Solution set for Homework (due April 2)

– Partially based on Jaeho Lee's Homework –

Problem 5. *Show that the ring $H^*(\mathbb{R}P^\infty;\mathbb{Z}_m)$ is isomorphic to $\mathbb{Z}_m[x,y]/(2x,2y,x^2)$ if $m>2$, where $|x|=1$ and $|y|=2$.* (**Warning :** This seems to be true only for $m\equiv 0 \mod 4$ is even. See the remark in bold in the solution below.)

Solution. Consider the cellular cochain complex for $\mathbb{R}P^\infty$ with $\mathbb{Z}_m$ coefficients

$$0\to\mathbb{Z}_m\xrightarrow{0}\mathbb{Z}_m\xrightarrow{2}\mathbb{Z}_m\xrightarrow{0}\mathbb{Z}_m\xrightarrow{2}$$

We consider the cases $m=2k$ and $m=2k-1$ separately.

Case I, $m=2k$**:** From the above cochain complex (or by the Universal Coefficient Theorem), we obtain

$$\begin{aligned} H^0(\mathbb{R}P^\infty;\mathbb{Z}_{2k}) &\cong \mathbb{Z}_{2k}\\ H^n(\mathbb{R}P^\infty;\mathbb{Z}_{2k}) &\cong \begin{cases}\{0,k\}\cong\mathbb{Z}_2 & \text{for odd } n\geq 1\\ \mathbb{Z}_{2k}/2\mathbb{Z}_{2k}\cong\mathbb{Z}_2 & \text{for even } n\geq 2\end{cases}\end{aligned}$$

Denote by α and β generators of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_{2k})$, and $H^2(\mathbb{R}P^\infty;\mathbb{Z}_{2k})$ respectively. Then we obtain

$$2\alpha=0=2\beta$$

as $H^1\cong H^2\cong\mathbb{Z}_2$ by the above isomorphism.

Let $\pi:\mathbb{Z}_{2k}\to\mathbb{Z}_2$ be the canonical projection. Then π induces a chain map of cellular cochain complexes for $\mathbb{R}P^\infty$ with $\mathbb{Z}_{2k}$ and $\mathbb{Z}_2$ coefficients :

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_{2k} \arrow[r, "0"] \arrow[d, "\pi"] & \mathbb{Z}_{2k} \arrow[r, "2"] \arrow[d, "\pi"] & \mathbb{Z}_{2k} \arrow[r, "0"] \arrow[d, "\pi"] & \mathbb{Z}_{2k} \arrow[r, "2"] \arrow[d, "\pi"] & \mathbb{Z}_{2k} \arrow[d, "\pi"] \\ 0 \arrow[r] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2 \arrow[r, "0"] & \mathbb{Z}_2 \end{tikzcd}$$

If k is odd, π induces a ring isomorphism

$$\overline{\pi}:\widetilde{H}^*(\mathbb{R}P^\infty;\mathbb{Z}_{2k})\to\widetilde{H}^*(\mathbb{R}P^\infty;\mathbb{Z}_2).$$

Furthermore we must have $\alpha^2=\beta(=k\beta)$ as their images under $\overline{\pi}$ satisfy the relation. Hence we obtain the relations

$$2\alpha=0, 2\beta=0, \alpha^2-\beta=0.$$

Moreover α^n is a generator of $H^n(\mathbb{R}P^\infty;\mathbb{Z}_{2k})$. Now we consider the map $\mathbb{Z}_{2k}[x,y]\to H^*(\mathbb{R}P^\infty;\mathbb{Z}_{2k})$ mapping x to α and y to β which is surjective. Its kernel is precisely the ideal generated by

$$2x,\ 2y,\ x^2-y$$

and hence $H^*(\mathbb{R}P^\infty;\mathbb{Z}_{2k})\cong\mathbb{Z}_{2k}[x,y]/(2x,2y,x^2-y)$ with $|x|=1$, $|y|=2$. (**If one insists as in the answer in the textbook**, one can check that the ideal $(2x,2y,x^2-y)$ is the same as $(2x,2(x-y),(x-y)^2)$. Then setting $z=x-y$, one may say that the ring is isomorphic to $\mathbb{Z}_{2k}[x,z]/(2x,2z,z^2)$ with $|x|=1$ but $|z|$ is a mixture of degree 1 and degree 2 elements. But **I doubt** that the author meant this. So the problem seems to contain some error. See also the case $m=2k-1$ below.)

Now consider the case *k is even.* In this case, the induced homomorphism satisfies that $\overline{\pi}:H^n(\mathbb{R}P^\infty;\mathbb{Z}_{2k})\to H^n(\mathbb{R}P^\infty;\mathbb{Z}_2)$ is an isomorphism if $n\geq 2$ is even and zero if $n \geq 1$ is odd. Again $\alpha \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ and $\beta \in H^2(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ be as above. Now we consider the commutative diagram

$$\begin{tikzcd}
H^1(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \times H^1(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \arrow[r] \arrow[d, "\overline{\pi} \times \overline{\pi}"] & H^2(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \arrow[d, "\overline{\pi}"] \\
H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \times H^1(\mathbb{R}P^\infty; \mathbb{Z}_2) \arrow[r] & H^2(\mathbb{R}P^\infty; \mathbb{Z}_2)
\end{tikzcd} .$$

Therefore we have $\overline{\pi}(\alpha \cup \alpha) = (\overline{\pi}\alpha) \cup (\overline{\pi}\alpha) = 0 \cup 0 = 0$. Since $\overline{\pi}$ in H^2 is an isomorphism, we have $\alpha \cup \alpha = 0$.

Note that β^n is a generator of $H^{2n}(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ as $\overline{\pi}(\beta)$ is a generator of $H^2(\mathbb{R}P^\infty; \mathbb{Z}_2)$, $\overline{\pi}(\beta^n) = (\overline{\pi}(\beta))^n$ and we know from the isomorphism $H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[x]$ with $|x| = 1$.

Next we check that $\alpha\beta^n$ is a generator of $H^{2n+1}(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$. For this one, we can repeat the proof of the statement "*The cup product of a generator of $H^{2n}(P^{2n+1})$ with a generator of $H^1(P^{2n+1})$*" in the course of the proof of Theorem 3.12, with $\mathbb{Z}$ coefficients replaced by $\mathbb{Z}_{2k}$ coefficients. This proves that α, β generates $H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$ as a ring with the relation

$$2\alpha = 0,\ 2\beta = 0,\ \alpha \cup \alpha = 0$$

and hence the ring is isomorphic to $\mathbb{Z}_{2k}[x, y]/(2x, 2y, x^2)$.

Case $m = 2k + 1$: In this case, in the cellula cochain complex

$$0 \to \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{2} \mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_m \xrightarrow{2}$$

the map, multiplication by 2 is an isomorphism. Then it immediately follows from this that $H^n(\mathbb{R}P^\infty; \mathbb{Z}_{2k-1}) = \mathbb{Z}_{2k-1}$ for n even and 0 for odd n. Again a modification of the proof of Theorem 3.12 with $\mathbb{Z}$ coefficients replaced by $\mathbb{Z}_{2k}$ coefficients, and the statement "*The cup product of a generator of $H^{2n}(P^{2n+2})$ with a generator of $H^2(P^{2n+2})$*" by "*The cup product of a generator of $H^{2n}(P^{2n+2})$ with a generator of $H^2(P^{2n+2})$*" proves that

$$H^*(\mathbb{R}P^{2n}) = \mathbb{Z}_{2k-1}[x]/(x^{n+1}).$$

(**Check this!**) By letting $n \to \infty$, **we obtain** $H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k-1}) \cong \mathbb{Z}_{2k-1}[x]$ **with** $|x| = 2$.