DYNAMICAL DENSITY AND INVARIANT SETS WITH POSITIVE VOLUMES

YONGXIA HUA AND ZHIHONG XIA

ABSTRACT. We consider volume preserving diffeomorphisms on compact manifolds and study invariant sets with positive volumes. If an invaraint set has some form of hyperbolicity, say uniformly or non-uniformly hyperbolic or partially hyperbolic, then the invariant set must contain almost all of its stable manifold and unstable manifold. The main tool is the dynamical density point introduced in Xia [7]. This density is a simple version of the so-called Juliennes in Pugh and Shub [5]. We also present several applications of the main results.

1. INTRODUCTION

Let M be a compact Riemannian manifold and let $\mathrm{Diff}^r_m(M)$ be the set of all C^r diffeomorphisms preserving a smooth volume m on M. If r is not an integer and $r = k + \alpha$ for some positive integer k and $0 < \alpha < 1$, it is understood that the functions in $\mathrm{Diff}^r_m(M)$ are C^k functions with α-*Hölder* k-th derivatives. In the rest of this paper we will assume that f is $C^{1+\alpha}$.

An f-invariant set $\Lambda \subset M$ is said to be *(uniformly) partially hyperbolic* if for every $x \in \Lambda$ the tangent space at x admits an invariant splitting

$$T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x)$$

into *strongly stable* $E^s(x) = E^s_f(x)$, *central* $E^c(x) = E^c_f(x)$, and *strongly unstable* $E^u(x) = E^u_f(x)$ subspaces and there exist constants $c_0 > 1$ and

$$0 < \lambda_s < \lambda'_c \le 1 \le \lambda''_c < \lambda_u$$

such that for every $x \in \Lambda$ and all $i \in \mathbb{N}$,

$$\begin{aligned} v \in E^s(x) &\Rightarrow \|df^i_x(v)\| \le c_0\lambda^i_s\|v\|, \\ v \in E^c(x) &\Rightarrow c_0^{-1}(\lambda'_c)^i\,\|v\| \le \|df^i_x(v)\| \le c_0(\lambda''_c)^i\,\|v\|, \\ v \in E^u(x) &\Rightarrow c_0^{-1}\lambda^i_u\|v\| \le \|df^i_x(v)\|. \end{aligned}$$

If the center bundle in the above splitting is zero, Λ is said to be *uniformly hyperbolic*.

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Given a point $x \in \Lambda$, one can construct local strongly stable and strongly unstable manifolds at x. They can be characterized by follows:

$$W^s_\delta(x) = \{y \in M|\ d(y,x) < \delta, d(f^n(x), f^n(y)) \leq c_0\lambda_s^n d(y,x)\ for\ all\ n \geq 0\}$$
$$W^u_\delta(x) = \{y \in M|d(y,x) < \delta, d(f^n(x), f^n(y)) \geq c_0^{-1}\lambda_u^n d(y,x)\ for\ all\ n \geq 0\}$$

where δ is some positive constant. We stress that the sizes of the local strongly stable and strongly unstable manifolds are uniformly bounded from below.

We define the strongly stable and unstable global manifolds by

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}W^s_\delta(f^n x)$$
$$W^u(x) = \bigcup_{n\geq 0} f^{n}W^u_\delta(f^{-n} x)$$

The strongly stable and unstable global manifolds are absolutely continuous foliations(see Anosov [1], Brin and Pesin [3], Push and Shub [5]).

We will also discuss the non-uniformly hyperbolic and nonuniformly partially hyperbolic invariant sets. For this, let

$$\lambda(x,v) = \lim_{i\to\infty} \frac{1}{i} \ln \|d_x f^i v\|.$$

the Lyapunov exponents of a nonzero vector v at x. The Multiplicative Ergodic Theorem of Oseledec states that almost every point $x \in M$ is a Lyapunov regular point, i.e., (1). the limit exists for all tangent vectors $v \in T_xM$; (2). there are at most $k = \dim M$ distinct values of $\lambda(x,v)$.

Let $k(x)$ be the number of distinct values of $\lambda(x,v)$ at x for $v \in T_xM$, with tangent vectors $v^j \in T_xM$ for $1 \leq j \leq k(x)$ giving distinct values,

$$\lambda_j(x) = \lambda(x, v^j),$$

with

$$\lambda_1(x) < \lambda_2(x) < ... < \lambda_{k(x)}(x)$$

An invariant set Λ is said to be *nonuniformly hyperbolic* if none of these $k(x)$ distinct $\lambda_j(x)$ is zero, for a.e. $x \in \Lambda$. It is well-known (see Pesin[4]) that such a set Λ with $m(\Lambda) > 0$ can be decomposed into countably many disjoint invariant ergodic components, each with positive measure. On each ergodic component, the Lyapunov exponents are constants almost everywhere. Moreover there exist Lyapunov exponents with different signs. This is true because of the following simple proposition:

Proposition 1.1. *If all Lyapunov exponents of $\mu \in M_{erg}$ are non-zero and have the same sign then μ is concentrated on a single orbit.*

On an ergodic component Λ^l, assume the Lyapunov exponents of f are

$$\lambda_1 < ... < \lambda_r < 0 < \lambda_{r+1} < ... < \lambda_k.$$

By multiplicative ergodic theorem, for almost every point $x \in \Lambda^l$, the tangent space at x can be written as an increasing set of subspaces

$$\{0\} = V_x^0 \subset V_x^1 \subset ... \subset V_x^k = T_x M$$

such that (1). for all $v \in V_x^j \setminus V_x^{j-1}$, $\lambda(x,v)$ equals λ_j; (2). the bundle of the subspaces $\{V_x^j\}$ are invariant in the sense that $df_x V_x^j = V_{f(x)}^j$ for all $1 \leq j \leq k$. Let $E_x^- = V_x^r$. This is the stable subspace at x. Similarly the unstable subspace E_x^+ can be determined by using f^{-1}. The splitting of the tangent space by means of the stable and unstable subspaces, $T_x M = E_x^- \oplus E_x^+$ for almost every point $x \in \Lambda^l$, is measurable but not necessarily continuous. This type of structure is called a nonuniformly hyperbolic structure. For almost every point $x \in \Lambda^l$, one can construct stable and unstable local manifolds $W_\delta^s(x)$ and $W_\delta^u(x)$, which are tangent to E_x^- and E_x^+ respectively (see Pesin [4]). We stress that in this case, the diameter of the local manifolds varies with the base point x. Before stating the stable (unstable) manifold theorem, we will first introduce the definition of Pesin set which is independent of the measures.

Definition 1.2. *Given $\lambda,\ \lambda^{'} \gg \epsilon > 0$, and for all $k \in \mathbb{N}$, define Λ_k to be all points $x \in M$ for which there is a splitting $T_x M = E_x^- \oplus E_x^+$ with the invariance property $df_x E_x^- = E_{f(x)}^-$ and $df_x E_x^- = E_{f(x)}^+$ and satisfying:*

(1). $\|df^n | E_{f^m x}^-\| \leq e^{\epsilon k} e^{-(\lambda-\epsilon)n} e^{\epsilon|m|}$, for all $m \in \mathbb{Z}$, $n \geq 1$;

(2). $|df^{-n} | E_{f^m x}^+\| \leq e^{\epsilon k} e^{-(\lambda^{'}-\epsilon)n} e^{\epsilon|m|}$, for all $m \in \mathbb{Z}$, $n \geq 1$;

(3). $tan(Angle(E_{f^m x}^-, E_{f^m x}^+)) \geq e^{-\epsilon k} e^{-\epsilon|m|}$, for all $m \in \mathbb{Z}$.

Definition 1.3. $\Lambda(\lambda, \lambda^{'}; \epsilon) = \cup_{k=1}^{+\infty} \Lambda_k$ *is a Pesin set.*

It is immediate from the definitions that

$$\Lambda_1 \subseteq \Lambda_2 \subseteq \dots \Lambda_k \subseteq \Lambda_{k+1} \subseteq \dots$$

Furthermore, while each set Λ_k may not be f-invariant, we have that

$$f(\Lambda_k), f^{-1}(\Lambda_k) \subseteq \Lambda_{k+1}$$

Finally the Pesin set itself is f-invariant, i.e. $f(\Lambda) = \Lambda$.

Pesin [4] proved that with the choice of $\lambda = |\lambda_r|$, $\lambda^{'} = \lambda_{r+1}$ and any ϵ such that $\lambda,\ \lambda^{'} > \epsilon$, where λ_r, λ_k are the Lyapunov exponents as above, then $m(\Lambda^l) = m(\Lambda(\lambda, \lambda^{'}; \epsilon))$. That is, each ergodic component is a Pesin set up to a set of measure zero.

The stable and unstable local manifolds for a Pesin set $\Lambda(\lambda, \lambda^{'}; \epsilon)$ can be characterized by follows: for each $x \in \Lambda_k$

$$W_\delta^u(x) = \{y \in M | d(f^{-n}x, f^{-n}y) \leq e^{\epsilon k} e^{-(\lambda^{'}-\epsilon)n} d(x,y), n \geq 0\}$$
$$W_\delta^s(x) = \{y \in M | d(f^{n}x, f^{n}y) \leq e^{\epsilon k} e^{-(\lambda-\epsilon)n} d(x,y), n \geq 0\},$$

where $\delta = \epsilon_0 e^{-\epsilon k}$ for some constant $\epsilon_0 > 0$.

One can define stable and unstable global manifolds by

$$W^s(x) = \bigcup_{n\geq 0} f^{-n} W^s_\delta(f^n x)$$

$$W^u(x) = \bigcup_{n\geq 0} f^{n} W^s_\delta(f^{-n} x)$$

The families of stable and unstable manifolds are absolutely continuous on each Pesin set (see Pesin[4]).

An invariant set Λ is said to be *nonuniformly partially hyperbolic* if there exist at least two distinct $\lambda_j(x)$ with different signs, for almost every point $x \in \Lambda$.

We denote by $r(x)$ and $r'(x)$ the largest natural number such that $\lambda_{r(x)} < 0$ and the smallest natuarl number such that $\lambda_{r'(x)} > 0$ respectively. For integers $s \geq t \geq 1$ and $j \geq k \geq 1$, set

$$\Lambda_{s,t,j,k} = \{x \in \Lambda : \frac{t-1}{s} < e^{\lambda_{r(x)}(x)} \leq \frac{t}{s} \leq \frac{j}{k} \leq e^{\lambda_{r'(x)}(x)} < \frac{j}{k-1}\}$$

where s and j are the smallest numbers satisfying these inequalities for a certain t and k respectively. From the definition it is obviously that $\Lambda_{s,t,j,k}$ are measurable and f-invariant. Moreover

$$\bigcup_{s>t\leq 1, j>k\geq 1} \Lambda_{s,t,j,k} = \Lambda$$

up to a zero set and if $s_1 \neq s_2$ or $t_1 \neq t_2$ or $j_1 \neq j_2$ or $k_1 \neq k_2$, then

$$\Lambda_{s_1,t_1,j_1,k_1} \cap \Lambda_{s_2,t_2,j_2,k_2} = \emptyset$$

With the choice of $\lambda = -\ln \frac{t}{s}$ and $\lambda' = \ln \frac{j}{k}$ such a decomposition component $\Lambda_{s,t,j,k}$ is a Pesin set up to a zero set . We remark that in the definition of Pesin set here, $E^-_x \oplus E^+_x$ is not necessarily to be T_xM. Therefore although the stable and unstable theorem holds and the stable and unstable manifolds are absolutely continuous on each component, the stable and unstable manifolds are not necessarily transversal foliations. For more details, see Pesin [4].

Let

$$\mu = \min\{m(Tf), m(Tf^{-1})\}$$

where m is the minimum norm (or called conorm).

If Λ is a uniformly (partially) hyperbolic invariant set, let

$$\nu = \max(\lambda_s, \lambda_u^{-1}).$$

In the nonuniformly (partially) hyperbolic cases, consider each (ergodic) component. Let

$$\nu = \max(e^{\lambda+\epsilon}, e^{-\lambda'+\epsilon})$$

where λ, λ' are the λ, λ' defined in the Pesin set. Then $0 < \mu \leq \nu < 1$.

The behavior of f in the stable direction is a contraction whose strength is between μ and $c_0\nu$, the behavior of f in the unstable direction is an expansion whose strength is between $1/\mu$ and $c_0^{-1}1/\nu$, where c_0 depends on the point x, in the nonuniformly (partially) hyperbolic case.

In the next section, all the results are for ergodic components of Λ when Λ is nonuniformly (partially) hyperbolic. Therefore in the nonuniform case, we assume that Λ is itself ergodic. It should be clear when these results can be extended to countable unions of ergodic components.

2. Statements of Results and Proofs

In these section, we introduce a new and dynamically defined density basis and density points. Recall that a point $x \in M$ is said to be a *Lebesgue density point* for a set Λ if

$$\lim_{r\to 0} \frac{m(B_r(x)\cap\Lambda)}{m(B_r(x))} = 1,$$

where $B_r(x)$ is a ball of radius r centered at x. The collection of the balls centered around x is called the *Lebesgue density basis*. Lebesgue Density Theorem states that almost every point in Λ is a Lebesgue density point.

Unfortunately, the Lebesgue density basis, or the regular balls does not behave well under the map f. We need to replace it with a new set of density basis and we will construct the new basis from stable and unstable manifolds.

Let $W^u_\delta(x)$ be the local unstable manifold of a point $x \in \Lambda$. Let m_u and m_s respectively be the induced measures on the unstable and stable leaves. For any integer k, let $B^u_k(x)$ be a subset of $W^u(x)$ defined by $B^u_k(x) = f^{-k}(W^u_\delta(f^k(x)))$. Since in the nonuniformly (partially) hyperbolic case δ varies with the base point, we use $B^u_k(x) = f^{-k}(W^u_{\delta_k}(f^k(x)))$ to denote the difference(for uniformly (partially) hyperbolic cases, $\delta \equiv \delta_k \equiv \epsilon_0$ for all $k \in \mathbb{N}$). Clearly, the balls $B^u_k(x)$, $k \in \mathbb{N}$ shrink to the point x as $k \to \infty$. We call the collection of the sets $\{B^u_k(x)|\ k \in \mathbb{N}, x \in \Lambda\}$ the unstable density basis. The idea of constructing density basis goes back to the work of Pugh and Shub [5]. The definition here can be regarded as a simplified version and it was introduced in Xia [7]. Moreover, we have the following properties for the density basis which are called scaling and engulfing. Before stating the scaling and engulfing properties, first we need to prove a lemma which gives us an estimate of the restriction of a n iterate of f to a subset of W^u_δ.

Lemma 2.1. *Fix some point $x \in \Lambda$ and let U be a small disk in $W^u_\delta(x)$. If the disc is so small that*

$$f^n(U) \in W^u_{\delta_n}(f^n(x))$$

then there exist constants C_1, C_2, independent of x and n, such that

$$C_1 \le \frac{m_u(f^nU)}{|det(df^n_x|E^u_x)|m_u(U)} \le C_2$$

Proof. Let m_u be the dimension of the leaves of the unstable foliation. Local $W^u_\delta(x)$ can be identified with a cube in $E^U_x = \mathbb{R}^{m_u}$ by the exponential map from E^u_x to $W^u_\delta(x)$. since the leaves of the unstable foliation are smooth(as smooth as the map which is at least C^1), m_u is smoothly equivalent to the stable Lebesgue measure m on $\mathbb{R}^n$, i.e. for any point $x \in \Lambda$, there is a smooth function $g_u(y)$ defined for $y \in W^u_\delta(x)$ on the local unstable manifold , uniformly bounded away from zero and infinity such that $m_u(U) = \int_U g_u dm$ where m is the standard Lebesgue measure in $\mathbb{R}^n$.

Let $J_u(y) = |det(df_y|E^u_y)|$ be the Jacobi of the map f at y restricted on the unstable manifold of y. Since f is $C^{1+\alpha}$, J_u is α-$H\ddot{o}lder$, i.e., there is a constant $c_1 > 0$ such that $|J_u(y) - J_u(z)| \leq c_1|x-y|^\alpha$. Here $|y-x|$ denotes the distance between y and z.

$$\begin{aligned} m_u(f(U)) &= \int_{f(U)} g_u dm = \int_U J_u(g_u \circ f)dm \\ &= \int_U J_u(y)(g_u \circ f)(y) g_u^{-1}(y) dm_u, \end{aligned}$$

Then

$$\frac{m_u(f(U))}{J_u(x)m_u(U)} = \frac{1}{m_u(U)} \int_U \frac{J_u(y)(g_u \circ f)(y)g_u^{-1}(y)}{J_u(x)(g_u \circ f)(x)g_u^{-1}(x)} dm_u.$$

since the functions g_u and g_u^{-1} are smooth on any unstable manifold and f is $C^{1+\alpha}$, there is a constant $c_2 > 0$ such that

$$|J_u(y)(g_u \circ f)(y)g_u^{-1}(y) - J_u(x)(g_u \circ f)(x)g_u^{-1}(x)| \leq c_2|y-x|^\alpha,$$

for all $y \in U$. Therefore

$$1 - c_2(\text{diam}U)^\alpha \leq \frac{m_u(f(U))}{J_u(x)m_u(U)} \leq 1 + c_2(\text{diam}U)^\alpha$$

Let $\epsilon_j(x) = \text{diam} f^j(U)$ and apply the above argument to $f(U)$, we have

$$1 - c_2(\epsilon_1)^\alpha \leq \frac{m_u(f^2(U))}{J_u(f(x))m_u(f(U))} \leq 1 + c_2(\epsilon_1)^\alpha$$

By induction and by the chain rule that $|det(df^n_x|E^u_x)| = \prod_{j=0}^{n-1} J_u(f^j(x))$, we have

$$\prod_{j=0}^{n-1}(1 - c_2(\epsilon_j(x))^\alpha) \leq \frac{m_u(f^n U)}{|det(df^n_x|E^u_x)|m_u(U)} \leq \prod_{j=0}^{n-1}(1 + c_2(\epsilon_j(x))^\alpha)$$

In the uniformly(partially) hyperbolic case, since $f^n(U) \in W^u_\delta(f^n(x))$, by the Local Stable and Unstable Manifold Theorem, there exist positive real numbers $c_0 > 0$ and $\nu > 1$ such that $\text{diam} f^{n-1}(U) \leq c_0\nu^{-1}\delta$ and $\text{diam} f^i(U) \leq c_0\nu^{-(n-i)}\delta$, for $i = 0, 1, ..., n-1$. In the non-uniformly(partially) hyperbolic case, since $f^n(U) \in W^u_{\delta_n}(f^n(x))$, by the Local Stable and Unstable Manifold

Theorem, there exist real numbers $\epsilon_0 > 0$ and $\nu > 1$ such that

$$\begin{aligned} \operatorname{diam} f^{n-1}(U) &\leq e^{\epsilon(k+n)}\nu^{-1}\delta_n \\ &= e^{\epsilon(k+n)}\nu^{-1}\epsilon_0 e^{-\epsilon(k+n)} \\ &= \epsilon_0\nu^{-1} \end{aligned}$$

where k is the integer such that $x \in \Lambda$. And $\operatorname{diam} f^i(U) \leq \epsilon_0\nu^{-(n-i)}$, for $i = 0, 1, ...n-1$. Let C_0 be a constant larger than $c_0\delta$ andϵ_0. Then

$$\prod_{j=0}^{infty}(1 - c_2(C_0\nu^{-n+j})^\alpha) \leq \frac{m_u(f^nU)}{|det(df_x^n|E_x^u)|m_u(U)} \leq \prod_{j=0}^{\infty}(1 + c_2(C_0\nu^{-n+j})^\alpha)$$

Since $\nu > 1$, then $\nu^\alpha > 1$, the infinite products on both sides converge. Therefore we have

$$C_1 \leq \frac{m_u(f^nU)}{|det(df_x^n|E_x^u)|m_u(U)} \leq C_2$$

for some constants $C_1 > 0$ and $C_2 > 0$.

This proves the lamma. □

In order to have a result similar to the Lebesgue Density Theorem for our new density basis, we need to prove two basic properties. These two properties are called *scaling* and *engulfing* properties. These are the contents of the next two lemmas.

Lemma 2.2. *[Scaling]. Fix an integer $k \geq 1$, then there exists a constant $C = C(k) \geq 1$ such that for every point $x \in \Lambda$ and all positive integer n*

$$1 \leq \frac{m_u(B_n^u(x))}{m_u(B_{n+k}^u(x))} \leq C$$

Proof. The lower bound of 1 is trivial, since the balls $B_n^u(x)$ nest down to x as n goes to infinity.

We first assume $k = 1$. let $Y_n(x) = f^n(B_n^u(x))$ and $Y_{n+1}(x) = f^n(B_{n+1}^u(x))$. Then $Y_n(x) = W_{\delta_n}^u(f^n(x))$ and $Y_{n+1}(x) = f^{-1}(W_{\delta_{n+1}}^u(f^{n+1}(x)))$. Notice that

$$W^u(f^n(x), \mu\delta_{n+1}) \subset Y_{n+1}(x) \subset W^u(f^n(x), c_0\nu\delta_{n+1})$$

and $\delta_{n+1} = \delta_n e^{-\epsilon}$, we have, for some constant C_3,

$$\frac{m_u(Y_n(x))}{m_u(Y_{n+1}(x))} \leq C_3.$$

Iterating $B_n^u(x)$ and $B_{n+1}^u(x)$ by f^n and applying Lemma 2.1, we see that

$$\begin{aligned} \frac{m_u(B_n^u(x))}{m_u(B_{n+1}^u(x))} &\leq \frac{m_u(Y_n(x))}{C_1|det(df_x^n|E_x^u)|} \times \frac{C_2|det(df_x^n|E_x^u)|}{m_u(Y_{n+1}(x))} \\ &\leq C_4 \end{aligned}$$

where $C_4 = C_3C_2/C_1$ is a constant. Therefore we finish the proof when $k=1$. For general k, the choice $C = C_4^k$ give

$$\frac{m_u(B_n^u(x))}{m_u(B_{n+k}^u(x))} \leq C.$$

This proves the lemma. □

The next lemma shows the engulfing property.

Lemma 2.3. *[Engulfing]. There is a uniform integer L such that for all integer $n \geq 1$, and for any $x, y \in \Lambda$, if $B_{n+L}^u(x) \cap B_{n+L}^u(y) \neq \emptyset$,*

$$B_{n+L}^u(x) \cup B_{n+L}^u(y) \subset B_n^u(x).$$

Proof. Choose an integer $L > 0$ such that $c_o\nu^L < 1/3$. Now consider the f^n images of $B_{n+L}^u(x)$ and $B_{n+L}^u(y)$. Let $Y_n = f^n(B_{n+L}^u(x))$ and $Y_n' = f^n(B_{n+L}^u(y))$, then

$$Y_n = f^{-L}(W_{\delta_{n+L}}^u(f^{n+L}(x))) \subset W^u(f^n(x), c_0\nu^L\delta_{n+L}) \subset W(f^n(x), \frac{1}{3}\delta_n)$$

$$Y_n' = f^{-L}(W_{\delta_{n+L}}^u(f^{n+L}(y))) \subset W^u(f^n(y), c_0\nu^L\delta_{n+L}) \subset W^u(f^n(y), \frac{1}{3}\delta_n).$$

Moreover since $B_{n+L}^u(x) \cap B_{n+L}^u(y) \neq \emptyset$, $Y_n \cap Y_n' \neq \emptyset$. Therefore Y_n and Y_n' are both contained in $W_{\delta_n}^u(f^n(x))$. Iterating Y_n and Y_n' and $W_{\delta_n}^u(f^n(x))$ by f^{-n}, we have $B_{n+L}^u(x), B_{n+L}^u(y) \subset B_n^u(x)$.

This proves the lemma. □

Now, we can similarly define the stable density basis $\{B_k^s(x) |\ k \in \mathbb{N}, x \in \Lambda\}$ by taking

$$B_k^s(x) = f^k(W_{\delta_k}^u(f^{-k}(x))).$$

Lemmas 2.2 and 2.3 can be proved in the same way for the stable density basis by considering f^{-1}.

The following definitions of density points on the stable and unstable foliations are simplified versions of juliennes density points defined by Push and Shub [5] and were introduce in Xia [7]. A point $x \in \Lambda$ is said to be a density point on the unstable foliations if

$$\lim_{k\to\infty} \frac{m_u(B_k^u(x) \cap \Lambda)}{m_u(B_k^u(x))} = 1$$

Similarly one can define the density points on the stable foliations.

Finally, we have the following proposition.

Proposition 2.4. *The set of points on Λ that are both density points on the unstable foliations and stable foliations has the full measure in Λ.*

Proof. Assume

$$A = \{x \in \Lambda :\ x \textit{ is not a density point on the unstable foliations }\}$$

has positive measure. By the absolute continuity of the unstable foliations, $m_u(A \cap W_\delta^u(y)) > 0$ for almost every point $y \in A$. From the construction of

B_k^u, one can easily see that k goes to infinity if $m_u(B_k^u)$ goes to zero. And we have proved that the scaling and engulfing properties. All these properties are the same as assumption $(i)(ii)^*$ of Stein [6]. As a consequence of the corollary of Stein [6], we have that for any measurable set in $W^u(y)$, m_u-a.e. point is a density point on the unstable foliations. This a contraction to $m_u(A \cap W_\delta^u(y)) > 0$. The proof of the full measure of density points on the stable foliations is the same. □

We now state our main result.

Theorem 2.5. *Let* Λ *be invariant set for* $f \in Diff_m^{1+\alpha}(M)$, $\alpha > 0$. *Assume that* $m(\Lambda) > 0$ *and* Λ *is uniformly hyperbolic, or uniformly partially hyperbolic, or non-uniformly hyperbolic, or onon-uniformly partially hyperbolic. Then for almost every point* $x \in \Lambda$, $W^s(x) \subset \Lambda$ *modulo a* m_s*-measure zero set in* $W^s(x)$ *and* $W^u(x) \subset \Lambda$ *modulo a* m_u*-measure zero set in* $W^u(x)$.

Proof. For any fixed positive integer k and positive number $\eta > 0$, let $\Lambda(\eta, k) \subset \lambda$ be the set such that

$$\frac{m_u(B_i^u(x) \cap \Lambda)}{m_u(B_i(x))} > (1-\eta), \;\; for \;\, all \;\, i \geq k.$$

Since almost every point of Λ is a density point of Λ, for any $\eta > 0$,

$$\lim_{k \to \infty} m_u(\Lambda(\eta, k)) = m_u(\Lambda).$$

Let $x \in \Lambda(\eta, k)$ and $B_0^i(x) = f^{-i}(W_{\delta_i}^u(f^i(x)))$ with $i \geq k$. Then

$$m_u(B_0^i(x) \setminus \Lambda) < \eta m_u(B_0^i(x)).$$

Applying Lemma 2.1 to $B_0^i(x)$ and $B_0^i(x) \setminus \Lambda$, we have

$$\begin{aligned} m_u(f^i(B_0^i(x) \setminus \Lambda)) &\leq C_2 |det(df_x^n|E_x^u)| m_u(B_0^i(x) \setminus \Lambda) \\ &< C_2 \eta |det(df_x^n|E_x^u)| m_u(B_0^i(x)) \\ &\leq C_2 C_1^{-1} \eta |det(df_x^n|E_x^u)| |det(df_x^n|E_x^u)|^{-1} m_u(f^i(B_0^i(x))), \\ &= C_2 C_1^{-1} \eta m_u(f^i(B_0^i(x))) \end{aligned}$$

where $C_1, \; C_2$ are the constants as stated in Lemma 2.1. Therefore

$$m_u(f^i(B_0^i(x)) \cap \Lambda) \geq (1 - C_2 C_1^{-1} \eta) m_u(f^i(B_0^i(x)))$$

That is

$$m_u(W_{\delta_i}^u(f^i(x)) \cap \Lambda) \geq (1 - C_2 C_1^{-1} \eta) m_u(W_{\delta_i}^u(f^i(x)))$$

By *Poincaré* recurrence theorem, for a.e. $x \in \Lambda(\eta, k)$, there exists a sequence of integers $n_i \to \infty$ such that $f^{-n_i}(x) \in \Lambda(\eta, k)$. Now specially apply the above argument to $f^{-n_i}(x)$ with $n_i \geq k$, we have

$$m_u(W_\delta^u(x) \cap \Lambda) \geq (1 - C_2 C_1^{-1} \eta) m_u(W_\delta^u(x))$$

Since the above estimate is independent of k, it must hold for almost all $x \in \cup_{k=1}^{\infty}\Lambda(\eta, k)$, and since $m_u(\cup_{k=1}^{\infty}\Lambda(\eta, k)) = m_u(\Lambda)$ for any fixed $\eta > 0$, this implies that for a.e. $x \in \Lambda$

$$m_u(W_\delta^u(x) \cap \Lambda) \geq (1 - C_2 C_1^{-1}\eta) m_u(W_\delta^u(x))$$

Since this is true for all $\eta > 0$, we have that for a.e. $x \in \Lambda$

$$m_u(W_\delta^u(x) \cap \Lambda) = m_u(W_\delta^u(x))$$

This result on the stable foliation can be proved in the same way by considering f^{-1}.

This proves the Theorem. □

3. Applications

In this section we will give two applications of the main theorem in the last section. The second result on uniformly (partially) hyperbolic systems is known. The original proof involved the well-known *Hopf argument*, introduced by E. Hopf to prove the ergodicity of the goedesic flow of a compact negatively curved surface. The Hopf argument is based on Birkhoff ergodic theorem. Our main point is that, in almost all cases involving the hyperbolicity, the Hopf arguments were unneccessary. As an example, Xia [7] gave a simple and direct proof of ergodicity of the Anosov diffeomorphisms without the use of Hopf argument or Birkhoff ergodic theorem.

3.1. us-saturations.

A set $A \subset M$ is said to be *almost u-saturated* if except for a measure zero set, it consists of whole unstable manifolds. That is, $A = A_0 \cup B$ such that B is a zero set, and if $p \in A_0$, then $W^u(p) \setminus A$ has leaf measure zero. Likewise, we can define *almost s-saturated* for stable leaves. We say that A is *almost us-saturated* if it's both almost u-saturated and almost s-saturated.

In many papers, the Hopf argument are used to get the almost us-saturation of (partially) hyperbolic invariant sets. Actually one can easily get the almost us-saturation as an immediate corollary of Theorem 2.5.

Corollary 3.1. *Let f be a $C^{1+\alpha}$ volume-preserving diffeomorphism of a compact smooth Riemannian manifold and Λ is a uniformly (partially) hyperbolic invariant set for f, then Λ is almost us-saturated.*

Proof. Notice that in the (partially) hyperbolic case, the size of local stable manifolds and unstable manifolds, δ, is a fixed positive constant, uniform for all points. Therefore, the above corollary is an immediate consequence of Theorem 2.5.

We remark that the Corollary 3.1 is not necessarily true for nonuniformly (partially) hyperbolic invariant sets.

3.2. **expanding or contracting center foliations.** A partially hyperbolic diffeomorphism with invariant measure μ is said to have *negative central exponents (on a set Λ)* if for μ-a.e. x (in the set Λ) we have $\lambda(x,v) < 0$ for all nonzero $v \in E^c(x)$. The definition of positive central exponents is similar.

The following theorem is due to K. Burns, D. Dolgopyat and Y. Pesin and it is the main key result in [2]. The proof, as one expects, uses the Hopf arguments. Here we give a simple and direct proof.

Theorem 3.2. *Let f be a $C^{1+\alpha}$ volume-preserving diffeomorphism of a compact smooth Riemannian manifold. Assume that there exists an invariant subset $\Lambda \subset M$ with $m(\Lambda) > 0$ such that f has negative (or positive) central exponents on Λ. Then Λ is open (mod a measure zero set). Moreover if f is topologically transitive on Λ, then f is ergodic on Λ.*

Proof. We assume that f has negative central exponents on Λ. The case with positive central exponents can be dealt in the same way. The set Λ is a nonuniformly hyperbolic invariant set with all Liapunov exponents non-zero. The set Λ can be decomposed into countably many ergodic components. For the first part of the theorem, on openness, we may assume that Λ is actually ergodic.

For almost every point $x \in \Lambda$, the tangent space at x decomposes into (non-uniformly) contracting and expanding direction, $T_xM = E^+(x) \oplus E^-(x)$, where

$$E^+(x) = E^u(x)$$

and

$$E^-(x) = E^s(x) \oplus E^c(x).$$

Let $W^s_{\delta'}(x)$ be the local stable manifold tangent to $E^-(x)$ at x. By Pesin theory, these manifolds exists for almost every point x. However, unlike the uniform case, δ' depends on x, or more precisely, depend on Λ_k as in Definition 1.2., if $x \in \Lambda_k$. However, the size of the local unstable manifold $W^u_\delta(x)$ is independent of point x, since we have uniform expansion on $E^u(x)$. Let

$$N(x) = \bigcup_{y \in W^s_{\delta'}(x)} W^u_\delta(y)$$

Then $N(x)$ is a neighborhood of x. From Theorem 2.5,

$$m_u(W^u_\delta(x) \cap \Lambda) = m_u(W^u_\delta(x))$$

and

$$m_u(W^s_{\delta'}(x) \cap \Lambda) = m_u(W^s_{\delta'}(x))$$

together with the absolute continuity of the strongly unstable local manifolds and stable local manifolds ensure that

$$N(x) \subset \Lambda \ (mod\ 0)$$

If f is furthermore assumed to be topologically transitive, any nonempty open set is dense. Since Λ is open (mod 0) with $m(\Lambda) > 0$, Λ is dense. $f|\Lambda$ is ergodic because there are no nontrivial subset $A \subset \Lambda$ with $A = f(A)$. If there is such a set A, by the argument above, A and $\Lambda \setminus A$ are both open sets (mod 0). f is topologically transitive, therefore there exists some natural number n such that

$$f^n(A) \cap (\Lambda \setminus A) \neq \emptyset$$

and topologically transitive, any nonempty open set is dense. Since Λ is open (mod 0) with $m(\Lambda) > 0$, Λ is dense. $f|\Lambda$ is ergodic because there are no nontrivial subset $A \subset \Lambda$ with $A = f(A)$. If there is such a set A, by the argument above, A and $\Lambda \setminus A$ are both open sets (mod 0). f is topologically transitive, therefore there exists some natural number n such that

$$f^n(A) \cap (\Lambda \setminus A) \neq \emptyset$$

and

$$m(f^n(A) \cap (\Lambda \setminus A)) \neq \emptyset$$

That's impossible since $f^n(A) = A$.

This proves the Theorem. $\square$

In the above thorem, if f is further assumed to be essentially accessible (see [5]), one can easily show that $A = M$ (mod 0) and f is ergodic on M.

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Department of Mathematics, Northwestern University, Evanston, Illinois 60208

E-mail address: xia@math.northwestern.edu

E-mail address: hua@math.northwestern.edu