second projection is fixed along the characteristic curve $\Psi_{(Q,P)}$ at $(q,p) = (Q,P)$ associated to each trajectory $z_G^{(Q,P)}$. We also recall that we consider engulfable family satisfying (11.9) and so the image of the path $s \mapsto z_G^{(Q,P)}(s)$ is contained in the Darboux chart $U_{(Q,P)}$ at (Q,P) in the given Darboux family Φ. In particular, we can connect its initial and final points by a short line segment from the final point $\phi_G^1(Q,P)$ to the initial point (Q,P) and close it up. We denote by $\ell_G^{(Q,P)}$ the inverse image of the line segment under the chart $\Phi_{(Q,P)}$, and denote the loop obtained by concatenating $z_G^{(Q,P)}$ and $\ell_G^{(Q,P)}$ in M by $w_G^{(Q,P)}$. By construction, the diameter of $\Phi_{(Q,P)} \circ w_G^{(Q,P)} \subset \mathbb{R}^{2n}$ is not be bigger than

$$\mathrm{Lip}(\Phi_{(Q,P)}) \cdot \overline{d}(\phi_G, id) \leq C_\Phi \overline{d}(\phi_G, id)$$

for all $(Q,P) \in M$. We consider the family of (locally defined) one-forms on M

$$\alpha_{(q,p)} = -\frac{1}{2}\left((q - Q)\, dP + (P - p) dQ\right)$$

for each fixed (q,p). We note

$$d\alpha_{(q,p)} = -\omega \tag{11.15}$$

on a given Darboux chart at (q,p) for all $(q,p) \in M$.

Then the one-form $(\mathrm{pr} \circ \Psi)^* \eta'$ is nothing but α with substitutions of (Q,P) into (q,p) and $\phi_G^s(Q,P)$ into (Q,P). Therefore we derive

$$\int_{\Psi_{(Q,P)}} \eta' = \int_{z_G^{(Q,P)}} \alpha_{(Q,P)} = \int_{z_G^{(Q,P)}} \alpha_{(Q,P)} = \int_{w_G^{(Q,P)}} \alpha_{(Q,P)} - \int_{\ell_G^{(Q,P)}} \alpha_{(Q,P)}. \tag{11.16}$$

We now evaluate each summand in this sum separately. We start with evaluating the second term

Lemma 11.8.

$$\left| \int_{\ell_G^{(Q,P)}} \alpha_{(Q,P)} \right| \leq C_\Phi^2 d_{C^0}(\phi_G^1, id)^2$$

where $C_\Phi > 0$ is the constant given in (11.11), *which depends only on* (M, ω) *and the Darboux family* Φ.

Proof. We parameterize $\ell_G^{(Q,P)}$ by the standard linear interpolation

$$\begin{aligned} \ell_G^{(Q,P)}(s) &= \Phi_{(Q,P)}^{-1}\left((1-s)\Phi_{(Q,P)}(\phi_G^1(Q,P)) + s\Phi_{(Q,P)}(Q,P)\right) \\ &= \Phi_{(Q,P)}^{-1}\left((1-s)\Phi_{(Q,P)}(\phi_G^1(Q,P))\right). \end{aligned}$$

Note that the speed of the curve $\Phi_{(Q,P)} \circ \ell_G^{(Q,P)}$ is given by $|\Phi_{(Q,P)}(\phi_G^1(Q,P))|$ in the flat metric. Moreover, we also have the upper bound of the norm of α at (Q,P)

$$\left|\alpha_{(Q,P)}\right|_{z_G^{(Q,P)}} \leq |\Phi_{(Q,P)}(\phi_G^1(Q,P))|$$

again in the flat metric. Therefore we obtain

$$\left| \int_{\ell_G^{(Q,P)}} \alpha_{(Q,P)} \right| \leq \int_0^1 |\Phi_{(Q,P)}(\phi_G^1(Q,P))|^2 \, dt \leq C_\Phi^2 \overline{d}(\phi_G^1, id)^2$$

This finishes the proof. ∎

To estimate the first integral of (11.16), we use the following lemma. This is a key lemma, together with Lemma 11.8, in that converting the line integral into the symplectic area and homology invariance of the symplectic area enables us to use C^0-smallness of the boundary curve in terms of the geometric area of the small disc of radius not bigger than $\overline{d}(\phi_G, id)$. We would like to recall readers that *the length of a C^0-small curve could have very large geometric length.*

We first note that since $w_G^{(Q,P)}$ is C^0-small, it is contractible and so carries the unique homotopy class of the bounding disc $\psi_G^{(Q,P)} : D^2 \to M$ with $\partial \psi_G^{(Q,P)} = w_G^{(Q,P)}$.

Lemma 11.9. *Let* $\psi_G^{(Q,P)}$ *be as above with* $\partial\psi_G^{(Q,P)} = w_G^{(Q,P)}$. *Then*

$$\int_{w_G^{(Q,P)}} -\alpha_{(Q,P)} = \int_{\psi_G^{(Q,P)}} \omega.$$

In particular, we have

$$\left|\int_{w_G^{(Q,P)}} \alpha_{(Q,P)}\right| \leq \text{Area}\, D_G^{(Q,P)} = \pi(C_\Phi \overline{d}(\phi_G, id))^2. \tag{11.17}$$

where $D_G^{(Q,P)} \subset \Phi_{(Q,P)}(U_{(Q,P)}) \subset \mathbb{R}^2$ *is the disc centered at* $(q,p) = (Q,P)$ *with radius* $C_\Phi \overline{d}(\phi_G, id)$.

Proof. Since (q,p) is fixed, we obtain

$$d\alpha_{(q,p)} = -dQ \wedge dP = -\omega.$$

Then by Stokes' formula, we obtain

$$\int_{w_G^{(Q,P)}} -\alpha_{(Q,P)} = \int_{\psi_G^{(Q,P)}} \omega.$$

This proves the first equality.

For the inequality, we first note that $\text{Im}\,\psi_G^{(Q,P)} \subset D_G^{(Q,P)}$ by definition of $D_G^{(Q,P)}$ and the fact that $d_{\text{H}}\left((q,p), \text{Im}\,\psi_G^{(Q,P)}\right) \leq \overline{d}(\phi_G, id)$. Then (11.17) follows from the fact that the two form ω has comass 1 with respect to the flat metric. □

Now combining Lemma 11.8 and 11.9, we have obtained the following proposition.

Proposition 11.10.

$$\left|\int_{z_G^{(Q,P)}} \alpha_{(Q,P)}\right| \leq (1+\pi) C_\Phi^2 (\overline{d}(\phi_G, id))^2.$$

Finally combining this with (11.14), we derive the following general inequality, which is of independent interest:

$$\left|\int_{R_\Psi} (-\eta' \wedge \Omega_2)\right| \leq (1+\pi) C_\Phi^2 \overline{d}(\phi_G, id))^2\, \text{Area}(M). \tag{11.18}$$

Remark 11.4. Here we would like to emphasize that when we do integration along the fiber, it is crucial to define pr_G as

$$\text{pr}_G = \text{pr}_1 \circ \Psi,$$

especially using the first projection, not the second, by two reasons:

(1) It makes the corresponding coordinate of the second factor (q,p) become constant.
(2) The projection $z_G^{(Q,P)}(s)$ has the form

$$z_G^{(Q,P)}(s) = \phi_G^s(Q,P)$$

which is the symplectic image of the point (Q,P).

Since we do not have any control of the derivative of ϕ_{G_i} for the approximating sequence, the above whole setting-up is crucial in our proof in that the derivative terms of the flow or the Hamiltonian do not appear in the inequality in Proposition 11.10.

Finally, we apply the main inequality (11.18) to the given sequence G_i with $\overline{d}(\phi_{G_i}, id) \to 0$ and completes the proof of Theorem 11.5 for the two dimensional case. $\square$

12. Homotopy invariance of spectral invariants

We recall that the Hamiltonian spectral invariant $\rho^{ham}(H;a)$ of the Hamiltonian is unambiguously defined, whether H is normalized or not. To associate a spectral invariant of the Hamiltonian path $\lambda = \phi_H$, we recall that the spectral invariant of the path λ is defined to be

$$\rho^{ham}(\lambda; a) = \rho^{ham}(\underline{H}; a).$$

as given in (1.19) for $a = 1$.

In this section, we prove Theorem 1.1 for an $\mathcal{U}$-engulfable topological Hamiltonian loop ϕ_F hamiltonian homotopic to the constant identity path, where $\mathcal{U} = \mathcal{U}(V_\Delta)$. Denote by $\underline{F}$ the normalization of F. The main goal of this section is to show $\rho^{ham}(\underline{F}; 1) = 0$. Here we continue our discussion starting at section 2.

We rewrite

$$\rho^{ham}(\underline{F}; 1) = \rho^{ham}_{\mathcal{U}}(\underline{F}; 1_0) + \big(\rho^{ham}(\underline{F}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F}; 1_0)\big)$$

as in section 2 and divide this section into two subsections containing the proofs of $\rho^{ham}_{\mathcal{U}}(\underline{F}; 1_0) = 0$ and $\rho^{ham}(\underline{F}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F}; 1_0) = 0$ respectively.

12.1. **Vanishing of local invariant.** As in the previous section, we fix a Darboux neighborhood $V_\Delta \subset \overline{V}_\Delta \subset U_\Delta$ and a cut-off function $\chi : M \times M \to \mathbb{R}$ so that $\chi \equiv 1$ on V_Δ and $\operatorname{supp} \chi \subset U_\Delta$. Consider the pull-back of the Hamiltonian $H_i(s,t,x)$ to $U_\Delta \subset M \times M$ under the first projection $M \times M \to M$. This itself is not supported in U_Δ but we can multiply a cut-off function χ of $U_\Delta \supset o_\Delta$ and consider

$$\mathbb{H}((s,t,(x,y)) = (\chi(H \oplus 0))(x,t,(x,y)) = \chi(x,y)H(s,t,x) \tag{12.1}$$

for any given Hamiltonian H on M.

By a slight abuse of notation, for the simplicity of notations, we denote

$$\underline{\mathbb{H}}(s,t,(x,y)) = (\chi(\underline{H} \oplus 0))(x,t,(x,y)) = \chi(x,y)\underline{H}(s,t,x). \tag{12.2}$$

We would like to emphasize that the Hamiltonian deformations of $\phi^t_{\mathbb{H}_i(s)}(o_\Delta)$ and their Hamiltonians are unchanged on the union

$$\bigcup_{s\in[0,1]} \phi^t_{\mathbb{H}_i(s)}(o_\Delta)$$

when we cut-off by multiplying χ. Obviously $\phi_{\mathbb{H}_i(s)} \equiv id$ on a tubular neighborhood of o_{Δ_B} in $T^*\Delta$ if $H_i(s) \equiv c(s,t)$ on B for a function $c = c(s,t)$ depending only on (s,t).

Here we recall the equality

$$\rho^{ham}_{\mathcal{U}}(\underline{H_i}; 1) = \rho^{lag}_{V_\Delta}(\underline{\mathbb{H}_i}; 1) = \rho^{lag}(\underline{\mathbb{H}_i}; 1)$$

from Corollary 10.3. Therefore this subsection will be devoted to the proof of

$$\rho^{lag}(\underline{\mathbb{H}}; 1) = \lim_{i\to\infty} \rho^{lag}(\underline{\mathbb{H}_i}; 1) = 0 \tag{12.3}$$

whose proof entirely uses the study of Lagrangian spectral invariants developed in sections 3-6 and section 11.

The Hamiltonian $\mathbb{H}_i(1)$ belongs to $\mathcal{PC}^\infty_{(\Delta_B;e_i)}$ introduced in Definition 4.4 with the constants

$$e_i = -\int_0^1 c_i(t)\,dt \tag{12.4}$$

and satisfy

$$\phi^t_{\mathbb{H}_i(1)}(o_\Delta) \in \mathfrak{Iso}_{o_{\Delta_B}}(o_\Delta, T^*\Delta).$$

Furthermore since $(\operatorname{supp}\Lambda_i) \cap B = \emptyset$, it follows

$$\phi^1_{\mathbb{H}_i(s)}(o_\Delta) \cap o_\Delta \supset o_{\Delta_B}$$

for all $s \in [0,1]$, which is a requirement needed in Definition 4.4. Therefore Proposition 4.5 gives rise to

$$\rho^{(\Delta_B;e_i)}(L_i;1) = \rho^{lag}(\mathbb{H}_i(1);1), \quad L_i = \phi^1_{\mathbb{H}_i(1)}(o_\Delta)$$

given in Definition 4.5 at the end of section 4. We also note

$$\rho^{lag}(\underline{\mathbb{H}_i}(1);1) = \rho^{lag}(\mathbb{H}_i(1);1) + \mathrm{Cal}_U(H_i(1)).$$

Since $\overline{d}(\phi^1_{H_i(s)}, id) \to 0$ as $i \to \infty$ uniformly over $s \in [0,1]$, we have

$$\lim_{i\to\infty} \mathrm{osc}_{C^0}\left(\phi^1_{\mathbb{H}_i(s)}; o_\Delta\right) = 0$$

uniformly over s, where we recall the definition

$$\mathrm{osc}_{C^0}(\phi^1_{\mathbb{H}_i(s)}; o_\Delta) = \max\left\{d_{\mathrm{H}}\left(\phi^1_{\mathbb{H}_i(s)}(o_\Delta), o_\Delta\right), d_{\mathrm{H}}\left(\left(\phi^1_{\mathbb{H}_i(s)}\right)^{-1}(o_\Delta), o_\Delta\right)\right\}$$

from (1.26) applied to $H = \mathbb{H}_i(1)$ and $N = \Delta$.

Therefore Theorem 8.1, applied to the Hamiltonians $\underline{\mathbb{H}_i}(1)$, implies

$$\lim_{i\to\infty}\left(\rho^{lag}(\underline{\mathbb{H}_i}(1);1) - \rho^{lag}(\underline{\mathbb{H}_i}(1);[pt]^\#\right) = 0. \tag{12.5}$$

Now we improve this vanishing result to the following

Proposition 12.1. *Assume* $\dim M = 2$. *We have*

$$\lim_{i\to\infty} \rho^{lag}(\underline{\mathbb{H}_i}(1);1) = 0.$$

Or more succinctly,

$$\rho^{lag}(\underline{\mathbb{H}}(1);1) = 0.$$

Proof. We start with the following crucial lemma. This is the lemma that bridges between the measure theory and the study of symplectic topology in a fundamental way. A measure theoretic consideration involving our particular choice of density ρ_Δ and the measurable map $\varphi^{\underline{\mathbb{H}_i}(1)}$ with its uniform C^0-convergence to the identity (almost everywhere) enables us to reduce its proof to the vanishing result for the sequence $\widetilde{h}_{\underline{\mathbb{H}_i}}$ (Theorem 11.1). In this paper, we will restrict ourselves to the two dimensional case.

We recall the definition (6.6) of the basic phase function for the topological Hamiltonian in general.

Lemma 12.2. *Assume* $\dim M = 2$. *Consider the basic phase function* $f_{\underline{\mathbb{H}_i}(1)}$ *of* $\underline{\mathbb{H}_i}(1)$. *Put a density* ρ_Δ *as in Theorem 11.1 induced by the form* $\omega^n \oplus 0$. *Then*

$$\int_\Delta f_{\underline{\mathbb{H}}(1)}\,\rho_\Delta = \lim_{i\to\infty}\int_\Delta f_{\underline{\mathbb{H}_i}}(1)\,\rho_\Delta = 0.$$

Proof. We first recall the vanishing result, Theorem 11.1, which is

$$\lim_{i\to\infty}\int_\Delta \widetilde{h}_{\underline{\mathbb{H}_i}(1)}(\mathbf{q})\,\rho_\Delta = 0. \tag{12.6}$$

(We would like to note that unlike the case of $f_{\underline{\mathbb{H}_i}}(1)$, $\widetilde{h}_{\underline{\mathbb{H}_i}(1)}$ may not converge for any approximating sequence H_i of a given topological Hamiltonian H, even in L^1. Therefore the expression like $\widetilde{h}_{\underline{\mathbb{H}}(1)}$ or $\int \widetilde{h}_{\underline{\mathbb{H}}(1)}\,\rho_\Delta$ will not make sense at all for the limit topological Hamiltonian $\underline{\mathbb{H}}(1)$.)

We derive $f_{\underline{\mathbb{H}_i}(1)} \to f_{\underline{\mathbb{H}}(1)}$ in C^0 from the convergence $H_i \to H$ in $L^{(1,\infty)}$-topology by Corollary 6.5. Therefore $f_{\underline{\mathbb{H}_i}(1)}$ have the uniform bound

$$-E^+(\underline{H}) - \frac{1}{2} \le f_{\underline{\mathbb{H}_i}(1)} \le E^-(\underline{H}) + \frac{1}{2} \tag{12.7}$$

for all sufficiently large i's.

We note that $\varphi^{\underline{\mathbb{H}_i}(1)}$ is one-one but not onto M in general. With a slight abuse of notation, we denote by $\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)^{-1}$ the corresponding inverse map whose domain is the subset

$$U(\varphi^{\underline{\mathbb{H}_i}(1)}) := \operatorname{Im}\varphi^{\underline{\mathbb{H}_i}(1)} \subset \Delta$$

and whose image is the full domain Δ. Since $\overline{d}(\phi^1_{H_i(1)}; id) \to 0$, $\operatorname{osc}_{C^0}(\phi_{\mathbb{H}_i} \oplus id, \Delta) \to 0$ and so

$$\overline{d}(\varphi^{\underline{\mathbb{H}_i}(1)}, id) \to 0$$

(in the L^∞-sense on Δ). Since $\varphi^{\underline{\mathbb{H}_i}(1)}$ is ρ_Δ-measurable, this implies the weak convergence $(\varphi^{\underline{\mathbb{H}_i}(1)})_*\rho_\Delta \to \rho_\Delta$ of measure as $i \to \infty$, which is equivalent to the convergence

$$\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)^{-1}_*\left(\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right) \to \rho_\Delta$$

on Δ in weak topology of measures. (We would like to note that the push-forward family $\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)^{-1}_*\left(\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right)$ is a sequence of measures defined on Δ.) In particular, we have

$$\rho_\Delta\left(\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})\right) \to 0. \tag{12.8}$$

Here we denote by $\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}$ the restriction of the measure ρ_Δ to $U(\varphi^{\underline{\mathbb{H}_i}(1)})$.

Then combining these convergences with the boundedness (12.7) of $f_{\underline{\mathbb{H}_i}(1)}$, we derive

$$\lim_{i\to\infty}\int_\Delta f_{\underline{\mathbb{H}_i}(1)}\,\rho_\Delta = \lim_{i\to\infty}\int_\Delta f_{\underline{\mathbb{H}_i}(1)}\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)^{-1}_*\left(\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right).$$

Now recalling the identity $f_{\underline{\mathbb{H}_i}(1)} = \widetilde{h}_{\underline{\mathbb{H}_i}(1)} \circ \varphi^{\underline{\mathbb{H}_i}(1)}$ from (6.8), we compute

$$\begin{aligned}
&\int_\Delta f_{\underline{\mathbb{H}_i}(1)}\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)^{-1}_*\left(\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right)\\
= &\int_\Delta \left(\widetilde{h}_{\underline{\mathbb{H}_i}(1)} \circ \varphi^{\underline{\mathbb{H}_i}(1)}\right)\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)^{-1}_*\left(\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right)\\
= &\int_{U(\varphi^{\underline{\mathbb{H}_i}(1)})} \widetilde{h}_{\underline{\mathbb{H}_i}(1)}\left(\rho_\Delta|_{U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right)\\
= &\int_\Delta \widetilde{h}_{\underline{\mathbb{H}_i}(1)}\,\rho_\Delta - \int_{\Delta\setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})} \widetilde{h}_{\underline{\mathbb{H}_i}(1)}\left(\rho_\Delta|_{\Delta\setminus U\left(\varphi^{\underline{\mathbb{H}_i}(1)}\right)}\right).
\end{aligned} \tag{12.9}$$

By (12.6), the first integral converges to 0 as $i \to \infty$.

It remains to show the second integral also converges to 0. For this purpose, we recall the definition of $\widetilde{h}_{\underline{\mathbb{H}_i}(1)}$

$$\widetilde{h}_{\underline{\mathbb{H}_i}(1)}(\mathbf{q}) = \int \left(z^{\mathbf{q}}_{\underline{\mathbb{H}_i}(1)}\right)^* \Theta - \int_0^1 \underline{\mathbb{H}_i}(1)\left(t, z^{\mathbf{q}}_{\underline{\mathbb{H}_i}(1)}(t)\right)\, dt$$

and $\|\underline{H_i}\| \leq 2\|\underline{H}\|$ for all sufficiently large i's since $H_i \to H$ in $L^{(1,\infty)}$. In particular

$$\left|\int_0^1 \underline{\mathbb{H}_i}(1)\left(t, z^{\mathbf{q}}_{\underline{\mathbb{H}_i}(1)}(t)\right)\, dt\right| \leq 2\|\underline{H}\|$$

for all $\mathbf{q} \in \Delta$, and hence

$$\begin{aligned} &\left|\int_{\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})} \int_0^1 \underline{\mathbb{H}_i}(1)\left(t, z^{\mathbf{q}}_{\underline{\mathbb{H}_i}(1)}(t)\right)\, dt\, \rho_\Delta\right| \\ \leq\ & 2\|\underline{H}\| \cdot \rho_\Delta\left(\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)}\right) = 2\|\underline{H}\| \cdot \mathrm{vol}_\omega\left(M \setminus \pi_1(U(\varphi^{\underline{\mathbb{H}_i}(1)}))\right). \end{aligned} \tag{12.10}$$

On the other hand, in 2 dimension, both Proposition 11.10 and the pushforward formula in Lemma 11.7 apply to any V_Δ-engulfable Hamiltonian and so to $\mathbb{G} = \underline{\mathbb{H}_i}(1)$. Therefore we obtain the upper bound

$$\begin{aligned} &\left|\int_{\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})} \int (z^{\mathbf{q}}_{\underline{\mathbb{H}_i}(1)})^*\Theta\ \left(\rho_\Delta|_{\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right)\right| \\ =\ & \left|\int_{M \setminus \pi_1(U(\varphi^{\underline{\mathbb{H}_i}(1)}))} \left(\int_{\Psi_{(Q,P)}} \eta'\right) \omega^n\right| \\ \leq\ & \int_{M \setminus \pi_1(U(\varphi^{\underline{\mathbb{H}_i}(1)}))} \left|\int_{\Psi_{(Q,P)}} \eta'\right| \omega^n \\ =\ & (1+\pi)C_\Phi^2 \overline{d}(\phi_{H_i(1)}, id))\, \mathrm{vol}_\omega\left(M \setminus \pi_1\left(U(\varphi^{\underline{\mathbb{H}_i}(1)})\right)\right) \\ \leq\ & (1+\pi)C_\Phi^2 \,\mathrm{diam}\, M\, \mathrm{vol}_\omega\left(M \setminus \pi_1\left(U(\varphi^{\underline{\mathbb{H}_i}(1)})\right)\right). \end{aligned} \tag{12.11}$$

Combining (12.10) and (12.11), we have obtained

$$\begin{aligned} &\left|\int_{\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})} \widetilde{h}_{\underline{\mathbb{H}_i}(1)}\ \left(\rho_\Delta|_{\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)})}\right)\right| \\ \leq\ & \left(2\|\underline{H}\| + (1+\pi)C_\Phi^2 \,\mathrm{diam}\, M\right) \mathrm{vol}_\omega\left(M \setminus \pi_1\left(U(\varphi^{\underline{\mathbb{H}_i}(1)})\right)\right). \end{aligned}$$

Since $\mathrm{vol}_\omega(M \setminus \pi_1(U(\varphi^{\underline{\mathbb{H}_i}(1)})) = \rho_\Delta(\Delta \setminus U(\varphi^{\underline{\mathbb{H}_i}(1)}) \to 0$, this proves the second integral of (12.9) also converges to 0.

This finishes the proof of Lemma 12.2. □

In particular, for any given $\varepsilon > 0$ we have $\max f_{\underline{\mathbb{H}_i}(1)} \geq -\varepsilon$ for all sufficiently large i because

$$\max f_{\underline{\mathbb{H}_i}(1)} \geq \frac{1}{\mathrm{vol}(\rho_\Delta)} \int_\Delta f_{\underline{\mathbb{H}_i}(1)}\, \rho_\Delta.$$

Since Theorem 7.1 implies $\rho^{lag}(\underline{\mathbb{H}_i}(1); 1) \geq \max f_{\underline{\mathbb{H}_i}(1)}$, we have obtained

$$\rho^{lag}(\underline{\mathbb{H}_i}(1); 1) \geq -\varepsilon.$$

Since this holds the case for any $\varepsilon > 0$, we have obtained

$$\lim_{i\to\infty} \rho^{lag}(\underline{\mathbb{H}_i}(1); 1) \geq 0 \tag{12.12}$$

By replacing the role of λ_i and λ_i^{-1} in the above proof, we also derive

$$\lim_{i\to\infty} \rho^{lag}(\widetilde{\underline{\mathbb{H}_i}(1)}; 1) \geq 0. \tag{12.13}$$

On the other hand we have the identity

$$\rho^{lag}(\widetilde{\underline{\mathbb{H}_i}(1)}; 1) = -\rho^{lag}(\underline{\mathbb{H}_i}(1); [pt]^\#)$$

from (4.8) and so

$$\rho^{lag}(\underline{\mathbb{H}_i}(1); 1) + \rho^{lag}(\widetilde{\underline{\mathbb{H}_i}(1)}; 1) \to 0$$

by (12.5). Combined with (12.12), (12.13), this implies

$$\lim_{i\to\infty} \rho^{lag}(\underline{\mathbb{H}_i}(1); 1) = 0 = \lim_{i\to\infty} \rho^{lag}(\widetilde{\underline{\mathbb{H}_i}(1)}; 1).$$

This finishes the proof of the proposition. □

12.2. **Vanishing of difference from global invariant.** Here we recall that we continue our discussion starting at section 2.

We denote $F_i = H_i(1) = H_i(1, t, x)$. By definition, we recall

$$\underline{F_i}(s, x) = F(s, x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M F_i(s, x)\omega^n$$

and so

$$\rho^{ham}(\phi_{F_i}; 1) = \rho^{ham}(\underline{F_i}; 1) = \rho^{ham}(F_i; 1) + \mathrm{Cal}_U(F_i). \tag{12.14}$$

Similarly,

$$\rho^{ham}_{\mathcal{U}}(\underline{F}; 1_0) = \rho^{ham}_{\mathcal{U}}(F; 1_0) + \mathrm{Cal}_U(F)$$

We remind the readers that $\operatorname{supp} F_i \subset U = M - B$ while its normalization $\underline{F_i}$ satisfies

$$\underline{F_i}(s, x) \equiv -\int_M F_i(s, x)\, \omega^n = -\int_U F_i(s, x)\, \omega^n$$

on B. Since $\|\underline{H}(1) - \underline{F_i}\| \to 0$, we have

$$0 < \frac{c}{2} \leq \rho^{ham}(\underline{F_i}; 1) \leq E^-(\underline{F_i}) = E^-(\underline{H_i}(1)) < \eta \tag{12.15}$$

for all sufficiently large i's by (2.7) and Remark 2.1. Then using (12.14), we can rewrite the difference for $\underline{F_i}$ in terms of that of F_i itself, i.e., we have

$$\rho^{ham}(\underline{F_i}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0) = \rho^{ham}(F_i; 1) - \rho^{ham}_{\mathcal{U}}(F_i; 1_0). \tag{12.16}$$

The rest of this subsection, until at the very latest, of the proof will concern the functions F_i which satisfies $F_i \equiv 0$ on B.

We fix a normalized Morse function $f : M \to \mathbb{R}$ such that

$$|df|_{C^0} \leq \frac{c}{8\,\mathrm{diam}\, M}, \quad \mathrm{Crit}\, f \subset \mathrm{Int}\, B. \tag{12.17}$$

In particular

$$|f|_{C^0} \leq \|f\| \leq \frac{c}{8} < \frac{\eta}{4}$$

and there exists a constant $C' = C'(B, f) > 0$ depending only on B and f such that

$$|df(x)| \geq C'$$

for all $x \in X \setminus B$. The following lemma is an important point in any application of Ostrover's trick (see [Os, EP, U, Sey] for example.)

Lemma 12.3. *We have*

$$\mathrm{Fix}(\phi^1_{F_i\#f^1}) = \mathrm{Fix}(\phi^1_{F_i}\phi^1_f) = \mathrm{Fix}(\phi^1_f) \tag{12.18}$$

for all sufficiently large i's and all the periodic Hamiltonian trajectories associated to the fixed points of $F_i\#f$ are constant.

Proof. Recall $\overline{d}(\phi^1_{F_i}, id) = \overline{d}(\phi^1_{H_i(1)}, id) \to 0$ as $i \to \infty$. In particular, we will have

$$\overline{d}(\phi^1_{F_i}, id) < \frac{C'}{4}$$

for all sufficiently large i's. We also recall $\mathrm{supp}\, H_i \subset U = M \setminus B$. Now the proof is similar to the proof of Lemma 8.3, which is even easier, and so omitted. □

Then using $F_i \equiv 0$ on B, $\mathrm{Crit}\, f \subset B$ and spectrality of $\rho^{ham}(F_i\#f;1)$, there exist some $p_i \in \mathrm{Crit}\, f$ and a disc $w_i : (D^2, \partial D^2) \to M$ with $w_i|_{\partial D^2} \equiv p_i$ such that

$$\begin{aligned}\rho^{ham}(F_i\#f;1) &= \mathcal{A}_{F_i\#f}([c_{p_i}, w_i]) = \int w_i^*\omega - \int_0^1 F_i\#f(t,p_i)\,dt \\ &= \int w_i^*\omega - f(p_i).\end{aligned} \tag{12.19}$$

Since $|\rho(f;1)|$, $|\rho(f;1)| \le \|f\| = \mathrm{osc}(f)$ and

$$\rho^{ham}(F_i;1) - \rho^{ham}(-f;1) \le \rho^{ham}(F_i\#f;1) \le \rho^{ham}(F_i;1) + \rho^{ham}(f;1)$$

and by the choice of f, we obtain

$$\begin{aligned}\left|\int w_i^*\omega\right| &\le |\rho^{ham}(F_i\#f;1)| + \|f\| \le |\rho^{ham}(F_i;1)| + 2\|f\| \\ &\le |\rho^{ham}(\underline{F_i};1)| + |\mathrm{Cal}_U(F_i)| + 2\|f\| \\ &\le \eta + \frac{\Sigma_\omega}{4} + \frac{\eta}{2} < \frac{3\Sigma_\omega}{8}.\end{aligned}$$

Therefore, by definition of Σ_ω, we must have $\int w_i^*\omega = 0$. Then (12.19) is reduced to

$$\rho^{ham}(F_i\#f;1) = -f(p_i). \tag{12.20}$$

On the other hand, by the triangle inequality and (12.20), we obtain

$$\rho^{ham}(F_i;1) \le \rho^{ham}(F_i\#f;1) + \rho^{ham}(-f;1) \le -f(p_i) + \|f\| \le \frac{c}{4}. \tag{12.21}$$

For the term $\rho^{ham}_{\mathcal{U}}(F_i;1_0)$, we use Proposition 10.1 to obtain

$$\rho^{ham}_{\mathcal{U}}(F_i;1_0) = \rho^{lag}(F_i \oplus 0;1). \tag{12.22}$$

Then using the triangle inequality of ρ^{lag} in Proposition 4.3, we obtain

$$\rho^{lag}(F_i \oplus 0;1) \ge \rho^{lag}((F_i\#f)\oplus 0;1) - \rho^{lag}(f \oplus 0;1).$$

The following lemma is another place where the fact that H_i is an approximating sequence of a hamiltonian homotopy $H = H(s,t,x)$ of a topological Hamiltonian loop $\lambda = \phi_F$ with $F = H(1)$ enters in a crucial way.

Lemma 12.4.

$$\rho^{lag}((F_i\#f)\oplus 0;1) = \rho^{lag}(f \oplus 0;1)$$

Proof. Consider the function $s \mapsto \rho^{lag}((H_i(s)\#f) \oplus 0; 1)$ where $H_i(s)(s,x) = H_i(s,t,x)$ and $H(1,t,x) = F_i(s,x)$, $H_i(0,t,x) = 0$. By the hypothesis, we have $H_i(s) \equiv 0$ on B and $\operatorname{Crit} f \subset B$. And $\overline{d}(\phi^1_{H_i(s)}, id) \to 0$ uniformly over $s \in [0,1]$ since H_i is an approximating sequence $H_i = H_i(s,t,x)$ of a homotopy of topological Hamiltonian *loop* $\phi_{H(s)}$. Therefore we have $\operatorname{Fix} \phi^1_{H_i(s)\#f} = \operatorname{Fix} \phi^1_f$ for all $s \in [0,1]$ by the same proof as that of Lemma 12.3, which in turn implies

$$\phi^1_{(H_i(s)\#f)\oplus 0}(o_\Delta) \cap o_\Delta = \phi^1_{f\oplus 0}(o_\Delta) \cap o_\Delta$$

in $T^*\Delta$ for all $s \in [0,1]$ and the associated Hamiltonian chords are constant. This implies

$$\operatorname{Spec}((H_i(s)\#f) \oplus 0; \Delta) = \operatorname{Spec}(f \oplus 0; \Delta)$$

for all $s \in [0,1]$ and so $\rho^{lag}((H_i(s)\#f) \oplus 0; 1) \in \operatorname{Spec}(f \oplus 0; \Delta)$. (Recall (3.6) for the definition of the Lagrangian action spectrum $\operatorname{Spec}(H; N)$ on general T^*N.) Since $\operatorname{Spec}(f \oplus 0; \Delta)$ is nowhere dense (in fact is a finite set in this case) and independent of s, the continuous function

$$s \mapsto \rho^{lag}((H_i(s)\#f) \oplus 0; 1)$$

must be constant and hence

$$\rho^{lag}(f \oplus 0; 1) = \rho^{lag}((H_i(1)\#f) \oplus 0; 1) = \rho^{lag}((F_i\#f) \oplus 0; 1).$$

(See [Os, U, Sey] for similar arguments.) This finishes the proof. □

Therefore we derive

$$\begin{aligned} \rho^{lag}(F_i \oplus 0; 1) &\geq \rho^{lag}((F_i\#f) \oplus 0; 1) - \rho^{lag}(f \oplus 0; 1) \\ &= \rho^{lag}(f \oplus 0; 1) - \rho^{lag}(f \oplus 0; 1) = 0. \end{aligned} \tag{12.23}$$

Now we go back to the mean-normalized Hamiltonian $\underline{F_i}$. Combining (12.16), (12.21) and (12.23), we obtain

$$\rho^{ham}(\underline{F_i}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0) = \rho^{ham}(F_i; 1) - \rho^{ham}_{\mathcal{U}}(F_i; 1_0) \leq \frac{c}{4} + 0 = \frac{c}{4}. \tag{12.24}$$

Substituting (12.24) and $\rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0) \to 0$ into (12.15), we derive

$$0 < \frac{c}{2} \leq \rho^{ham}(\underline{F_i}; 1) = \left(\rho^{ham}(\underline{F_i}; 1) - \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0)\right) + \rho^{ham}_{\mathcal{U}}(\underline{F_i}; 1_0) \leq \frac{c}{3}$$

for all sufficiently large i's, which is absurd.

This finishes the proof of Theorem 1.1. □

Finally we prove $\rho^{ham}(\lambda_0; a) = \rho^{ham}(\lambda_1; a)$ for all $a \in QH^*(M)$ when λ_0 and λ_1 are hamiltonian homotopic to each other.

Proof of Corollary 1.3. First note that if λ is a topological Hamiltonian loop contractible to the identity path, so is λ^{-1} and hence $\rho^{ham}(\lambda^{-1}; 1) = 0$. Suppose that λ_0 is hamiltonian-homotopic to λ_1 relative to the ends. Then $\lambda_0^{-1}\lambda_1$ is a topological hamiltonian loop hamiltonian-homotopic to the identity and so $\rho^{ham}(\lambda_0^{-1}\lambda_1; 1) = 0$ by Theorem 1.1.

Now we compare $\rho^{ham}(\lambda_0; a)$ and $\rho^{ham}(\lambda_1; a)$. By the triangle inequality,

$$\rho^{ham}(\lambda_1; a) - \rho^{ham}(\lambda_0; a) \leq \rho^{ham}(\lambda_1(\lambda_0)^{-1}; 1) = 0.$$

By changing the roles of λ_0, λ_1, we obtain the other inequality. This finishes the proof. □

13. Appendix

13.1. **Darboux family and symplectic exponential map.** We first recall the notion of Darboux family from [W1] and interpret it as a symplectic version of the exponential map (or rather its inverse) in Riemannian geometry. We provide our explanation more than what we need for the main purpose of the present paper, since we feel that they are natural continuations of the discussion and will be useful for other future purpose. A similar discussion was also previously made in section 8.1 [OZ] for a different purpose in a less systematic way.

By Darboux theorem, there exists a chart

$$\Phi_y = (q_1, \cdots, q_n, p_1, \cdots, p_n) : (U_y, \omega) \to (\mathbb{R}^{2n}, \omega_0)$$

centered at each point $y \in X$, i.e., $\Phi_y(y) = 0 \in \mathbb{R}^{2n}$ satisfying $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. By applying the parametric version thereof, we can obtain a smooth family of Darboux charts $\{(U_x, \Phi_x)\}_{x\in M}$ parameterized by M.

The following notion was introduced and exploited by Weinstein [W1] in his proof of Arnold's conjecture for C^0-small Hamiltonian diffeomorphisms.

Definition 13.1 (Darboux family)**.** We call a smooth family $\{(U_y, \Phi_y)\}_{y\in M}$ of Darboux charts a *Darboux family.*

Here the smoothness mentioned in the definition means that the map

$$\Phi : \bigcup_{y\in M} U_y \times \{y\} \subset M \times M \to \mathbb{R}^{2n} \times M$$

defined by $\Phi(x, y) = (\Phi_y(x), y)$ is smooth.

We note that when Φ_y is given, its derivative $d_x\Phi_y : (T_xM, \omega(x)) \to (\mathbb{R}^{2n}, \omega_0)$ induces a a natural isomorphism as a symplectic vector space at each point $x \in U_y$. Therefore we have a collection of natural local symplectic trivializations of TM

$$d\Phi_y : \bigcup_{x\in U_y} (TM|_{U_y}, \omega) \to \Phi_y(U_y) \times \mathbb{R}^{2n}.$$

By combining them all, we define a map

$$D\Phi : \bigcup_{y\in M} \bigcup_{x\in U_y} (TM|_{U_y}, \omega) \times \{y\} \subset TM \times M \to \mathbb{R}^{2n} \times M$$

by

$$D\Phi((x, v), y) = (d_x\Phi_y(v), y). \tag{13.1}$$

In particular, the collection of isomorphisms $\{d_x\Phi_x : T_xM \to \mathbb{R}^{2n}\}_{x\in M}$ provides a smooth choice of symplectic identification of the tangent space T_xM with $\mathbb{R}^{2n}$ at each point of $x \in M$.

We define a map

$$\exp_x^\Phi = \Phi_x^{-1} \circ d_x\Phi_x : (T_xM, \omega_x) \to (U_x, \omega) \subset (M, \omega)$$

which we call the *symplectic exponential map* at $x \in M$ associated to the Darboux family Φ. We would like to note that the symplectic exponential map $\exp_x^\Phi$ is a symplectic diffeomorphism onto its image, which is even better than the Riemannian exponential map in that the latter is not an isometry and which reflects the fact that all symplectic manifolds are locally isomorphic in same dimension.

Similarly we can define the globalized symplectic exponential map

$$\mathrm{Exp}^\Phi : \bigcup_{x\in M} (\exp_x^\Phi)^{-1}(U_x) \subset TM \to \bigcup_{x \in M} U_x \times \{x\} \subset M \times M$$

defined by

$$\mathrm{Exp}^\Phi(x,v) = (\exp_x^\Phi(x,v), x). \tag{13.2}$$

We note that as in the ordinary exponential map Exp^Φ maps the zero section of TM to the diagonal $\Delta \subset M \times M$. By the assumption that each chart Φ_y in the family is centered at y,

$$\begin{aligned} \{0\} \times M \quad &\subset \quad \bigcup_{x\in M} \Phi_x(U_x) \times \{x\} \subset \mathbb{R}^{2n} \times M \\ \Delta \quad &\subset \quad \bigcup_{x \in M} U_x \times \{x\} \subset M \times M. \end{aligned}$$

When M is compact, there exists some $\delta > 0$ independent of $x \in M$ such that

$$\Phi_x(U_x) \supset B^{2n}(\delta)$$

for all $x \in M$.

Definition 13.2 (Symplectic injectivity radius)**.** The *injectivity radius* of the Darboux family Φ is defined to be

$$\mathrm{inj}_\omega(\Phi) = \inf_{x\in M} \sup_\delta \{\delta > 0 \mid B^{2n}(\delta) \subset \Phi_x(U_x)\}, \tag{13.3}$$

and the *symplectic injectivity radius* of (M,ω) is

$$\mathrm{inj}(M,\omega) = \sup_\Phi \mathrm{inj}_\omega(\Phi). \tag{13.4}$$

In particular when M is compact, we can choose a Darboux-Weinstein neighborhood V_Δ of Δ in $M \times M$ so that

$$V_\Delta \subset \bigcup_{x\in M} U_x \times \{x\} \subset M \times M$$

and $\Phi_y^{-1}(B^{2n}(\delta)) \times \{y\} \subset V_\Delta$ for all $y \in M$ for some sufficiently small $0 < \delta < \mathrm{inj}_\omega(\Phi)$.

13.2. **Proof of Lemma 11.7.** We first note that $\Psi^*\Omega_2 = \Psi^* \mathrm{pr}_2^* \omega$ by definition. But we have $\mathrm{pr}_2 \circ \Psi = \pi_M$ since

$$\mathrm{pr}_2 \circ \Psi(s,(q,p)) = (q,p) = \pi_M(s,(q,p))$$

for any point $(s,(q,p))$. Therefore

$$\Psi^*\Omega_2 = \pi_M^*\omega. \tag{13.5}$$

Note that $(\pi_M)_! \Psi^*(\eta' \wedge \Omega_2)$ is a two-form on M and so we can write

$$(\pi_M)_! \Psi^*(\eta' \wedge \Omega_2) = g\,\omega$$

for some function $g : M \to \mathbb{R}$. It remains to prove $g(Q,P) = \int_{\Psi_{(Q,P)}} \eta'$. Let $\delta_{(Q,P)}$ be the Dirac delta current with support at (Q,P). Since π_M is a submersion, the pull-back $\pi_M^*\delta$ is well-defined as a current which is supported on $[0,1]\times\{(Q,P)\}\subset [0,1]\times M$. Then using the definition of the push-forward, we evaluate

$$\begin{aligned} g(Q,P) &= \int_M \delta\,(g\,\omega) = \int_M \delta\,(\pi_M)_!\Psi^*(\eta'\wedge\Omega_2) \\ &= \int_{[0,1]\times M} \pi_M^*\delta\,\Psi^*(\eta'\wedge\Omega_2) = \int_{[0,1]\times M} \pi_M^*\delta\,\Psi^*(\eta')\wedge\Psi^*(\Omega_2) \\ &= \int_{[0,1]\times M} \pi_M^*\delta\,\Psi^*(\eta')\wedge\pi_M^*(\omega) \\ &= \int_{[0,1]\times M} \Psi^*\eta'\wedge\pi_M^*(\delta\,\omega) = \int_M (\pi_M)_!(\Psi^*\eta')\wedge\delta\,\omega \\ &= (\pi_{(Q,P)})_!(\Psi^*\eta') = \int_0^1 \Psi_{(Q,P)}^*\eta' = \int_{\Psi_{(Q,P)}}\eta' \end{aligned}$$

where we use (13.5) for the fourth equality. We also use the identity $\Psi_{(Q,P)} = \Psi\circ i_{(Q,P)}$ which is nothing but the characteristic curve (11.13) on R_Ψ issued at (Q,P). This finishes the proof.

13.3. **Reparameterization.** In this appendix, we recall the precise details of the boundary flattening of Hamiltonians from [OM] and how the process suits well the $L^{(1,\infty)}$-approximation. As emphasized in [OM], this approximation result fails to hold in the stronger C^0 (or L^∞) topology.

We first recall the following definition from [OM].

Definition 13.3 (Definition 3.19 [OM])**.** We call the norm

$$\|\zeta\|_{ham} := \|\zeta\|_{C^0} + \|\zeta'\|_{L^1}$$

of a (smooth) function $\zeta:[0,1]\to\mathbb{R}$ the *hamiltonian norm* of the function ζ. Here ζ' denotes the derivative of the function ζ. We say that two smooth functions $\zeta_1,\,\zeta_2:[0,1]\to[0,1]$ are *hamiltonian-close* to each other if the norm

$$\begin{aligned} \|\zeta_1-\zeta_2\|_{ham} &:= \|\zeta_1-\zeta_2\|_{C^0} + \|\zeta_1'-\zeta_2'\|_{L^1} \\ &= \max_{t\in[0,1]}|\zeta_1(t)-\zeta_2(t)| + \int_0^1 |\zeta_1'(t)-\zeta_2'(t)|\,dt \end{aligned}$$

is small.

Recall that for a given Hamiltonian function H generating the Hamiltonian path ϕ_H, the reparameterized path $t\mapsto\phi_H^{\zeta(t)}$ is generated by the Hamiltonian function H^ζ defined by $H^\zeta(s,x)=\zeta'(t)H(\zeta(t),x)$, where ζ' again denotes the derivative of the reparameterization function $\zeta:[0,1]\to[0,1]$. The following lemma was proved in [OM] whose proof we refer readers to Appendix 7.2 thereof.

Lemma 13.1 (Lemma 3.20 [OM])**.** *Let $H:[0,1]\times M\to\mathbb{R}$ be a normalized smooth Hamiltonian function, and let $\zeta_1,\,\zeta_2:[0,1]\to[0,1]$ be two smooth reparameterization functions. Then*

$$\|H^{\zeta_1}-H^{\zeta_2}\| \le C\|\zeta_1-\zeta_2\|_{ham}, \tag{13.6}$$

where $C\le 2\max(\|H\|_{C^0},L)$ is a constant that depends only on the C^0-norm

$$\|H\|_{C^0} = \max_{(s,x)}|H(s,x)| < \infty$$

of H and a Lipschitz constant (in the time variable) L for H.

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