Math 340, Fall 2011
Lecturer: Yong-Geun Oh

Final Review problems

1. Consider the matrix
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$
   (1) **(5 pts)** Find the characteristic polynomial and eigenvalues of A.
   (2) **(10 pts)** Find an orthonormal basis for the eigenspace corresponding to each eigenvalue.
   (3) **(5 pts)** Find an orthogonal matrix P such that $P^T AP$ becomes a diagonal matrix.

2. **(5 pts each)** Consider the matrix
$$B = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix}$$
   (1) Find the inverse of the matrix B.
   (2) Consider the following matrix equation
$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 3 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$
   Find the matrix $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$.

3. **(5 pts each)** Consider the equation $A\vec{x} = \vec{b}$ with
$$A = \begin{pmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}$$
   Find the orthogonal projection of $\vec{b}$ onto Col A.

4. **(5 pts each)** Consider the matrix
$$A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}.$$
   (1) Find an invertible matrix P such that $P^{-1}AP$ becomes diagonal. Avoid fractional numbers in finding P.

(2) Compute A^n for positive integer n.

5. **(5pts each)** Let $\mathrm{Sym}_n(\mathbb{R})$ be the subset of $n \times n$ symmetric matrices in $M_{n\times n}(\mathbb{R})$ (i.e., the subset of $n \times n$ real matrices A satisfying $A = A^T$).
(1) Prove that $\mathrm{Sym}_n(\mathbb{R})$ is a subspace of $M_{n\times n}(\mathbb{R})$.
(2) Find the dimension of $\mathrm{Sym}_3(\mathbb{R})$.

6. **(5 pts each)** Consider the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^3$ defined by

$$T(\mathbf{p}) = \begin{pmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{pmatrix}$$

(1) Find the polynomial whose value under T is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.
(2) Find the matrix of T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}^2$ and the basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

for $\mathbb{R}^3$.

7. **(5 pts each)** Answer the following equations. You do not have to give the explantions.
(1) Find the inverse of orthogonal matrix U where

$$U = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2) Is the following set orthogonal

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

with respect to the inner product on $\mathbb{R}^3$ defined by

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + 3x_2y_2 + x_3y_3, \quad \text{where } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}?$$

(3) Interpret the Cauchy-Schwarz inequality for the inner product as given in (2).
(4) Check if the vector $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$ an eigenvector of $\begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{pmatrix}$? If so, find the corresponding eigenvalue.