

Solution set for Homework (due March 5)

– Partially based on Diane Holcomb’s Homework –

Problem 0. Let $s_0 \in S^1$ consider the pair $(S^1 \times Y, \{s_0\} \times Y)$. Prove that the connecting homomorphism

$$\delta : H^k(\{s_0\} \times Y; R) \rightarrow H^{k+1}(S^1 \times Y, \{s_0\} \times Y; R)$$

is trivial.

Solution. We start with the case $Y = \{pt\}$ as a warm-up, i.e., consider the pair $(S^1, \{s_0\})$. In this case, it is enough to prove $\delta : H^0(\{s_0\}; R) \rightarrow H^1(S^1, \{s_0\}; R)$ is trivial. Let $\varphi \in C^0(\{s_0\}; R) = H^0(\{s_0\}; R) \cong R$ be a cocycle. Let $\varphi(s_0) = r_0 \in R$. The extension $\bar{\varphi}$ is nothing but

$$\bar{\varphi} = \chi_{s_0} \cdot r_0.$$

Then by definition, we have

$$\delta[\varphi] = [\delta_{S^1}\bar{\varphi}] \in H^1(S^1; R).$$

To prove we have only to find a cochain $\psi \in C^0(S^1; R)$ such that

$$\psi|_{C^0(\{s_0\}; R)} = 0, \quad \delta_{S^1}\psi = \delta_{S^1}\bar{\varphi}.$$

Now we denote by $r_0\mathbf{1}$ the 0-cochain having constant value r_0 . Then we just consider the 0-cochain given by

$$\psi = \bar{\varphi} - r_0\mathbf{1}.$$

For any 0-simplex σ with its image contained $\{s_0\}$, we have

$$\psi(\sigma) = \bar{\varphi}(\sigma) - r_0\mathbf{1}(\sigma) = \varphi(s_0) - r_0 = r_0 - r_0 = 0.$$

Therefore $\psi \in C^0(S^1, \{s_0\}; R)$.

Next we note that *the cochain $r_0\mathbf{1}$ is a cocycle* : This is because for any 1-simplex $\sigma : [0, 1] \rightarrow S^1$, the value $r_0\mathbf{1}(\sigma(0)) = r_0 = r_0\mathbf{1}(\sigma(1))$. Therefore we have

$$\delta_{S^1}(\psi) = \delta_{S^1}\bar{\varphi} - \delta_{S^1}(r_0\mathbf{1}) = \delta_{S^1}\bar{\varphi}.$$

This proves $\delta_{S^1}\bar{\varphi}$ is indeed a coboundary in $C^1(S^1, \{s_0\}; R)$. This finishes the proof for the case $Y = \{pt\}$.

The above discussion can be immediately generalized to the case of general Y . Let $\varphi \in C^k(\{s_0\} \times Y; R)$ be a cocycle representing a cohomology class $\alpha \in H^k(\{s_0\} \times Y; R)$. We denote by $\bar{\varphi} : C^k(S^1 \times Y, \{s_0\} \times Y; R)$ the extension of φ by zero. Then by definition, we have

$$\delta\alpha = [\delta_{S^1 \times Y}(\bar{\varphi})].$$

To prove $\delta\alpha = 0$, it is enough to find a cocycle $\psi \in C^k(S^1 \times Y)$ such that

$$\psi|_{C^k(\{s_0\} \times Y)} = 0, \quad \delta_{S^1 \times Y}(\psi) = \delta_{S^1 \times Y}(\bar{\varphi}).$$

This $\bar{\varphi}$ can be written as

$$\bar{\varphi} = \chi_{s_0} \times \varphi_{s_0}$$

where $\varphi_{s_0} \in C^k(Y; R)$ defined by $\varphi_{s_0}(\sigma) = \varphi(\{s_0\} \times \sigma)$ for any chain $\sigma : \Delta^k \rightarrow Y$, and $\chi_{s_0} : S^1 \rightarrow \mathbb{Z}$ the characteristic function of $s_0 \in S^1$. In other words, $\varphi_{s_0} = i_{s_0}^*\varphi$ where $i_{s_0} : Y \rightarrow S^1 \times Y$ is the inclusion map $i_{s_0}(y) = (s_0, y)$. In particular φ_{s_0} is a cocycle on Y .

More explicitly, $\bar{\varphi}$ has its values given by

$$\bar{\varphi}(\sigma) = \begin{cases} \varphi(\sigma) & \text{if } \sigma \in C_k(\{s_0\} \times Y; R) \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $\pi : S^1 \times Y \rightarrow Y$ the projection. We consider a k -cochain given by

$$\pi^* \varphi_{s_0} = \pi^* i_{s_0}^* \varphi = (i_{s_0} \circ \pi)^* \varphi$$

which is a cocycle on $S^1 \times Y$. We check $\pi^* \varphi_{s_0}(\sigma) = \varphi(i_{s_0} \circ \pi(\sigma)) = \varphi(\sigma) = \bar{\varphi}(\sigma)$ whenever $\sigma \in C_k(\{s_0\} \times Y) \subset C_k(S^1 \times Y)$.

Therefore $\psi := \bar{\varphi} - \pi^* \varphi_{s_0}$ is a cocycle living in $C^k(S^1 \times Y, \{s_0\} \times Y; R)$ such that

$$\delta_{(S^1 \times Y)}(\bar{\varphi}) = \delta_{(S^1 \times Y)}(\psi).$$

This finishes the proof.

Problem 1. Assuming as know the cup product structure on the torus $S^1 \times S^1$, compute the cup product structure in $H^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to the wedge sum of g tori.

Solution: We have a quotient map from the surface of genus g to the wedge sum of g tori where $f : M_g \rightarrow \bigvee(S^1 \times S^1)$ induces a map on the first homology groups as follows. We have that $H_1(X)$ is the free abelian group on $2g$ generators for both M_g and $\bigvee(S^1 \times S^1)$. So if we have $\bar{\alpha}_i, \bar{\beta}_i$ for $1 \leq i \leq g$ are the generators for $H_1(M_g)$ and α_i, β_i the generators for the wedge sum then $f_* : H_1(M_g) \rightarrow H_1(\bigvee(T))$ is given by $\bar{\alpha}_i \mapsto \alpha_i$ and $\bar{\beta}_i \mapsto \beta_i$, since the standard representatives of these cycles (meridians and longitudes of the torus) are sent to each other by f . Thus also $f^*(\beta_i^*) = \bar{\beta}_i^*$ and $f^*(\alpha_i^*) = \bar{\alpha}_i^*$.

We have that $\alpha_i^* \smile \alpha_j^* = 0$, $\beta_i^* \smile \beta_j^* = 0$, and $\alpha_i^* \smile \beta_j^* = \delta_{ij} \gamma_i$, where γ_i is the generator of H^2 of the i th torus. But, since $f_*(\bar{\gamma}) = \sum_{i=1}^g \gamma_i$ (since both are the entire surface), we have that $f^*(\bar{\gamma}_i) = \bar{\gamma}_i$ for all i . Then, $\delta_{ij} \bar{\gamma}^* = \delta_{ij} f^*(\gamma_i^*) = f^*(\alpha_i^* \smile \beta_j^*) = f^*(\alpha_i^*) \smile f^*(\beta_j^*) = \bar{\alpha}_i^* \smile \bar{\beta}_j^*$ for all i, j . Similarly, we see that $\bar{\alpha}_i^* \smile \bar{\alpha}_j^* = 0 = \bar{\beta}_i^* \smile \bar{\beta}_j^*$ for all i, j .

Problem 2. Using the cup product $H^k(X, A; R) \times H^l(X, B; R) \rightarrow H^{k+l}(X, A \cup B; R)$, show that if X is the union of contractible open subsets A and B , then all cup products of positive dimensional classes in $H^*(X; R)$ are zero. Generalize to the situation that X is the union of n contractible open subsets, to show that all n -fold cup products of positive dimensional classes are zero.

Solution: If we consider the map

$$H^k(X, A; R) \times H^l(X, B; R) \rightarrow H^{k+l}(X, A \cup B; R)$$

we have that $H^{k+l}(X, A \cup B; R) = 0$ since $X = A \cup B$ and so the map is trivial. We also have that for $k > 0$ $H^k(X, A; R) \simeq H^k(X; R)$ and $H^l(X, B; R) \simeq H^l(X; R)$ since A and B are contractible. Furthermore we have that $C^n(X, A) \subseteq C^n(X)$, therefore for $\alpha \in H^k(X, R)$ we can choose some representative ϕ in $H^k(X, A; R)$. Similarly we can choose a representative φ in $H^l(X, B; R)$ therefore if we consider $\alpha \smile \beta$ with $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$ we can choose representatives $\phi \in H^k(X, A; R)$ and $\varphi \in H^l(X, B; R)$ so $\alpha \smile \beta = [\phi \smile \varphi] = 0$. Now, assume that X is the union of n contractible sets $\{A_i\}_{i=1}^n$, and consider any n -fold cup product of positive dimensional classes $\alpha_1 \smile \alpha_2 \smile \dots \smile \alpha_n$. Again, as above, we can choose

representatives of each α_i from $H^{k_i}(X, A_i)$. Then since $H^m(X, \cup_{i=1}^n A_i) = 0$ for any m , the n -fold cup product is also trivial.

Problem 3. (a) *Using the cup product structure, show there is no map $\mathbb{R}P^n \rightarrow \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m; \mathbb{Z}_2) \rightarrow H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}P^n \rightarrow \mathbb{C}P^m$?*

(b) *Prove the Borsuk-Ulam theorem using (a).*

Solution for (a): Let f be a map from $\mathbb{R}P^n \rightarrow \mathbb{R}P^m$, where $n > m$. We recall that $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$. For clarity, we write $H^*(\mathbb{R}P^m; \mathbb{Z}_2) \cong \mathbb{Z}_2[\beta]/(\beta^{m+1})$. Assume that f^* is non-trivial on H^1 . Then $f^*(\beta) = \alpha$. Hence, $0 = f^*(0) = f^*(\beta^m) = (f^*(\beta))^m = \alpha^m$, a contradiction since $\alpha^m \neq 0$ since $m < n$.

The corresponding result for $\mathbb{C}P^n$ is that no map $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^m$ can induce a nontrivial map $H^2(\mathbb{C}P^n; \mathbb{Z}) \rightarrow H^2(\mathbb{C}P^m; \mathbb{Z})$ if $m < n$. Since $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, the argument is precisely the same as above.

Solution for (b): We follow the suggestion in the problem. Suppose to the contrary that there exists a continuous map $f : S^n \rightarrow \mathbb{R}^n$ such that $f(x) \neq f(-x)$ for all $x \in S^n$. Then we define a map $g : S^n \rightarrow S^{n-1}$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

Then g defines a continuous map $g : S^n \rightarrow S^{n-1}$. Furthermore g satisfies $g(-x) = -g(x)$. This relation makes g induces a continuous map $\tilde{g} : \mathbb{R}P^n \rightarrow \mathbb{R}P^{n-1}$.

Denote by α_n a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}_2)$.

Claim: This map must induce a non-trivial homomorphism $g^* : H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \rightarrow H^1(\mathbb{R}P^n; \mathbb{Z}_2)$. This then will give rise to a contradiction to (a) and hence the proof.

It remains to prove the claim. Suppose to the contrary $g^*\alpha_n = 0$ for a generator of $H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$. We consider the map from $\mathbb{R}P^1$ to $\mathbb{R}P^n$ or $\mathbb{R}P^{n-1}$ such that $[x_1; x_2]$ is mapped to the first two coordinates of $\mathbb{R}P^n$ or $\mathbb{R}P^{n-1}$. Denote by i and j the corresponding map respectively. We note that both maps induces isomorphisms on π_1 . Therefore $j^*\alpha_n \neq 0$ in $H^1(\mathbb{R}P^1; \mathbb{Z}_2) \cong \text{Hom}(H_1(\mathbb{R}P^1), \mathbb{Z}_2)$. On the other hand we have $(g \circ i)^*\alpha_n = i^*(g^*\alpha_n) = 0$. Therefore $j^* \neq (g \circ i)^*$. In particular, $g \circ i : \mathbb{R}P^1 \cong S^1 \rightarrow \mathbb{R}P^n$ is not homotopic to $j : \mathbb{R}P^1 \rightarrow \mathbb{R}P^n$. Since $\pi_1(\mathbb{R}P_n) \cong \mathbb{Z}_2$ and j is not null-homologous, $g \circ i$ is homotopic to zero. In particular $g_* : \pi_1(\mathbb{R}P^n) \rightarrow \pi_1(\mathbb{R}P^{n-1})$ is the zero map. By the homotopy lifting theorem, g has a lifting $\tilde{g} : \mathbb{R}P^n \rightarrow S^{n-1}$ for the covering projection $\pi_{n-1} : S^{n-1} \rightarrow \mathbb{R}P^{n-1}$, i.e., $g = \pi_{n-1} \circ \tilde{g}$.

Composing with π_n , we obtain

$$g \circ \pi_n = \pi_{n-1} \circ \tilde{g} \circ \pi_n.$$

Recalling $g \circ \pi_n = \pi_{n-1} \circ g$ by definition of g , both g and $\tilde{g} \circ \pi_n$ are two liftings of the map

$$g \circ \pi_n : S^n \rightarrow \mathbb{R}P^{n-1}.$$

On the other hand, for $x \in S^n$, since $g(-x) = -g(x)$, either $\tilde{g} \circ \pi_n(x) = g(x)$ or $\tilde{g} \circ \pi_n(x) = -g(x) = g(-x)$ must hold, which implies that *the liftings g and $\tilde{g} \circ \pi_n$ agree with at least one point on S^n* . These two statements in italics imply $g \equiv \tilde{g} \circ \pi_n$. Since by hypothesis $g(-x) = -g(x)$, i.e., g is *odd* while $\tilde{g} \circ \pi_n(-x) = \tilde{g} \circ \pi_n(x)$, i.e., $\tilde{g} \circ \pi_n$ is *even*, this is a contradiction. This finishes the proof of Claim.

Problem 4. Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ has a fixed point if n is even, using the fact that $f^* : H^*(\mathbb{C}P^n; \mathbb{Z}) \rightarrow H^*(\mathbb{C}P^n; \mathbb{Z})$ is a ring homomorphism. When n is odd show that there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $H^2(\mathbb{C}P^n; \mathbb{Z})$.

Solution: Let $f : \mathbb{C}P^n \rightarrow \mathbb{C}P^n$ be any map. Again, $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$. Hence, $f^*(\alpha) = k\alpha$ for some integer k . Then $f^*(\alpha^n) = k^n\alpha^n$. Then we note that

$$\tau = \sum_i (-1)^i \text{trace}(f^i) = \sum_{i=0}^n k^i.$$

This number is clearly not zero if $k = 1$, and is equal to $(1 - k^{n+1})/(1 - k)$ if $k \neq 1$. This can only be zero if n is odd and $k = -1$. Thus $\tau \neq 0$ and f has a fixed point by the Lefschetz fixed point theorem, unless n is odd and $f^*(\alpha) = -\alpha$.