Solution set for Homework (due April 14)

– Partially based on Diane Holcomb's Homework –

Problem 10. *Show that the cross product map $H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \to H^*(X \times Y;\mathbb{Z})$ is not an isomorphism if X and Y are infinite discrete sets.*

Solution: Consider the structure of $H^*(X;\mathbb{Z})\otimes H^*(Y;\mathbb{Z})$ this has the form $\prod_{i=1}^{\infty} a_i \otimes \prod_{j=1}^{\infty} b_j$ which is the set of all finite sums of elements of the form $a_i \otimes b_j$. Then if we consider structure of $H^*(X\times Y;\mathbb{Z})$ then this contains the element $\sum_{i=1}^{\infty} a_i \times b_i$ which is and infinite sum, meaning that $\sum_{i=1}^{\infty} a_i \times b_i$ has no preimage under the map

$$H^*(X;\mathbb{Z}) \otimes H^*(Y;\mathbb{Z}) \xrightarrow{\times} H^*(X \times Y;\mathbb{Z})$$

and so the map is not an isomorphism.

Problem 11. *Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$, assuming $k > 0$ and $l > 0$.*

Solution: Let $f : S^{k+l} \to S^k \times S^l$ then f induces a map f^* on cohomology. By the Künneth formula we have that $H^*(S^k \times S^l;\mathbb{Z}) \simeq H^*(S^k;\mathbb{Z}) \otimes H^*(S^l;\mathbb{Z})$ which gives us that f^* may be understood as a map $f^* : H^*(S^k;\mathbb{Z}) \otimes H^*(S^l;\mathbb{Z}) \to H^*(S^{k+l};\mathbb{Z})$ that is

$$f^* : \mathbb{Z}[x]/x^2 \otimes \mathbb{Z}[y]/y^2 \to \mathbb{Z}[w]/w^2$$

where $|x| = k$, $|y| = l$ and $|w| = k + l$. It suffices to understand this map on the generators. Since x has order n and there are no elements of order n in $H^*(S^{k+l};\mathbb{Z})$ we have that $f^*(x) = 0$ and similarly $f^*(y) = 0$ which gives us that $f^*(xy) = 0$. Lastly if we consider the coefficients we have that $f^*(1) = 1$ for this to be a ring homomorphism.

Now to understand $f_* : H_{k+l}(S^{k+l}) \to H_{k+l}(S^k \times S^l)$ consider that $f^* : \text{Hom}\,(H_{k+l}(S^k \times S^l),\mathbb{Z}) \to hm(H_{k+l}(S^{k+l}),\mathbb{Z})$ is the 0 map, so $f^*\phi = \phi f_* = 0$ for all ϕ so $f_* = 0$.

Problem 12. *Show that the spaces $(S^1 \times \mathbb{C}P^\infty)/(S^1 \times \{x_0\})$ and $S^3 \times \mathbb{C}P^\infty$ have isomorphic cohomology rings with $\mathbb{Z}$ or any other coefficients.*

Solution: We begin by finding the cohomology rings of $S^3 \times \mathbb{C}P^\infty$ and $(S^1 \times \mathbb{C}P^\infty)/(S^1 \times \{x_0\})$. For $S^3 \times \mathbb{C}P^\infty$ we can use the Künneth formula to give us that

$$H^*(S^3 \times \mathbb{C}P^\infty;\mathbb{Z}) \simeq H^*(S^3;\mathbb{Z}) \otimes H^*(\mathbb{C}P^\infty;\mathbb{Z}).$$

Let α and β be the ring generator of $H^*(S^3;\mathbb{Z})$ and $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ respectively. We note that $|\alpha| = 3$ and $|\beta| = 2$ and so they commute each other in $H^*(S^3 \times \mathbb{C}P^\infty;\mathbb{Z})$, i.e., $\alpha \cup \beta = \beta \cup \alpha$. They satisfy $\alpha^2 = 0$. Therefore we have

$$\begin{aligned} H^*(S^3 \times \mathbb{C}P^\infty;\mathbb{Z}) &\simeq \mathbb{Z}[x]/x^2 \otimes \mathbb{Z}[y] \\ &\simeq \mathbb{Z}[x,y]/\langle x^2 \rangle. \end{aligned}$$

where $|x| = 3$ and $|y| = 2$ and $[x] = \alpha$, $[y] = \beta$.

Now for $(S^1 \times \mathbb{C}P^\infty)/(S^1 \times \{x_0\})$ we consider the pair $X = S^1 \times \mathbb{C}P^\infty$, $A = S^1 \times \{x_0\}$. Denote by θ a generator of $H^1(S^1 \times \mathbb{C}P^\infty)$. Again using the Künneth formula and β has even degree, we can get that

$$H^*(X, \mathbb{Z}) \simeq \mathbb{Z}[a, b]/\langle a^2 \rangle$$

with $|a| = 1$, $|b| = 2$ and $[a] = \theta$, $[b] = \beta$. Now we consider the map $f : X \to X/A$, which gives us a ring homomorphism $f^* : H^*(X/A; \mathbb{Z}) \to H^*(X; \mathbb{Z})$.

We will show that this homomorphism is injective. We know that both X/A and X are connected and so $H^0(X/A) \to H^0(X)$ is an isomorphism.

For $n \geq 1$, we look at the long exact sequence of the pair.

$$\tilde{H}^0(A) \longrightarrow H^1(X/A) \longrightarrow H^1(X) \longrightarrow H^1(A) \longrightarrow H^2(X/A) \longrightarrow H^2(X) \longrightarrow H^2(A) \longrightarrow$$

$$H^3(X, A) \longrightarrow H^3(X) \longrightarrow H^3(A) \longrightarrow H^4(X/A) \longrightarrow \cdots$$

We also know $H^n(A) = 0$ for $n \geq 2$ meaning that $f^* : H^n(X/A) \to H^n(X)$ is an isomorphism for H^n with $n \geq 3$, so the only point requiring work is $n = 1$ and 2. For $n = 1$, since $\tilde{H}^0(A) = 0$, the map $f^* : H^1(X/A) \to H^1(X)$ is injective. On the other hand, we show that the map $H^1(A) \to H^2(X, A) \cong H^2(X/A)$ is zero by considering an obvious cell decomposition of $X = S^1 \times \mathbb{P}^\infty$. Therefore this gives us that the long exact sequence of the form

$$0 \longrightarrow H^2(X/A) \longrightarrow H^2(X) \longrightarrow 0 \longrightarrow H^3(X/A) \longrightarrow \cdots$$

Which give us that f^* is an isomorphism in H^2. This gives us that $H^*(X/A; \mathbb{Z})$ is a subring of $H^*(X; \mathbb{Z}) = \mathbb{Z}[a, b]/\langle a^2 \rangle$ with $H^*(X/A; \mathbb{Z}) = H^*(X; \mathbb{Z})$ for $* \geq 2$. We also note that $[ab^k]$ is a generator of $H^{2k+1} \cong \mathbb{Z}$ for $k \geq 1$ and $[b^k]$ one of $H^{2k} \cong \mathbb{Z}$. Therefore $H^*(X/A; \mathbb{Z})$ is generated by ab and b as a ring.

We also note that $[a]$ and $[b]$ *commutative*, not just graded commutative, in $H^*(X/A; \mathbb{Z})$ as $|b| = 2$ even.

Now we consider the map $\mathbb{Z}[u, v] \to H^*(X/A; \mathbb{Z})$ defined by

$$u \mapsto [ab],\ v \mapsto [b].$$

Then its kernel is the ideal generated by u^2 and so we have proven $H^*(X/A; \mathbb{Z}) \simeq \mathbb{Z}[u, v]/\langle u^2 \rangle$ with $|u| = 3$ and $|v| = 2$. An isomorphism between $(S^1 \times \mathbb{C}P^\infty)/(S^1 \times \{x_0\})$ and $S^3 \times \mathbb{C}P^\infty$ is given by $\alpha \mapsto [ab]$ and $\beta \mapsto [b]$.

Problem 16. Show that if X and Y are finite CW complexes such that $H^*(X; \mathbb{Z})$ and $H^*(Y; \mathbb{Z})$ contain no elements of order a power of a given prime p, then the same is true for $X \times Y$.

Solution: Since X and Y are finite CW complexes we can use the fundamental theorem of finitely generated abelian groups to get that $H^*(X; \mathbb{Z}) = \mathbb{Z}^r \oplus_{i=1}^{N} \mathbb{Z}_{n_i}^{m_i}$ and similarly

$H^*(Y;\mathbb{Z}) = \mathbb{Z}^s \oplus_{j=1}^M \mathbb{Z}_{a_j}^{b_j}$ so if there are no elements of order a power of a given prime p then we have that there is no a_j or n_i a power of p. Therefore if we consider the tensor product of these two things we have

$$(\mathbb{Z}^r \oplus_{i=1}^N \mathbb{Z}_{n_i}^{m_i}) \otimes (\mathbb{Z}^s \oplus_{j=1}^M \mathbb{Z}_{a_j}^{b_j}) = \mathbb{Z}^r \otimes \mathbb{Z}^s \oplus (\oplus_{j=1}^M (\mathbb{Z}^r \otimes \mathbb{Z}_{a_j}^{b_j})) \oplus (\oplus_{i=1}^N (\mathbb{Z}^s \otimes \mathbb{Z}_{n_i}^{m_i}) \oplus (\oplus_{i,j} (\mathbb{Z}_{n_i}^{m_i} \otimes \mathbb{Z}_{a_j}^{b_j})).$$

Here we note that

$$\mathbb{Z}^r \otimes \mathbb{Z}_{a_j}^{b_j} \cong \mathbb{Z}_{a_j}^{rb_j}, \quad \mathbb{Z}^s \otimes \mathbb{Z}_{n_i}^{m_i} \cong \mathbb{Z}_{n_i}^{sm_i}$$

and

$$\mathbb{Z}_{n_i} \otimes \mathbb{Z}_{a_j} \cong \mathbb{Z}_{(n_i,a_j)}$$

where (n_i, a_j) is the g.c.d of n_i and a_j **(Prove this last fact!)** Therefore we have shown

$$(\mathbb{Z}^r \oplus_{i=1}^N \mathbb{Z}_{n_i}^{m_i}) \otimes (\mathbb{Z}^s \oplus_{j=1}^M \mathbb{Z}_{a_j}^{b_j}) = \mathbb{Z}^r \otimes \mathbb{Z}^s \oplus (\oplus_{j=1}^M (\mathbb{Z}_{a_j}^{rb_j}) \oplus (\oplus_{i=1}^N (\mathbb{Z}_{n_i}^{sm_i}) \oplus (\oplus_{i,j} (\mathbb{Z}_{(n_i,b_j)}^{m_i bj}).$$

This gives us that there can be no element of order a power of p.

Remark : *Note that when we write $\mathbb{Z}[x,y]$, we presume that the variables x, y commute. So to say a cohomology is isomorphic to a quotient of $\mathbb{Z}[x,y]$, you need to mention that the corresponding generators in cohomology commute each other. Otherwise identifying the cohomology with a quotient of $\mathbb{Z}[x,y]$ does not quite make sense.*