Loop Products and Closed Geodesics

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Loop Products and Closed Geodesics

joint with Mark Goresky

(* preliminary version of manuscript now available on Mark's homepage *)

M^n = compact Riemannian manifold

$\Lambda = \Lambda M = \text{Maps}(S^1, M)$.

Chas-Sullivan product

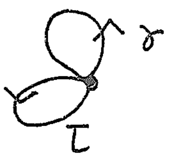
$$H_j(\Lambda M) \times H_k(\Lambda M) \to H_{j+k-n}(\Lambda M)$$

- What is it?
- What does it have to do with closed geodesics?
- What does it have to do with the rate of growth of the index of a closed geodesic?
- Define related cohomology product

$$H^j(\Lambda, \Lambda^0) \times H^k(\Lambda, \Lambda^0) \to H^{j+k+n-1}(\Lambda, \Lambda^0)$$

- Properties of both products.

Note: If $\gamma, \tau \in \Lambda$ have the same basepoint, we can form the product $\gamma \circ \tau \in \Lambda$.



But there is no good way to get a product $\Lambda \times \Lambda \to \Lambda$.

Intuitive, original def of CS product:

$A^j, B^k \subset \Lambda$. Assume $eA \pitchfork eB$, $e: \Lambda \to M$, $e(\gamma) = \gamma(0)$

Then $A * B = \{\gamma \circ \tau \mid \gamma \in A, \tau \in B$ <u>and</u> $\gamma(0) = \tau(0)\}$ has dim $j+k-n$

"pairs of composable loops"

$$[A] \underset{\overset{\uparrow}{CS}}{\bullet} [B] \equiv [A * B].$$

More rigorous definition (R Cohen, M Schwarz, ...)

$$\Lambda \times \Lambda \hookleftarrow \mathcal{F} \hookrightarrow \Lambda$$

codimension n embeddings

$\mathcal{F}$ = figure 8 space $= \{\gamma \in \Lambda \mid \gamma(0) = \gamma(\tfrac{1}{2})\}$

$$H_j(\Lambda) \times H_k(\Lambda) \xrightarrow[\text{Kunneth}]{} H_{j+k}(\Lambda \times \Lambda) \longrightarrow H_{j+k}(\Lambda \times \Lambda, \Lambda \times \Lambda \setminus \mathcal{F})$$

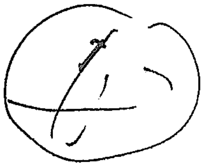
$$\xrightarrow[\cap \tau]{} H_{j+k-n}(\mathcal{F}) \xrightarrow[i_*]{} H_{j+k-n}(\Lambda)$$

Abbondandolo-Schwarz: $(H_*(\Lambda), CS) \simeq (\underset{\text{hom}}{\text{Floer}}(T^*M), \text{pair of pants product})$

Cohen-Klein-Sullivan: homotopy invariant

Example A^0 = 1 great circle γ on S^n

B^{2n-1} = All parameterized great circles on the sphere S^n



$(A \circ B)^{n-1} = \{$ loops $\gamma \circ \tau \mid \tau$ is a great circle starting at $\gamma(0)$. $\}$

$(B \circ B)^{3n-2} = \{$ loops $\sigma \circ \tau \mid \sigma, \tau$ great circles beginning at the same point.

Example Let $\Lambda^0 \approx M$ be the constant loops on M. Then $[\Lambda^0]$ is the identity element of the Chas Sullivan ring.

Geometry: CS product models geometry of closed geodesics with slowest possible index growth.

Background: $E : \Lambda \to \mathbb{R}$ $\quad E(\gamma) = \int |\dot{\gamma}|^2 dt$

Critical points of E = closed geodesics on M

index = Morse index

Morse theory : Topology of Λ $\rightsquigarrow$ Closed geodesics on M

homology in dim λ $\longrightarrow$ closed geodesics index λ

Difficult program! ($\exists$ infinitely many closed geodesics for any metric on S^2 (Birkhoff, Lusternik-Schnirelmann, Bangert, Grayson, Franks) ; S^3??)

Difficulty / Beautiful Structure : $O(2)$ action on Λ

Iterations $\gamma^m(t) = \gamma(mt)$

Each geometric closed geodesic $\Leftrightarrow$ Infinite sequence of $O(2)$-orbits of critical points in Λ.

Bott :
$$\underset{\lambda_m^{\min}}{\underset{\uparrow}{m\lambda_1 - (m-1)(n-1)}} \leq \underset{\lambda_m}{\underset{\uparrow}{\mathrm{Index}(\gamma^m)}} \leq \underset{\lambda_m^{\max}}{\underset{\uparrow}{m\lambda_1 + (m-1)(n-1)}}$$

<u>Q</u> Is there an algebraic operation on $H_*(\Lambda)$ $\Leftrightarrow$ iteration?

CS product

$$x \in H_{\lambda}(\Lambda) \Rightarrow x \circ x \in H_{2\lambda-n}(\Lambda),\ x\circ x\circ x \in H_{3\lambda-2n} \quad ???$$

Does not work.

But $x \in H_{\lambda+1}(\Lambda) \Rightarrow x\circ x \in H_{\lambda_2^{min}+1}(\Lambda), \dots, \underbrace{x\circ x\circ\cdots\circ x}_{m} \in H_{\lambda_m^{min}+1}(\Lambda)$

$\Rightarrow$ models slowest index growth

You can see the nonzero CS products in manifolds

S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, CaP^2 – metrics with all geodesics closed.

These manifolds have a nondegerate critical manifold $\approx T_1M$ of critical points.

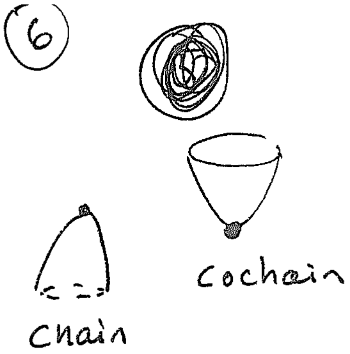
Index + nullity has minimal growth.

Question: Where is the other product?

Geometry suggests there should be another product modeling fastest possible growth.

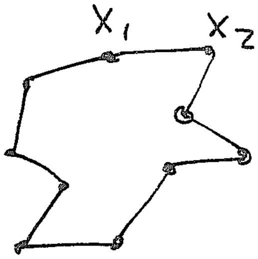
Intuitive, original definition:

CS product on Morse cochains



Finite dim. approx

$$A_N^a = \{(x_1, .., x_N) \mid |x_1 - x_N| = |x_i - x_{i+1}| \leq \tfrac{a}{N}\}$$



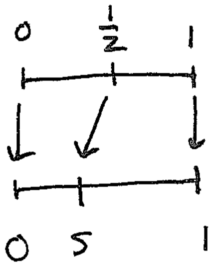
$$H^*(\Lambda) \xleftrightarrow[\text{duality}]{\text{Poincaré}} \text{Morse cochains}$$

Rigorous definition:

$$\begin{tikzcd} \Lambda \times \Lambda & \mathcal{F} \arrow[l, hook'] \arrow[r, hook] & \Lambda & \Lambda \times I \arrow[l, "J"'] \end{tikzcd}$$

$$J : \Lambda \times I \to \Lambda$$

$$J(\gamma, s) = \gamma \circ \theta_{\frac{1}{2} \to s}$$



$$H^j(\Lambda, \Lambda^0) \times H^k(\Lambda, \Lambda^0) \to H^{j+k}(\Lambda \times \Lambda, \Lambda \times \Lambda \setminus \Lambda^{>0} \times \Lambda^{>0}) \to H^{j+k}(\tilde{\mathcal{F}}, \tilde{\mathcal{F}} - \tilde{\mathcal{F}}^{>0,>0})$$

$$\xrightarrow[\cup \tau]{} H^{j+k+n}(\Lambda\ \Lambda \setminus \tilde{\mathcal{F}}^{>0,>0}) \xrightarrow{} H^{j+k+n}(\Lambda \times I, \Lambda \times \partial I \cup \Lambda^0 \times I)$$

$$J^*$$

$$\xrightarrow[\sim]{} H^{j+k+n-1}(\Lambda, \Lambda^0).$$

$$\left(\Lambda^0 = \text{trivial loops} \approx M.\right)$$

Where does the extra dimension come from?

In A_N^a, $\sqrt{E} \longleftrightarrow -\sqrt{E}$

CS product "independent of N"

But A_N^a has singularities

Spurious critical points

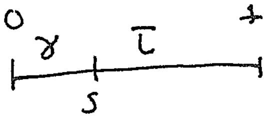
Better FDA (Morse):

$$\mathcal{M}_N^a = \{(x_1,\dots,x_N) \mid \sqrt{N \sum |x_i - x_{i+1}|^2} \le a\} \simeq \Lambda^{\le a}$$

But in $\mathcal{M}_N^a$, $A * B \sim 0$ $\forall$ cochains A, B!

$$[0,1] \times \mathcal{F}^{>0,>0} \longrightarrow \Lambda$$

$$(s, \gamma \cdot \tau) \longmapsto \gamma \cdot_s \tau$$



$$\gamma \cdot_s \tau = \gamma \cdot \tau \circ \theta_{s \to \frac{1}{2}}$$

$\gamma \cdot_{\frac{1}{2}} \tau = \gamma \cdot \tau$. But $\gamma \cdot_0 \tau$ and $\gamma \cdot_1 \tau$ have $\sqrt{E} = \infty$

(if γ, τ have $E > 0$) so trivial.

Define $[A] \bullet [B] = [\,[0,1] \times A * B\,]$.

$$\underset{\substack{\uparrow \\ \text{Morse} \\ \text{cochain}}}{[A]} \; \underset{\substack{\uparrow \\ \text{coh.} \\ \text{product}}}{\bullet} \; \underset{\substack{\uparrow \\ \text{Morse} \\ \text{cochain}}}{[B]}$$

Properties

Family of Products $\Lambda^{\le a} = \{\gamma \in \Lambda \mid \sqrt{E(\gamma)} \le a\}$

$$H_j(\Lambda^{\le a}) \times H_k(\Lambda^{\le b}) \to H_{j+k-n}(\Lambda^{\le a+b})$$

$$H^j(\Lambda, \Lambda^{< a}) \times H^k(\Lambda, \Lambda^{< b}) \to H^{j+k+n-1}(\Lambda, \Lambda^{< a+b})$$

Critical values

$x \in H_*(\Lambda)$ $\quad cr(x) = \inf\{a \in \mathbb{R} \mid x \in \text{Image } H_*(\Lambda^{\le a})\}$

$x \in H^*(\Lambda, \Lambda^0)$ $\quad cr(x) = \sup\{a \in \mathbb{R} \mid x$ is supported on $\Lambda^{\ge a}\}$

then $cr(x)$ is a critical value of $\sqrt{E}$

$cr(x \bullet y) \le cr(x) + cr(y)$ (homology)

$cr(x \bullet y) \ge cr(x) + cr(y)$ (cohomology)

Question of Eliashberg

Given M^n, $d(t) \equiv \text{Max}\left\{k \,\middle|\, \underset{\neq 0}{\text{Image}(H_k(\Lambda^{\le t})) \subseteq H_k(\Lambda)}\right\}$

(maximal degree of an essential homology class at level t)

Does $\exists$ $C \in \mathbb{R}$, indep of metric, $\ni$:

$$d(t_1+t_2) \le d(t_1)+d(t_2)+C \quad ?$$

Ans Yes if $H^*(\Lambda, \Lambda^0)$ is finitely generated as ring.

Examples: S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, CaP^2

$$H_*(\Lambda, \Lambda^0)$$

$$\underset{\text{Associated Graded Ring}}{\rightsquigarrow} H_*(T,M)[T]$$

$$H^*(\Lambda, \Lambda^0)$$

$$\underset{\text{Associated Graded Ring}}{\rightsquigarrow} H^*(T,M)[T]$$

Critical values $\ell, 2\ell, 3\ell, 4\ell, \dots$

Filter homology, cohomology at these levels

Rk. Ring structure $H_*(\Lambda)$ computed by Cohen, Jones, Yan for S^n, P^n

Another property: "Level nilpotence":

$\eta \in H_*(\Lambda)$ $\quad cr(\eta^N) \leq N \cdot cr(\eta)$

η is "level nilpotent" if $cr(\eta^N) < N \cdot cr(\eta)$

some $N < \infty$.

$\eta \in H^*(\Lambda)$ $\quad cr(\eta^N) \geq N \cdot cr(\eta)$

η is "level nilpotent" if $cr(\eta^N) > N \cdot cr(\eta)$.

some $N < \infty$.

Prop If all closed geodesics on M are nondegenerate, then every homology class is level nilpotent

every cohomology class is level nilpotent.

(generic condition)

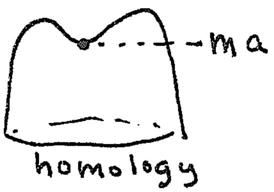
the NON-nilpotent case.

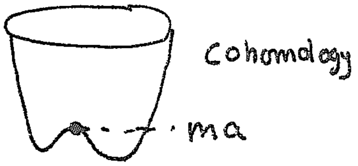
Let γ be an isolated closed geodesic of length a.

"Level" homology ring $\left(\bigoplus_{m\geq 1} H_*(\Lambda^{\leq ma}, \Lambda^{<ma}), \bullet\right)$

"Level" cohomology ring $\left(\bigoplus_{m\geq 1} H^*(\Lambda^{\leq ma}, \Lambda^{<ma}), \bullet\right)$

Old theorems restated:





Assume $x \in \begin{cases} H_*(\Lambda^{\leq a}, \Lambda^{<a}) \\ H^*(\Lambda^{\leq a}, \Lambda^{<a}) \end{cases}$ is non nilpotent.

then $\forall \varepsilon > 0$, if $m \in \mathbb{Z}$ is sufficiently large, there is a critical value of $\sqrt{E}$ in $\begin{cases} (ma, ma+\varepsilon) \\ (ma-\varepsilon, ma) \end{cases}$.

As a consequence, M has infinitely many closed geodesics.

Related products

Homology bracket

$\{x, y\} \in H_{j+k-n+1}(\Lambda)$

Cohomology bracket

$\{x, y\} \in H^{j+k+n-2}(\Lambda, \Lambda^0)$

Equivariant homology product

$H_j^{\mathbb{T}}(\Lambda) \times H_k^{\mathbb{T}}(\Lambda) \to H^{\mathbb{T}}_{j+k+2-n}(\Lambda)$

Equivariant cohomology product

$H^j_{\mathbb{T}}(\Lambda, \Lambda^0) \times H^k_{\mathbb{T}}(\Lambda, \Lambda^0) \to H^{j+k+n-2}_{\mathbb{T}}(\Lambda, \Lambda^0)$

$\mathbb{Q}$ coefficients!

Proof $d(t_1+t_2) \leq d(t_1)+d(t_2)+2n+g-2$

if all generaters of $H^*(\Lambda, \Lambda^0)$ have degree $\leq g$.

Let $t_1, t_2, t_3 \in \mathbb{R}^+$; $d_i = d(t_i)$. Will show:

$$d_3 > d_1+d_2+2n+g-2 \Rightarrow t_3 > t_1+t_2.$$

By def, $\exists$ a class $z \in H_{d_3}(\Lambda, \Lambda^0)$ in image of $H_{d_3}(\Lambda^{\leq t_3}, \Lambda_0)$

$\exists$ $Z \in H^{d_3}(\Lambda, \Lambda^0)$: $[Z, z] \neq 0$.

Write $Z = U_1 \bullet U_2 \bullet \cdots \bullet U_g +$ ~~other terms~~

$\deg(U_i) \leq g$

$= X \bullet Y$, $\deg X > d_1$

$\deg Y > d_2$

$$\Rightarrow \begin{tikzcd} \hat{X} \in H^*(\Lambda, \Lambda^{\leq t_1}) \arrow[d] & \hat{Y} \in H^*(\Lambda, \Lambda^{\leq t_2}) \arrow[d] \\ X \in H^*(\Lambda, \Lambda^0) \arrow[d] & Y \in H^*(\Lambda, \Lambda^0) \\ 0 \in H^*(\Lambda^{\leq t_1}, \Lambda^0) & \end{tikzcd}$$

$$\begin{tikzcd} \hat{X} \bullet \hat{Y} = \hat{Z} \in H^*(\Lambda, \Lambda^{\leq t_1+t_2}) \arrow[d] \\ Z \in H^*(\Lambda, \Lambda^0) \end{tikzcd}$$

$\Rightarrow Z \equiv 0$ on $\Lambda^{\leq t_1+t_2}$

$\Rightarrow t_3 > t_1+t_2$