Solution set for Homework 2 (due February 5)

– Partially based on *Diane Holcomb*'s HW –

1. Let G, H be abelian groups. By an *extension* of G by H, we mean a short exact sequence

$$C : 0 \to G \xrightarrow{\alpha} J \xrightarrow{\beta} H \to 0$$

of abelian groups. Two extensions C and

$$C' : 0 \to G \xrightarrow{\alpha'} J' \xrightarrow{\beta'} H \to 0$$

of G by H are said to be *equivalent* iff there exists a homomorphism $h : J \to J'$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \longrightarrow & J & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & G & \longrightarrow & J' & \longrightarrow & H & \longrightarrow & 0 \end{array}$$

is commutative. Prove that h must be an isomorphism and that this relation is an equivalence relation. Then construct a bijective correspondence between the elements of $\mathrm{Ext}(H, G)$ and the equivalence classes of the extensions of G by H. Solution. Assume the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & H & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & G & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & H & \longrightarrow & 0 \end{array}$$

and that h is a homomorphism. Then we have the that $\beta = \beta' \circ h$ and $\alpha' = h \circ \alpha$. To show that h is and isomorphism we will first show that it is surjective, and then that it is injective.

Surjective: Let $x' \in J'$, if $x' \in \ker(\beta')$ this implies that $x' \in \mathrm{Im}\ (\alpha') = \mathrm{Im}\ (h \circ \alpha)$ it follows that $x' \in \mathrm{Im}\ (h)$.

If $x' \notin \ker(\beta')$ then $\beta'(x') \in \Im(\beta') = H$. Therefore $\beta'(x') = \beta(x)$ for some $x \in J$. Therefore $x' - h(x) \in \mathrm{Im}\ (\alpha') = \ker(\beta') \subseteq \mathrm{Im}\ (h)$ by the above calculation. This gives us that $x' - h(x) \in \mathrm{Im}\ (h)$ and we know that $h(x) \in \mathrm{Im}\ (h)$ therefore $x' \in \mathrm{Im}\ (h)$.

Injective: Let $h(x) = 0$ this gives us that $\beta'(h(x)) = 0$ which implies that $\beta(x) = 0$ or $x \in \ker(\beta) = \mathrm{Im}\ (\alpha)$. Since $x \in \mathrm{Im}\ (\alpha)$ we have that $x = \alpha(y)$ for some $y \in G$. Therefore we have that $h(\alpha(y)) = 0$, so $\alpha'(y) = 0$ and since α' is injective we get that $y = 0$ and so $x = 0$ meaning that $\ker(h) = \{0\}$. and so h is injective.

Therefore this relation is and equivalence relation because isomorphism is an equivalence relation.

Now to construct a bijective correspondence between the elements of $\mathrm{Ext}(H, G)$ and the equivalence classes of the extensions of G by H. For the convenience of notation, we denote by $\mathrm{Ext}'(H, G)$ the set of equivalence classes of the extension of G by H.

We start with an extension of G by H, and a free resolution of H.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & H & \longrightarrow & 0 \\
 & & & & & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & H & \longrightarrow & 0
\end{array}$$

Since C_0 is free we can pick a basis $\{a_\alpha\}$ and consider $\partial_0(a_\alpha) \in H$. If we look at the set of preimages $\beta^{-1}(\partial(a_\alpha) \subset J$ we can pick some element x_α in the preimage for all α. Then we define a map $f : C_0 \to J$ by $f(a_\alpha) = x_\alpha$.

Now similarly since C_1 is free we can pick a basis $\{b_\gamma\}$. If we look at $f(\partial_1(b_\gamma))$ we need to show that it is contained in $\text{Im }(\alpha) = \ker(\beta)$. We have that $\partial_0(\partial_1(b_\gamma)) = 0$ therefore $\partial_1(b_\gamma) \in \ker \partial_0$ and by construction this gives us $f(\partial_1(b_\gamma)) \in \ker(\beta) = \text{Im }(\alpha)$. This tells us that there exists $y_\gamma \in G$ such that $\alpha(y_\gamma) = f(\partial_1(b_\gamma))$ so we can define a map $F : C_1 \to G$ by $F(b_\gamma) = y_\gamma$. This gives us the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & H & \longrightarrow & 0
\end{array}$$

We can then identify F with its equivalence class in $\text{Ext}(G, H) = \text{Hom }(C_1 : G)/\text{Im }\partial_1^*$. This is well defined because the only choice made in determining F was the choice of x_α but $J/\text{Im }(\alpha) \simeq H$ so all possible choices of x_α would be identified meaning that F is well defined as an element of $\text{Ext}(G, H)$. Furthermore if we have two extension of G by H we can construct the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha} & J & \xrightarrow{\beta} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & G & \xrightarrow{\alpha'} & J' & \xrightarrow{\beta'} & H & \longrightarrow & 0
\end{array}$$

which gives us that F is also identified with equivalent extensions by commutativity of the diagram. This finishes a construction of a map

$$\Phi : \text{Ext}'(H, G) \to \text{Ext}(H, G).$$

Now we construct a map $\Psi : \text{Ext}(H, G) \to \text{Ext}'(H, G)$. Fix a free resolution

$$0 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} H \to 0$$

and recall the exact sequence

$$0 \to \text{Hom}(H, G) \to \text{Hom}(C_0, G) \to \text{Hom}(C_1, G) \xrightarrow{\delta} \text{Ext}(H, G) \to 0.$$

Let $e \in \mathrm{Ext}(H, G)$. By the surjectivity of δ, there exists some $F \in \mathrm{Hom}(C_1, G)$ with $\delta(F) = e$. This induces a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & H & \longrightarrow & 0\, . \\ & & \downarrow F & & \downarrow ? & & \downarrow id & & \\ 0 & \longrightarrow & G & \xrightarrow{?} & ?? & \xrightarrow{?} & H & \longrightarrow & 0 \end{array}$$

Here we need to fill in all the question marks. We put the 'push-out' of the map $f : C_1 \to G$ under $\partial_1 : C_1 \to C_0$: We define J to the quotient group

$$J = C_0 \oplus G/N$$

where $N \subset C_0 \oplus G$ is the subgroup defined by

$$N = \{(c_0, g) \in C_0 \oplus G \mid c_0 = \partial_1(c_1),\ g = -F(c_1),\ c_1 \in C_1\}.$$

Then we define $\alpha : G \to J$, $\beta : J \to H$ and $f : C_0 \to J$ by

$$\alpha(g) = [(0, g)],\ \beta([(c_0, g)]) = \partial_0(c_0),\ f(c_0) = (c_0, 0).$$

Check that these maps are well-defined and define an extension

$$0 \to G \to J \to H \to 0$$

and the downward arrow $f : C_0 \to J$. We denote this extension by C_f. **Verify** that if $\delta(f') = \delta(f) = e$, the corresponding extension $C_{f'}$ and C_f are equivalent. Then we define

$$\Psi(e) = [C_f].$$

It remains to show that $\Phi \circ \Psi(e) = e$ for all $e \in \mathrm{Ext}(H, G)$. But we have $\Phi \circ \Psi(e) = \Phi([C_f])$. Then by examining the construction of Φ in the first step, we find $\Phi([C_f]) = [f] \in \mathrm{Coker}\, \partial_1^* = e$. The other identity $\Psi \circ \Phi = id$ can be proved similarly. This finishes the proof of bijective correspondence.

2. Consider the map $n : H \to H$ defined by multiplication by n. This map can easily be lifted resulting in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & H & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n & & \\ 0 & \longrightarrow & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & H & \longrightarrow & 0 \end{array}$$

And by applying Hom $(\cdot, G)$ and using the definition of $\mathrm{Ext}(H, G)$, we get

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{Hom}\ (H, G) & \xrightarrow{\partial_0^*} & \mathrm{Hom}\ (C_0, G) & \xrightarrow{\partial_1^*} & \mathrm{Hom}\ (C_1, G) & \xrightarrow{\delta} & \mathrm{Ext}(H, G) & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n & & \downarrow ? & & \\ 0 & \longrightarrow & \mathrm{Hom}\ (H, G) & \xrightarrow{\partial_0^*} & \mathrm{Hom}\ (C_0, G) & \xrightarrow{\partial_1^*} & \mathrm{Hom}\ (C_1, G) & \xrightarrow{\delta} & \mathrm{Ext}(H, G) & \longrightarrow & 0 \end{array}$$

Which clearly gives us a map $\text{Ext}(\bar{H}, G) = \text{Hom }(\bar{C_1})/\text{Im } \delta_1 \to \text{Ext}(H, G) = \text{Hom }(C_1)/\text{Im } \delta_1$ defined by $[\phi] \mapsto [n\phi] = n[\phi]$.

Similarly consider the map $n : G \to G$. This map naturally induces a map

$$\text{Hom }(F, G) \to \text{Hom }(F, G)\, ;\, \varphi \mapsto n \cdot \varphi$$

which is nothing but multiplication by the integer n. Therefore we again have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Hom }(H,G) & \xrightarrow{\partial_0^*} & \text{Hom }(C_0,G) & \xrightarrow{\partial_1^*} & \text{Hom }(C_1,G) & \xrightarrow{\delta} & \text{Ext}(H,G) & \longrightarrow & 0 \\ & & \downarrow n & & \downarrow n & & \downarrow n & & \downarrow ? & & \\ 0 & \longrightarrow & \text{Hom }(H,G) & \xrightarrow{\partial_0^*} & \text{Hom }(C_0,G) & \xrightarrow{\partial_1^*} & \text{Hom }(C_1,G) & \xrightarrow{\delta} & \text{Ext}(H,G) & \longrightarrow & 0 \end{array}$$

and hence the induced map is again the same as $[\phi] \mapsto [n\phi] = n[\phi]$. This finishes the proof.

3. We begin by constructing a free resolution of $\mathbb{Z}_2$.

$$\cdots \longrightarrow \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \longrightarrow 0$$

By taking the dual we get

$$0 \longrightarrow \text{Hom }_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \longrightarrow \text{Hom }_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \longrightarrow \text{Hom }_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \longrightarrow \text{Hom }_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \longrightarrow \cdots$$

which is exactly

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \cdots$$

Which when we take Ext of this gives us $\text{Ext}^0 = \mathbb{Z}_2, \text{Ext}^1 = \mathbb{Z}_2, \text{Ext}^2 = \mathbb{Z}_2, \text{Ext}^3 = \mathbb{Z}_2, \ldots.$ And so Ext is always non-zero.