Math 340: Midterm II Solutions

– Fall 2008 : Instructor ; Yong-Geun Oh –

Problem 1. Answer the following questions. Do **not** explain your answers for these problems **unless** you are explicitly to do so.

(a) **(3 pts)** Write down the zero vector of the vector space $M_{2\times 2}(\mathbb{R})$ (= the set of 2×2 matrices).

Solution: $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(b) **(5 pts)** Find the dimension of $\mathbb{R}^3$. Is the following set of vectors a basis of $\mathbb{R}^3$? **Give the reason why.**

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Solution: $\dim \mathbb{R}^3 = 3$. The set is not a basis since it has 4 elements in it while $\dim \mathbb{R}^3 = 3$.

(c) **(6 pts)** Find the dimension of $M_{2\times 2}(\mathbb{R})$. Does the following set of vectors span $M_{2\times 2}(\mathbb{R})$? **Give the reason why.**

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Solution: $\dim M_{2\times 2}(\mathbb{R}) = 4$. The set cannot span $M_{2\times 2}(\mathbb{R})$ as it contains only 3 elements while at least 4 elements are needed to span $M_{2\times 2}(\mathbb{R})$.

(d) **(5 pts)** Let $S = \{\vec{b}_1, \vec{b}_2\}$ be a basis of a vector space V. Let $\vec{x} = 2\vec{b}_1 + 3\vec{b}_2$. Find the coordinate vector $[\vec{x}]_S$ relative to S.

Solution:

$$[\vec{x}]_S = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(e) **(5 pts)** Let $S = \{1+t, 1-t\}$ be a basis of $\mathbb{P}_1$ = the set of polynomials of degree less than or equal to 1. Find the polynomial p in $\mathbb{P}_1$ written **in its standard form** whose coordinate vector $[p]_S$ is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Solution: $p = 2(1+t) + 3(1-t)$ by definition of the coordinate vector $[p]_S$. Hence we have $p = 5 - t$.

(f) **(6 pts)** Let A be the matrix

$$\begin{pmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{pmatrix}$$

with a row echelon matrix given by

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the bases of Col(A) and Row(A) respectively and the rank A.

Solution: Note the the pivot columns are the 1st, the 2nd and the 4th ones. Therefore a basis for $Col(A)$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and a basis for $Row(A)$ is

$$\{(1, 0, 2, 0, 1), (0, 1, 2, 9, -1), (0, 0, 0, 1, 2)\}$$

and so $rank(A)$ is 3.

Problem 2. Let the matrix A be

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

(a) (**10 pts**) Solve equation $A\mathbf{x} = \mathbf{0}$.

Solution: We find the solution set

$$x_1 = -t,\ x_2 = -t,\ x_3 = -5t,\ x_4 = t, \quad t \text{ arbitrary}.$$

(b) (**10 pts**) Find a basis for the solution space of the above equation.

Solution: Writing the solutions above in a vector form, we get

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ -5t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ -5 \\ 1 \end{pmatrix}.$$

Hence a basis for the solution space is

$$\left\{ \begin{pmatrix} -1 \\ -1 \\ -5 \\ 1 \end{pmatrix} \right\}.$$

(c) (**5 pts**) Find the nullity of A.

Solution: $\text{nullity} A = 1$.

Problem 3. Let P_2 be the vector space of all polynomials of degree ≤ 2.

(a) (**10 pts**) Given a set $S = \{t-1, t+1\}$ in P_2. Verify that S is linearly independent.

Solution: We consider the defining equation for the linear independence

$$c_1(t-1) + c_2(t+1) = \vec{0}$$

for c_1, c_2. Writing this into the standard form, we get

$$(c_1 + c_2)t + (c_2 - c_1) = \vec{0}$$

we obtain the equation

$$c_1 + c_2 = 0,\ c_2 - c_1 = 0.$$

This equation has only the trivial solution $c_1 = c_2 = 0$ and so S is linearly independent.

(b) (**5 pts**) Extend S to a basis for P_2.

Solution: For example, $\{t-1, t+1, t^2\}$ is a basis of P_2 extending the set S. **Please check** whether this is really a basis!

Problem 4. (**10 pts**) In the standard inner product space $\mathbb{R}^3$, let $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$.

Use the Gram-Schmidt process to obtain an orthonormal basis of the subspace

$$W = \text{Span} S = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}.$$

Solution: Set

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ u_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

Then

$$v_1 = u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ v_2 = u_2 - \frac{(u_2, v_1)}{(v_1, v_1)} v_1.$$

But we have $(u_2, v_1) = 3$ and $(v_1, v_1) = 3$ and so

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

By normalizing them, we obtain an orthonormal basis

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Problem 5. Let $M_{22}(\mathbb{R})$ be the real vector space of all 2×2 matrices.

(a) (**10 pts**) Prove that the subset W in M_{22}, defined by

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a + b + c + d = 0, \ a, \ b, \ c, \ d \text{ real numbers} \right\},$$

is a subspace of $M_{22}(\mathbb{R})$.

Solution: Let $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ be in W. Then by definition of W we have

$$a_1 + b_1 + c_1 + d_1 = 0 = a_2 + b_2 + c_2 + d_2.$$

Therefore we have

$$(a_1 + a_2) + (b_1 + b_2) + (c_1 + d_2) + (d_1 + d_2) = (a_1 + b_1 + c_1 + d_1) + (a_2 + b_2 + c_2 + d_2) = 0 + 0 = 0.$$

On other other hand we have

$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}$$

which shows $A_1 + A_2$ satisfies the defining equation for W and so lies in W.

Next for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and a real number λ we have

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

On the other hand we have

$$(\lambda a) + (\lambda b) + (\lambda c) + (\lambda d) = \lambda(a + b + c + d) = \lambda 0 = 0$$

and so λA lies in W if A does. This finishes the proof.

(b) (**10 pts**) Find a basis for W. Then express $\begin{pmatrix} 5 & 1 \\ 2 & -8 \end{pmatrix}$ as a linear combination of the vectors in the basis.

Solution: We solve the defining equation

$$a + b + c + d = 0.$$

We set $b = t$, $c = s$, $d = u$ with t, s, u arbitrary and then solve for a to get

$$a = -t - s - u, \quad t,\ s,\ u \text{ arbitrary.}$$

Then any element of W can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -t - s - u & t \\ s & u \end{pmatrix} = t \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + s \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} + u \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore a basis of W is given by

$$\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Now we note that

$$\begin{pmatrix} 5 & 1 \\ 2 & -8 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} - 8 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$