EXTENSION OF CALABI HOMOMORPHISM AND NONSIMPLENESS OF THE AREA-PRESERVING HOMEOMORPHISM GROUP OF D^2

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Abstract. The group $Hameo(M,\omega)$ consisting of *Hamiltonian homeomorphisms* (or *hameomorphisms*) and the notion of topological Hamiltonian flows are previously introduced by Müller and the author. In this paper, we introduce the notion of *hamiltonian homotopy* of topological Hamiltonian paths. We then prove that the Alexander isotopy of topological Hamiltonian loop is a hamiltonian homotopy to the constant identity path. Combining this with the homotopy invariance of spectral invariants of topological Hamiltonian paths, whose proof is given in a sequel to this paper, we extend the well-known Calabi homomorphism defined on the area preserving diffeomorphism group to the subgroup $Hameo(D^2, \partial D^2)$ of area-preserving homeomorphism group of D^2. As a corollary, we prove $Hameo(D^2, \partial D^2)$ is a proper normal subgroup of the group of compactly supported area preserving homeomorphisms in Int D^2 and hence the latter group is *not simple*. We also prove the same properness of $Hameo(D^{2n}, \partial D^{2n})$ in the group of symplectic homeomorphisms in high dimension.

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Contents

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1. Introduction and the main results

1.1. Simpleness question of $Homeo^\Omega(D^2, \partial D^2)$. About 30 years ago, the algebraic structure of the groups of volume preserving diffeomorphisms [T], symplectic diffeomorphisms [B], or measure preserving homeomorphisms [Fa1] were well studied thanks to the work of Thurston, Banyaga and Fathi respectively. There is one case which has remained unsettled since then. This is the case of area preserving homeomorphisms in dimension 2, especially for the cases of S^2 or of $(D^2, \partial D^2)$, where the symplectic geometry and the area preserving geometry meet.

In [C], Calabi introduced the so called *Calabi invariants* on the group of symplectic diffeomorphisms. When restricted to the two dimensional compact surface, this symplectic construction meets the area preserving dynamical systems. Let Σ be a closed oriented surface and Ω be an area form on Σ normalized so that $\int_\Sigma \Omega = 1$. Denote by $Diff_0^\Omega(\Sigma)$ the group of smooth diffeomorphisms on Σ isotopic to the identity. When Σ has a boundary, we denote by $Diff_0^\Omega(\Sigma, \partial\Sigma)$ those that is assumed in addition to be the identity near $\partial\Sigma$.

When further restricted to the disc (D^2, Ω) with the standard area form Ω on $D^2 \subset \mathbb{C}$, the group $Diff_0^\Omega(D^2, \partial D^2)$ is contractible. This enables one to define the following type of Calabi invariants

$$\text{Cal} : Diff_0^\Omega(D^2, \partial D^2) \to \mathbb{R}$$

defined as follows: We normalize the area form Ω so that $\int_{D^2} \Omega = 1$. For given $\phi \in Diff_0^\Omega(D^2, \partial D^2)$, we consider the integral

$$\text{Cal}(\phi) = \int_0^1 \int_\Sigma H_t \, \Omega \, dt \tag{1.1}$$

where $H : [0, 1] \times D^2 \to \mathbb{R}$ is a smooth time-dependent Hamiltonian function whose time-one map is ϕ. Then one proves that the right-hand side integral depends only on the time-one map. Since Cal defines a nontrivial homomorphism, $\ker \text{Cal}$ is a proper normal subgroup of $Diff_0^\Omega(D^2, \partial D^2)$ which immediately implies that the latter group is not simple. But Banyaga proved that $\ker \text{Cal}$ itself is simple. We refer to [Fa1, GG1, GG2] for a different construction of Calabi homomorphism. (On general closed surface Σ, there is another invariant, also called Calabi invariant which has the form

$$\text{Cal}_{H^1} : Diff_0^\Omega(\Sigma) \to H^1(\Sigma, \mathbb{R})$$

as studied in [C], [B]. Banyaga [B] proved that

$$\ker(\mathrm{Cal}_{H^1}) \subset Diff_0^\Omega(\Sigma)$$

is a simple group. When restricted to S^2, we have $\ker(\mathrm{Cal}_{H^1}) = Diff_0^\Omega(S^2)$ which implies that $Diff_0^\Omega(S^2)$ is simple.)

Our main interest in this paper is to study a topological analog to this latter simpleness question on the disc D^2. We equip the group $Homeo(D^2, \partial D^2)$ of homeomorphisms with the standard C^0-metric given by

$$\overline{d}(\phi, \psi) = \max\{d_{C^0}(\phi, \psi), d_{C^0}(\phi^{-1}, \psi^{-1})\}. \tag{1.2}$$

We denote by

$$Homeo_0^\Omega(D^2, \partial D^2)$$

the group of Ω-area preserving homeomorphisms isotopic to the identity by an isotopy compactly supported in $\mathrm{Int}(D^2)$.

The following question is commonly attributed to Mather [M3]. (see e.g., [Fa1], section 7 [GG2]).

Question 1.1 (Mather)**.** Is $Homeo_0^\Omega(D^2, \partial D^2)$ simple?

There have been a few construction of normal subgroups of $Homeo_0^\Omega(D^2, \partial D^2)$ but attempts to prove properness of any of those subgroups have not been successful. (See [Gh], [leR] for some examples.)

Quite recently, motivated by Eliashberg-Gromov's C^0-rigidity theorem [El], Müller and the present author [OM] defined the group of symplectic homeomorphisms by the C^0-completion of $Symp(M, \omega)$ in $Homeo(M)$. It is denoted by $Sympeo(M, \omega)$. Then they constructed a normal subgroup $Hameo(M, \omega)$ of $Sympeo(M, \omega)$ exploiting the techniques of Hofer's geometry in symplectic topology [H].

The following theorem is an immediate consequence of a smoothing result of area preserving homeomorphisms which seems to have been a folklore among the experts. The detail of the proof is given in [Oh9], [Si].

Theorem 1.1 (Smoothing)**.** *We have*

$$Sympeo(\Sigma, \omega) = Homeo^\Omega(\Sigma).$$

for any symplectic form ω regarding it also as the area form $\Omega = \omega$.

Therefore the above mentioned conjecture in [OM] is equivalent to whether $Hameo(D^2, \partial D^2)$ is a proper subgroup of $Homeo^\Omega(D^2, \partial D^2)$ and hence closely related to Question 1.1.

1.2. **Review of algebra of Hamiltonians.**

A time-dependent Hamilton's equation on a symplectic manifold (M, ω) is the first order ordinary differential equation

$$\dot{x} = X_H(t, x)$$

where the time-dependent vector field X_H associated to a function $H : \mathbb{R} \times M \to \mathbb{R}$ is given by the defining equation

$$dH_t = X_{H_t} \rfloor \omega. \tag{1.3}$$

Therefore if we consider functions H *that are $C^{1,1}$ so that one can apply the existence and uniqueness theorem of solutions of the above Hamilton's equation*, the flow $t \mapsto \phi_H^t$, an isotopy of diffeomorphisms, is uniquely determined by the Hamiltonian H. We will always assume

(1) the Hamiltonians are normalized by $\int_M H_t \, d\mu = 0$ for the Liouville measure $d\mu$ of (M,ω) if M is closed,
(2) and they are compactly supported in Int$M$ if M is open.

We call such Hamiltonian functions *normalized*. Throughout the paper, we will assume that the Hamiltonian is mean-normalized on closed (M,ω), unless otherwise said explicitly.

We denote by $C_m^\infty(M)$ the set of normalized smooth functions on M and by $\mathcal{P}(C_m^\infty(M)) = C_m^\infty([0,1]\times M)$ the set of time-dependent normalized Hamiltonian functions. We will also denote the Hamiltonian isotopy generated by H by

$$\phi_H : t \mapsto \phi_H^t.$$

Conversely if a smooth isotopy λ of Hamiltonian diffeomorphisms is given, we can obtain the corresponding normalized Hamiltonian H by differentiating the isotopy and then solving (1.3). Therefore *in the smooth category* this correspondence is bijective. And it is a standard fact that the 'inverse' Hamiltonian $\bar H$ defined by

$$\bar H(t,x) = -H(t, \phi_H^t(x))$$

generates the flow ϕ_H^{-1}. The Hamiltonian denoted by $\widetilde H$ is defined by

$$\widetilde H(t,x) = -H(1-t,x)$$

generates a flow

$$\phi_{\widetilde H} : t \mapsto \phi_H^{1-t} \circ (\phi_H^1)^{-1}.$$

This flow also satisfies $\phi_{\widetilde H}^1 = (\phi_H^1)^{-1}$ and is path-homotopic to the inverse flow ϕ_H^{-1} relative to the end point. (See Lemma 5.2 [Oh6] for its proof.) We will use both $\bar H$ and $\widetilde H$ to generate the inverse $(\phi_H^1)^{-1}$ depending on the circumstances. And the 'product' Hamiltonian $H\#F$ defined by

$$(H\#F)(t,x) = H(t,x) + F(t, (\phi_H^t)^{-1}(x))$$

generates the product flow $t \mapsto \phi_H^t \circ \phi_F^t$. Following [OM, Oh10], we denote by

$$\mathcal{P}^{ham}(Symp(M,\omega), id)$$

the set of smooth Hamiltonian paths $\lambda : [0,1] \to Symp(M,\omega)$ with $\lambda(0) = id$, and consider the map

$$\text{Dev} : \mathcal{P}^{ham}(Symp(M,\omega), id) \to C_m^\infty([0,1]\times M, \mathbb{R}) \tag{1.4}$$

which is the assignment of the Hamiltonian associated to a Hamiltonian path. In other words, we define $\text{Dev}(\lambda) = H$ if $\lambda = \phi_H$.

Hofer's $L^{(1,\infty)}$ norm of Hamiltonian diffeomorphisms is defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\|$$

where $H \mapsto \phi$ means that $\phi = \phi_H^1$ is the time-one map of Hamilton's equation $\dot x = X_H(t,x)$ and the norm $\|H\|$ is defined by

$$\|H\| = \int_0^1 \text{osc } H_t \, dt = \int_0^1 (\max_x H_t - \min_x H_t)\, dt. \tag{1.5}$$

We would like to point out

$$\begin{aligned} \mathrm{Dev}(\phi_H^{-1}\phi_F) &= \bar{H}\#F = -H(t,\phi_H^t(x)) + F(t,\phi_H^t(x)) \\ \mathrm{Dev}(\phi_H\phi_F^{-1}) &= H\#\bar{F} = H(t,x) - F(t,\phi_F^t(\phi_H^t)^{-1}(x)) \\ &= -\bar{H}(t,\phi_{\bar{H}}^t(x)) + \bar{F}(t,\phi_{\bar{H}}^t(x). \end{aligned} \tag{1.6}$$

Therefore we have the identity

$$\|\,\mathrm{Dev}(\phi_H^{-1}\phi_F)\| = \|F-H\|, \quad \|\,\mathrm{Dev}(\phi_H\phi_F^{-1})\| = \|\bar{F}-\bar{H}\|. \tag{1.7}$$

Note that $\|F-H\| \neq \|\bar{F}-\bar{H}\|$ in general.

1.3. **Review of $L^{(1,\infty)}$ topological Hamiltonian flows.** In this section, we recall the construction of the group $Hameo(M,\omega)$ of Hamiltonian homeomorphisms, succinctly called *hameomorphisms*, introduced in [OM].

We give the compact-open topology on $Homeo(M)$, which is equivalent to the metric topology induced by the metric

$$\overline{d}(\phi,\psi) = \max\{d_{C^0}(\phi,\psi), d_{C^0}(\phi^{-1},\psi^{-1})\}$$

on a compact manifold M.

The following definition of the group of *symplectic homeomorphisms* is given in [OM] which is motivated by Eliashberg-Gromov's C^0-symplectic rigidity theorem [El].

Definition 1.2 (Symplectic homeomorphism group)**.** Define $Sympeo(M,\omega)$ to be

$$Sympeo(M,\omega) := \overline{Symp(M,\omega)}$$

the C^0-closure of $Symp(M,\omega)$ in $Homeo(M)$ and call $Sympeo(M,\omega)$ the *symplectic homeomorphism group.*

Hofer's norm (1.5) can be regarded as the Finsler length of the Hamiltonian path ϕ_H and will be also denoted by $\mathrm{leng}(\phi_H)$. For two given continuous paths $\lambda,\ \mu : [a,b] \to Homeo(M)$, we define their distance by

$$\overline{d}(\lambda,\mu) = \max_{t\in[a,b]} \overline{d}(\lambda(t),\mu(t)).$$

We equip $\mathcal{P}^{ham}(Symp(M,\omega), id)$ with the metric

$$d_{ham}(\lambda,\mu) := \overline{d}(\lambda,\mu) + \mathrm{leng}(\lambda^{-1}\mu)$$

where $\overline{d}$ is the C^0 metric on $\mathcal{P}(Homeo(M), id)$. (See Proposition 3.10 [OM].)

Definition 1.3 ($L^{(1,\infty)}$ topological Hamiltonian flow)**.** A continuous map $\lambda : \mathbb{R} \to Homeo(M)$ is called a topological Hamiltonian flow if there exists a sequence of smooth Hamiltonians $H_i : \mathbb{R}\times M \to \mathbb{R}$ satisfying the following:

(1) $\phi_{H_i} \to \lambda$ locally uniformly on $\mathbb{R}\times M$.
(2) the sequence H_i is Cauchy in the $L^{(1,\infty)}$-topology and so has a limit H_∞ lying in $L^{(1,\infty)}$.

We call any such ϕ_{H_i} or H_i an *approximating sequence* of λ. We call a continuous path $\lambda : [a,b] \to Homeo(M)$ a *topological Hamiltonian path* if it satisfies the same conditions with $\mathbb{R}$ replaced by $[a,b]$, and the limit $L^{(1,\infty)}$-function H_∞ called a $L^{(1,\infty)}$ *topological Hamiltonian* or just a *topological Hamiltonian.*

We denote by $\mathcal{H}_m([0,1]\times M,\mathbb{R})$ the set of (normalized) topological Hamiltonians.

It is not difficult to check (see [BS] for a proof) that there exists a natural embedding

$$\mathcal{H}_m([0,1]\times M,\mathbb{R}) \hookrightarrow L^1([a,b],C^0(M))$$

where $L^1([a,b],C^0(M))$ is the Banach space consisting of L^1-functions with values in $C^0(M)$. We would like to emphasize that $\mathcal{H}_m([0,1]\times M,\mathbb{R})$ is *not* a linear subspace of the latter. From now on, we will simply call any element in $\mathcal{H}_m([0,1]\times M,\mathbb{R})$ a *continuous (time-dependent) Hamiltonian.*

We denote by $ev_1 : \mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega),id) \to Sympeo(M,\omega),id)$ the evaluation map defined by

$$ev_1(\lambda) = \lambda(1).$$

By the uniqueness theorem of Buhovsky-Seyfaddini [BS], we can extend the map Dev (1.4) to

$$\overline{\text{Dev}} : \mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega),id) \to \mathcal{H}_m([0,1]\times M,\mathbb{R})$$

in an obvious way. Following the notation of [OM, Oh10], we denote the topological Hamiltonian path $\lambda = \phi_H$ when $\overline{\text{Dev}}(\lambda) = H$.

Definition 1.4 (Hamiltonian homeomorphism group)**.** We define

$$Hameo(M,\omega) = ev_1(\mathcal{P}^{ham}_{[0,1]}(Sympeo(M,\omega),id))$$

and call any element therein a *Hamiltonian homeomorphisms*

One basic theorem proved in [OM] is that $Hameo(M,\omega)$ forms a path-connected normal subgroup of $Sympeo(M,\omega)$.

1.4. **Statement of main results.** The following is the main result of the present paper, which resolves a conjecture stated in [OM].

Theorem 1.2 (Main Theorem)**.** *We equip D^2 with the standard area form $\Omega = \omega$. Denote by $Homeo^\Omega(D^2,\partial D^2)$ the group of area preserving homeomorphisms of D^2 with being identity near the boundary. Then $Hameo(D^2,\partial D^2)$ is a non-trivial proper normal subgroup of $Homeo^\Omega(D^2,\partial D^2)$.*

In fact, the scheme of our properness proof can be also applied to prove the same properness in arbitrary dimensions whose proof we will include in Appendix.

Theorem 1.3. *Equip $D^{2n} = D^{2n}(1) \subset \mathbb{C}^n$ with the standard symplectic form ω. Then $Hameo(D^{2n},\partial D^{2n})$ is a non-trivial proper normal subgroup of $Sympeo(D^{2n},\partial D^{2n})$.*

Non-simpleness of $Homeo^\Omega(D^2,\partial D^2)$ is an immediate corollary of Main Theorem. This in particular resolves Mather's question negatively.

Theorem 1.4. *The group $Homeo^\Omega(D^2,\partial D^2)$ is not simple.*

In the point of view of the study of deeper structure of area preserving dynamics on D^2, the following problem seems to be of much interest.

Question 1.5. Study the structure of the quotient group

$$Homeo^\Omega(D^2,\partial D^2)/Hameo(D^2,\partial D^2).$$

Our proof of Theorem 1.2 is based on the following extension of the Calabi invariant to this group $Hameo(D^2,\partial D^2)$.

Theorem 1.5. *There exists an extension of* Cal *to a homomorphism*

$$\overline{\mathrm{Cal}} : Hameo(D^2, \partial D^2) \to \mathbb{R}$$

which is continuous with respect to hamiltonian topology.

We like to emphasize that $\overline{\mathrm{Cal}}$ may not be extended to a homomorphism to the whole group $Homeo^\Omega(D^2, \partial D^2)$ of area preserving homeomorphisms. (See Corollary 2.3 [GG1] which reads that Cal is not continuous in C^0-topology.)

We will show that the above stated properness is due to the *non-approximability* of the wild homeomorphisms by a sequence of Hamiltonian paths with tame behavior on their associated Hamiltonians: More precisely speaking, it holds that for any sequence of smooth isotopies λ_i of area preserving diffeomorphisms on D^2 such that

(1) λ_i converges on $[0,1] \times D^2$ in the C^0-topology
(2) $\lambda_i(1)$ converges to the given wild area preserving homeomorphism in the C^0-topology,

the sequence of the associated Hamiltonians $\mathrm{Dev}(\lambda_i) =: H_i$ *cannot be pre-compact* in the $L^{(1,\infty)}$-topology. In other words, it must the case that either the Hofer length $\mathrm{leng}(\lambda_i)$ blows up or the Hamiltonian $\mathrm{Dev}(\lambda_i)$ behave somewhat chaotically as $i \to \infty$.

Existence of the homomorphism $\overline{\mathrm{Cal}} : Hameo(D^2, \partial D^2) \to \mathbb{R}$ is crucial to prove that this untame behavior of the Hamiltonians must occur for *any* choice of the sequence of smoothing isotopies. In this sense, our non-simpleness result is due to *topological Hamiltonian dynamics* advocated in [OM, Oh10]. This further indicates existence of topological Hamiltonian dynamics, which helps one to understand the true nature of the area preserving dynamics in two dimension. It would be interesting to relate the topological Hamiltonian dynamics with the well-established Aubry-Mather theory of area preserving dynamics in two dimension [AD], [M1, M2].

The following is the topological analog to Banyaga's nonsimpleness theorem of $Diff^\Omega(D^2, \partial D^2)$, which also answers negatively to the question in Problem (4) of p 211 [OM].

Corollary 1.6. $\ker \overline{\mathrm{Cal}}$ *is a proper normal subgroup of* $Hameo(D^2, \partial D^2)$ *and hence* $Hameo(D^2, \partial D^2)$ *is not simple.*

It is well-known (see Corollary 2.3 [GG1]) that

$$\mathrm{Cal} : Diff^\Omega(D^2, \partial D^2) = Ham(D^2, \partial D^2) \to \mathbb{R}$$

is *not* continuous in C^0 topology, although it is continuous in C^∞ (or even C^1) topology.

Using this theorem, we apply a construction of *infinite repetition* which is reminiscent of those used in the geometric topology (see [Maz], [GG1] for example) and construct an area preserving homeomorphism not contained in $Hameo(D^2, \partial D^2)$. This will finish the proof of Theorem 1.2 modulo the following vanishing result whose proof is given in a sequel [Oh11] to this paper.

Theorem 1.7 ([Oh11])**.** *Assume* (M, ω) *is rational. Let* λ *be a topological Hamiltonian loop hamiltonian homotopic to the identity via homotopy supported in* $U = M \setminus B$ *where* B *is a closed ball in* M *and* H *be the corresponding topological Hamiltonian supported in* $M \setminus B$. *Suppose that* λ *is hamiltonian homotopic to the identity* *path in* $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$*. Consider its normalization* $\underline{H}$ *defined by*

$$\underline{H}_t(x) := H_t(x) - \frac{1}{\mathrm{vol}_\omega(M)} \int_M H(t,x)\, \omega^n.$$

Then $\rho(\underline{H}; 1) = 0$.

We would like to note that S^2 (or more generally $\mathbb{C}P^n$), which is the main interest of this paper, is rational.

The proof of this theorem is the most fundamental ingredient and the step that explicitly uses the *hard symplectic* method (in Gromov's term [Gr]) in its proof. The proof is based on the chain level Floer theory in full throttle both in the Lagrangian context developed in [Oh1, Oh2] and in the Hamiltonian context developed in [Oh3, Oh5, Oh6, Oh8], and has its own independent interest. For these reasons, we isolate the proof of this theorem into a separate paper [Oh11]. By doing so, we can completely avoid using the Floer theory in the main proof of the nonsimpleness given in the current paper except the usage of some basic axiomatic properties of spectral invariants $\rho(\phi_H; 1)$ which we summarize in section 2.

Remark 1.6. In fact nonsimpleness of $Homeo^\Omega(D^2, \partial D^2)$ is an immediate corollary of Theorem 1.5 and its corollary above: If $Hameo(D^2, \partial D^2) = Homeo^\Omega(D^2, \partial D^2)$, then $\ker \overline{\mathrm{Cal}}$ will provide a proper normal subgroup of $Homeo^\Omega(D^2, \partial D^2)$. On the other hand, if $Hameo(D^2, \partial D^2)$ is a proper subset of $Homeo^\Omega(D^2, \partial D^2)$, again we conclude that $Homeo^\Omega(D^2, \partial D^2)$ is not simple since we know $Hameo(D^2, \partial D^2)$ is a normal subgroup thereof. Theorem 1.5 proves a stronger theorem in that it rules out the first possibility mentioned above.

Extension of Calabi homomorphism $\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id) \to \mathbb{R}$ exploits the natural embedding

$$i^+ : \mathcal{P}^{ham}(Symp(D^2, \partial D^2), id) \to \mathcal{P}^{ham}(Symp(S^2), id) \tag{1.8}$$

obtained via the embedding $D^2 \hookrightarrow S^2$ as the upper hemisphere, and the Calabi property of Entov-Polterovich's spectral quasimorphism [EP1] in an essential way.

1.5. **Outline of the proof of Main Theorem.** In this subsection we provide the chronological outline of the main ingredients established in this paper towards the proof of non-simpleness of $Homeo^\Omega(D^2, \partial D^2)$ assuming Theorem 1.7.

(1) We introduce the notion of *hamiltonian homotopy* of topological Hamiltonian paths.
(2) In a sequel [Oh11] to this paper, we prove Theorem 1.7 which in turn proves that spectral invariant $\rho(\lambda; 1)$ is preserved under the hamiltonian homotopy of $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ with a suitable support condition spelled out in Theorem 1.7 on any closed symplectic manifolds.
(3) Applying the smoothing theorem from [Oh9], [Si] of area preserving homeomorphisms on two dimensional surfaces and using the uniqueness result of topological Hamiltonian by Buhovsky-Seyfaddini [BS] (see also Viterbo [V] for the L^∞-context), we adapt Entov-Polterovich's construction and define a quasimorphism

$$\overline{\mu}^{path} : \mathcal{P}^{ham}(Sympeo(S^2), id) \to \mathbb{R}$$

on S^2 that extends the one on $\mathcal{P}^{ham}(Symp(S^2), id)$ and has the Calabi property in the sense of Entov-Polterovich [EP1].

(4) Using the homotopy invariance of the invariant $\rho(\lambda;1)$, we obtain homotopy invariance of $\overline{\mu}^{path}(\lambda)$.

(5) We establish the equality
$$\mu^{path}(\lambda) \leq \mathrm{vol}_\omega(M)\,\rho(\lambda;1).$$

(6) Via the embedding $i^+ : \mathcal{P}^{ham}(Symp(D^2,\partial D^2), id) \hookrightarrow \mathcal{P}^{ham}(Symp(S^2), id)$ we define the compactly supported analog of $\mathcal{P}^{ham}(Sympeo(D^2,\partial D^2), id)$.

(7) Using the Calabi property of $\overline{\mu}^{path}$ and the canonical embedding, we represent the Calabi homomorphism
$$\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2,\partial D^2), id) \to \mathbb{R}$$
defined in [Oh10] in terms of $\overline{\mu}^{path} \circ i^+$, i.e., $\overline{\mathrm{Cal}}^{path}(\lambda) = \overline{\mu}^{path}(i^+(\lambda))$.

(8) We prove that the Alexander isotopy of topological Hamiltonian loop in $\mathcal{P}^{ham}(Sympeo(D^2,\partial D^2), id)$ embedded into $\mathcal{P}^{ham}(Sympeo(S^2), id)$ via i^+ is a contractible hamiltonian homotopy in our definition.

(9) These give rise to the vanishing result $\overline{\mathrm{Cal}}^{path}(\lambda) = 0$ for any topological Hamiltonian loop in $\mathcal{P}^{ham}(Sympeo(D^2,\partial D^2), id)$. This implies that the value of
$$\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2,\partial D^2), id) \to \mathbb{R}$$
depends only on its final point and so descends to a homomorphism
$$\overline{\mathrm{Cal}} : Hameo(D^2,\partial D^2) \to \mathbb{R}$$
which extends the well-known Calabi homomorphism on $Diff^\Omega(D^2,\partial D^2)$.

(10) Using this extension theorem to $Hameo(D^2,\partial D^2)$ of the Calabi homomorphism, we prove that a compactly supported area preserving homeomorphism examined in [Oh10], which is constructed by an infinite repetition construction, cannot lie in $Hameo(D^2,\partial D^2)$.

(11) As a corollary, we prove that $Hameo(D^2,\partial D^2)$ is a proper normal subgroup of the group $Homeo^\Omega(D^2,\partial D^2)$ of compactly supported area preserving homeomorphisms in $\mathrm{Int}\, D^2$. Therefore $Homeo^\Omega(D^2,\partial D^2)$ is *not simple.*

We thank A. Fathi for introducing to the author the idea of construction of infinite repetition in the homeomorphism category and pointing out that extending Calabi homomorphisms to Hamiltonian homeomorphism group would be a key ingredient for the properness question during his visit of Korea Institute for Advanced Study in the summer of 2005 [Fa2]. We are especially grateful to S. Seyfaddini for showing us a counter example to the statement in the main theorem of the earlier version of the paper [Oh11] which led the author to revisit the entire framework of the the present paper and [Oh11].

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2. Review of spectral invariants $\rho(\lambda;1)$

We define the Novikov covering space of the set $\Omega_0(M)$ of contractible loops on M by the set of equivalence classes $[z,w]$ of the pairs (z,w) where z is one-periodic orbit of the Hamilton equation $\dot x = X_H(t,x)$ and $w : D^2 \to M$ a smooth map with $w|_{\partial D^2} = z$: We say $(z,w) \sim (z,w')$ if we have
$$c_1(\overline{w}\#w') = 0 = \omega(\overline{w}\#w').$$

We denote by Γ the deck transformation group of the covering $\widetilde\Omega_0(M) \to \Omega_0(M)$ which is isomorphic to

$$\Gamma = \frac{\pi_2(M)}{\ker c_1 \cap \ker[w]}.$$

We denote by Λ_ω the associated Novikov ring which is the (upward) completion of the group ring $\mathbb{Q}[\Gamma]$. We also denote by

$$\Gamma_\omega := \omega(\Gamma) = \omega(\pi_2(M))$$

which we call the (spherical) period group of (M,ω). The action functional $\mathcal{A}_H : \widetilde\Omega_0(M) \to \mathbb{R}$ is defined by

$$\mathcal{A}_H([z,w]) = -\int w^*\omega - \int_0^1 H(t, z(t))\, dt.$$

Note that $[z,w] \in \operatorname{Crit} \mathcal{A}_H$ if and only if z satisfies $\dot z = X_H(t,z)$. We then define

$$\operatorname{Spec}(H) = \{\mathcal{A}_H([z,w]) \mid [z,w] \in \operatorname{Crit} \mathcal{A}_H\}$$

Recall that Γ_ω is either a discrete or a countable dense subgroup of $\mathbb{R}$. The following was proven in [Oh3].

Proposition 2.1. *Let H be any periodic Hamiltonian.* $\operatorname{Spec}(H)$ *is a measure zero subset of* $\mathbb{R}$ *for any H.*

We note that when $H = 0$, we have

$$\operatorname{Spec}(H) = \Gamma_\omega.$$

Definition 2.1. We say that two normalized Hamiltonians H and F are *homotopic* if $\phi_H^1 = \phi_F^1$ and their associated Hamiltonian paths ϕ_H, $\phi_K \in \mathcal{P}(Ham(M,\omega), id)$ are path-homotopic relative to the boundary. In this case we denote $H \sim F$ and denote the set of equivalence classes by $\widetilde{Ham}(M,\omega)$.

The following lemma was proven in [Sc], (for the aspherical case) [Oh4] (for the general case).

Proposition 2.2. *Suppose that F, G are normalized. If $F \sim G$, we have*

$$\operatorname{Spec}(F) = \operatorname{Spec}(G)$$

as a subset of $\mathbb{R}$*.*

Assume that H is nondegenerate and $[z_i, w_i] \in \operatorname{Crit} \mathcal{A}_H$.

Definition 2.2. Consider a formal power series

$$\alpha = \sum_{i=1}^{\infty} a_i [z_i, w_i], \quad a_i \in \mathbb{Q}.$$

We define the support of α by

$$\operatorname{supp}(\alpha) = \{[z_i, w_i] \in \operatorname{Crit} \mathcal{A}_H \mid a_i \neq 0\}.$$

We say that α is a (Novikov) Floer chain if it satisfies the Novikov finiteness condition that for any $C \in \mathbb{R}$

$$\#\left(\operatorname{supp} \alpha \cap \mathcal{A}_H^{-1}([C,\infty))\right) < \infty.$$

We define the Floer chain module $CF_*(H)$ to be the set of Floer chains of H. $CF_*(H)$ is a naturally an (infinite dimensional) $\mathbb{Q}$-vector space and has the natural structure of Λ_ω-module.

Definition 2.3. The level of a Floer chain α, denoted by $\lambda_H(\alpha)$, is defined to be

$$\lambda_H(\alpha) = \max\{\mathcal{A}_H([z_i, w_i]) \mid [z_i, w_i] \in \operatorname{supp} \alpha\}.$$

The level function λ_H induces a natural $\mathbb{R}$-filtration on $CF_*(H)$ and so induces a natural topology thereon.

Now for each given generic one-periodic $J = \{J_t\}$, we consider the perturbed Cauchy-Riemann equation

$$\frac{\partial u}{\partial \tau} + J\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0 \tag{2.1}$$

and define a boundary map $\partial_{(H,J)} : CF_*(H) \to CF_{*-1}(H)$ by studying the moduli space of solutions of (2.1). Gromov-Floer compactness theorem implies that the map $\partial_{(H,J)}$ is continuous with respect to the above mentioned topology. (For the main purpose of this paper which deals with the case of S^2, we will only need this construction for the monotone case which was given in Floer's original paper [Fl2].)

Definition 2.4. A Floer chain α is called a cycle if $\partial_{(H,J)}(\alpha) = 0$ and a boundary if $\alpha = \partial_{(H,J)}(\beta)$ for a Floer chain β. We say two Floer cycles α, $\alpha' \in CF_*(H)$ are homologous (with respect to J) if $\alpha - \alpha' = \partial_{(H,J)}(\beta)$ for some β.

The associated homology is denoted by $HF_*(H, J)$, called the Floer homology of H whose isomorphism class does not depend on the choice of J. We call J H-regular if the pair (J, H) satisfies all the necessary transversality holds for the definition of Floer homology $HF_*(H, J)$. For each given (homogeneous) quantum cohomology class $a \in QH^*(M)$, we denote by $a^\flat = a^\flat_H \in HF_*(H, J)$ the image under the isomorphism $QH^*(M) \to HF_*(H, J)$. We denote by

$$i_\lambda : HF_*^\lambda(H, J) \to HF_*(H, J)$$

the canonical inclusion induced homomorphism.

Definition 2.5. Let H be a nondegenerate Hamiltonian, not necessarily normalized. For any given $0 \neq a \in QH^*(M)$, we consider Floer cycles $\alpha \in CF_*(H)$ of the pair (H, J) representing $a^\flat$. Then we define

$$\rho((H, J); a) := \inf_{\alpha; [\alpha] = a^\flat} \lambda_H(\alpha),$$

or equivalently

$$\rho((H, J); a) := \inf\{\lambda \in \mathbb{R} \mid a^\flat \in \operatorname{Im} i_\lambda \subset HF_*(H, J)\}.$$

We will mostly use the first definition in our exposition, which is more intuitive and flexible to use in practice. The following theorem was proved in [Oh5].

Theorem 2.3. *Suppose that H is nondegenerate and let $0 \neq a \in QH^*(M)$.*

1. *We have $\rho((H, J); a) > -\infty$ for any J.*
2. *The definition of $\rho((H, J); a)$ does not depend on the choice of J's. We denote by $\rho(H; a)$ the common value.*

We will be exclusively interested in the case of $a = 1$. Then we have the following triangle inequality

$$\rho(H_1 \# H_2; 1) \leq \rho(H_1; 1) + \rho(H_2; 1). \tag{2.2}$$

and $|\rho(H; 1)| \leq \|H\|$. (See [Oh1], [Sc], [Oh5] for its proof.) From now on, we use the one-one correspondence $H \leftrightarrow \phi_H$ for normalized Hamiltonian H and write

$$\rho(\lambda; 1) = \rho(\phi_H; 1) := \rho(H; 1), \quad \lambda \in \mathcal{P}^{ham}(Symp(M, \omega), id)$$

when $\lambda = \phi_H$. (See [Oh5], [Oh7] for further details on the study of spectral invariants.)

The following lemma turns out to be useful for the further discussion.

Lemma 2.4. *We have*

$$\begin{aligned} |\rho(\lambda_1; 1) - \rho(\lambda_2; 1)| &\leq \max\{\rho(\lambda_1^{-1}\lambda_2; 1), \rho(\lambda_2^{-1}\lambda_1; 1)\} \\ &\leq \max\{\|H_1 - H_2\|, \|\overline{H}_1 - \overline{H}_2\|\} \end{aligned} \tag{2.3}$$

where $H_i = \mathrm{Dev}(\lambda_i)$ *for* $i = 1, 2$.

Proof. By the triangle inequality, we have

$$\rho(\lambda_1; 1) - \rho(\lambda_2; 1) \leq \rho(\lambda_1 \lambda_2^{-1}; 1).$$

By switching H and F around, we also have

$$\rho(\lambda_2; 1) - \rho(\lambda_1; 1) \leq \rho(\lambda_2 \lambda_1^{-1}; 1).$$

Combining the two, we obtain

$$-\rho(\lambda_2 \lambda_1^{-1}; 1) \leq \rho(\lambda_1; 1) - \rho(\lambda_2; 1) \leq \rho(\lambda_1 \lambda_2^{-1}; 1)$$

which finishes the proof.

And the second follows from the first and the inequality $|\rho(\lambda; 1)| \leq \|H\|$. □

3. Flat family of Hamiltonians and hamiltonian homotopy

In this section, we introduce an important notion of *hamiltonian homotopy*. This notion is a crucial ingredient in our proof which enables us to prove that the well-known Alexander isotopy on the disc lives in the category of spectral Hamiltonian flows.

Let (M, ω) be a general symplectic manifold. In this section, for the simplicity of notations, we will always consider diffeomorphisms or homeomorphisms on M supported in $\mathrm{Int}(M)$ if M has a boundary. We will denote by $Sympeo(M, \omega)$ the set of symplectic homeomorphisms supported in $\mathrm{Int}(M)$ if M has a boundary.

To make our presentation most natural, we need to slightly generalize our spectral function ρ_1 to Hamiltonian paths λ issued at general element in $Symp(M, \omega)$. Note that ρ_1 is currently defined only on $\mathcal{P}^{ham}(Symp(M, \omega), id)$ i.e. for paths issued at the identity.

Definition 3.1. A *(smooth) Hamiltonian path* $\lambda : [0, 1] \to Symp(M, \omega)$ is a smooth map

$$\Lambda : [0, 1] \times M \to M$$

if we can write

$$\lambda(t) = \phi_H^t \circ \lambda(0), \quad \text{or equivalently,} \quad dH_t \circ \lambda(t) = \dot{\lambda}(t) \rfloor \omega.$$

We call such a function $H : \mathbb{R} \times M \to \mathbb{R}$ a *generating Hamiltonian* of λ. We denote by $\mathcal{P}^{ham}(Symp(M,\omega))$ the set of Hamiltonian paths $\lambda : [0,1] \to Symp(M,\omega)$. There is a natural evaluation maps

$$ev_0,\, ev_1 : \mathcal{P}^{ham}(Symp(M,\omega)) \to Symp(M,\omega)$$

defined by $ev_i(\lambda) = \lambda(i)$ for $i = 0,\, 1$.

If $\lambda(0) := \Lambda(0,\cdot) : M \to M$ is a Hamiltonian diffeomorphism in addition, and therefore $\lambda(t) = \Lambda(t,\cdot)$ is for all $t \in [0,1]$, we call it a Hamiltonian path lying in $Ham(M,\omega)$. We denote by $\mathcal{P}(Ham(M,\omega))$ the set of Hamiltonian paths lying in $Ham(M,\omega)$.

By definition, it follows

$$\begin{aligned}&(ev_0)^{-1}(Ham(M,\omega)) = \mathcal{P}(Ham(M,\omega))\\ &(ev_0)^{-1}(id) = \mathcal{P}^{ham}(Symp(M,\omega), id) = \mathcal{P}(Ham(M,\omega), id).\end{aligned}$$

It is also easy to check that a concatenation of two Hamiltonian paths are Hamiltonian again as long as the paths are constant near the boundary.

Let $\Lambda = \Lambda(s,t) : [0,1]^2 \to Ham(M,\omega)$ be a smooth two-parameter family of Hamiltonian diffeomorphisms satisfying

$$\Lambda(s,0) \equiv id. \tag{3.1}$$

We denote by $H = H(s,t,x)$ and $F = F(s,t,x)$ be the Hamiltonian functions satisfying

$$\frac{\partial \Lambda}{\partial t} \circ \Lambda^{-1} \quad = \quad X_H \tag{3.2}$$

$$\frac{\partial \Lambda}{\partial s} \circ \Lambda^{-1} \quad = \quad X_F \tag{3.3}$$

The following flatness lemma was proved in [P2], [Oh4] based on Banyaga's integrability condition

$$\frac{\partial X_{s,t}}{\partial s} = \frac{\partial Y_{s,t}}{\partial t} + [X_{s,t}, Y_{s,t}]$$

for the pair $\{X_{s,t}, Y_{s,t}\}$ of vector fields to integrate to a two-parameter family of diffeomorphisms [B]. We refer readers to [Oh4] for the proof of this lemma.

Lemma 3.1. *Let Λ and H, F be as above. Suppose that H satisfies one of the following two normalization conditions : For all t, $s \in [0,1]^2$, we have*

(1) $\int_M H_t^s \, d\mu = 0$ *or*
(2) $\mathrm{supp}(H_t^s) \subset \mathrm{Int}(M)$.

Then we have

$$\frac{\partial F}{\partial t} = \frac{\partial H}{\partial s} - \{F, H\}. \tag{3.4}$$

We call this property of the pair (H,F) flatness *or* integrability *condition for the pair.*

We state the following converse to this lemma

Proposition 3.2. *Let F, $H : [0,1]^2 \times M \to \mathbb{R}$ be two-parameter family of smooth functions satisfying (3.4). Then there exists a smooth family $\Lambda : [0,1]^2 \to Ham(M,\omega)$ that satisfies (3.2) and (3.3).*

Proof. We just integrate the Hamilton equation $\dot x = X_{H^s}(t,x)$ with the initial condition $\phi^0_{H^s} = id$. We define Λ by

$$\Lambda(s,t) = \phi^t_{H^s}.$$

A direct calculation of $\frac{\partial \Lambda}{\partial s}$ using the flatness of (H,F) proves (3.3) whose proof we omit. □

The following notion of hamiltonian homotopy of topological hamiltonian paths is an important ingredient in our study.

Definition 3.2. Let $\lambda_0,\, \lambda_1 \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$. A *hamiltonian homotopy* $\Lambda : [0,1]^2 \to Sympeo(M,\omega)$ between λ_0 and λ_1 based at the identity is the map such that

$$\Lambda(0,t) = \lambda_0(t),\, \Lambda(1,t) = \lambda_1(t), \tag{3.5}$$

and $\Lambda(s,0) \equiv id$ for all $t \in [0,1]$, and arises as follows: there is a sequence of smooth maps $\Lambda_j : [0,1]^2 \to Ham(M,\omega)$ that satisfy

(1) $\Lambda_j(s,0) = id$,
(2) $\Lambda_j \to \Lambda$ in C^0-topology
(3) Any 'horizontal' section $\Lambda_{j,s} : \{s\} \times [0,1] \to Ham(M,\omega)$ converges in hamiltonian topology in the following sense: If we write
$$\mathrm{Dev}\left(\Lambda_{j,s}\Lambda_{j,0}^{-1}\right) =: H_j(s),$$
then $H_j(s)$ converges in hamiltonian topology uniformly over $s \in [0,1]$. We call any such Λ_j an *approximating sequence* of Λ.

When $\lambda_0(1) = \lambda_1(1) = \psi$, a *hamiltonian homotopy relative to the ends* is one that satisfies $\Lambda(s,0) = id$, $\Lambda(s,1) = \psi$ for all $s \in [0,1]$ in addition.

We say that $\lambda_0,\, \lambda_1 \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ are *hamiltonian homotopic* (resp. relative to the ends), if there exists a hamiltonian homotopy (resp. a hamiltonian homotopy relative to the ends).

We emphasize that by the requirement (3),

$$H_j(0) \equiv 0 \tag{3.6}$$

in this definition, when H_j is mean-normalized.

We would like to emphasize that even when λ is a loop i.e., satisfies $\lambda(0) = \lambda(1) = id$, its approximation sequence of smooth Hamiltonian paths are not necessarily loops. This phenomenon is the heart of the difficulty lying in the study of *topological* Hamiltonian loops.

It is easy to check that hamiltonian homotopy is an equivalence relation. We call the equivalence class of a topological hamiltonian path λ a *hamiltonian homotopy class* of the path λ (relative to the ends). A variation of the above definition can be also applied to define a hamiltonian homotopy of 'loops' not necessarily based at the identity

Definition 3.3. Consider two loops $\lambda_0,\, \lambda_1 \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$, i.e., maps satisfying

$$\lambda_0(0) = \lambda_0(1) = \lambda_1(0) = \lambda_1(1) = id.$$

We say the λ_0, λ_1 are *hamiltonian homotopic* to each other, if there exists a hamiltonian-continuous family $\Lambda : [0,1]^2 \to Sympeo(M,\omega)$ such that

$$\Lambda(0,t) = \lambda_0(t),\, \Lambda(1,t) = \lambda_1(t) \tag{3.7}$$

and $\Lambda(s,0) = \Lambda(s,1) = id$ for all $s \in [0,1]$. We call such a Λ a hamiltonian homotopy of loops between λ_0 and λ_1. We denote by

$$\pi_1^{ham}(Hameo(M,\omega), id)$$

the set of equivalence classes of topological hamiltonian loops of (M,ω).

It is easy to check that $\pi_1^{ham}(Hameo(M,\omega), id)$ is a group. In fact we can prove

Proposition 3.3. *The group $\pi_1^{ham}(Hameo(M,\omega), id)$ is abelian.*

Proof. Let λ, μ be topological hamiltonian loops based at the identity. Using the group property, it is enough to prove

$$[\lambda][\mu][\lambda]^{-1}[\mu]^{-1} = [\lambda\mu\lambda^{-1}\mu^{-1}] = id$$

in $\pi_1^{ham}(Hameo(M,\omega), id)$. By definition, we need to construct a hamiltonian continuous family $\Lambda : [0,1]^2 \to Hameo(M,\omega)$ such that

$$\Lambda(0,t) = id,\ \Lambda(1,t) = \lambda(t)\mu(t)\lambda(t)^{-1}\mu(t)^{-1}.$$

In fact we claim that the family

$$s \mapsto \lambda(st)\mu(t)\lambda(st)^{-1}\mu(t)^{-1} =: \Lambda(s,t)$$

is such a family. Let H be the topological Hamiltonian associated to μ, i.e., $\mu = \phi_H$.

It follows $\Lambda(s,1) = \lambda(0)\mu(1)\lambda(0)^{-1}\mu(1)^{-1} = id$ and so each $\Lambda(s,\cdot)$ defines a loop. Furthermore the horizontal section

$$t \mapsto \Lambda(s,t)$$

is generated by the Hamiltonian

$$H(t, \lambda^{-1}(st)) - H(t, \mu(t)\lambda(st)\mu^{-1}(t)\lambda(st)^{-1}).$$

These and unraveling the definition of hamiltonian homotopy given in Definition 3.2 show that $\lambda\mu\lambda^{-1}\mu^{-1}$ is hamiltonian homotopic to the constant path id and so $[\lambda\mu\lambda^{-1}\mu^{-1}] = id$. This finishes the proof. □

With this definition of hamiltonian homotopy, the following question is an important question to ask in relation to the simpleness question of the area-preserving homeomorphism group of S^2.

Question 3.4. Is any continuous hamiltonian loop on $Hameo(S^2)$ hamiltonian homotopic either to the constant loop or to a loop of one-full rotation around an axis? In particular, is $\pi_1^{ham}(Hameo(S^2), id) \cong \mathbb{Z}_2$?.

4. Spectral Calabi quasimorphism on $\mathcal{P}^{ham}(Sympeo(S^2), id)$

In this section, we will restrict to the case of S^2 with the standard symplectic form ω_{S^2} with area 4π on it. Here we mostly borrow the materials presented in [Oh10] with some additional twists.

Omitting the symplectic form ω_{S^2} from their notations, we just denote the groups of topological Hamiltonian paths and of Hamiltonian homeomorphisms on S^2 respectively by $\mathcal{P}^{ham}(Sympeo(S^2), id)$ and $Hameo(S^2)$.

4.1. **Smooth Hamiltonian case.** We first recall some basic properties of $\rho(\cdot;1)$ essentially established by Entov and Polterovich [EP1]. The following important fact can be derived from the arguments used in [EP1]. (See the proof of Theorem 3.1 [EP1] in the context of the covering space $\widetilde{Ham}(S^2)$ but its proof equally applies to the context of the path space). For readers' convenience, we will give its proof in Appendix.

Lemma 4.1. *There exists a constant $R = R(S^2) > 0$ independent of H such that*

$$\rho(\phi_H^{-1};1) \leq -\rho(\phi_H;1) + R. \tag{4.1}$$

In fact, we will show in Appendix that we can take $R = 4\pi$ which is the area of S^2.

We prove the following quasimorphism property of $\lambda \mapsto \rho(\lambda;1)$ on S^2 using Lemma 4.1, which is the path space version of Theorem 3.1 [EP1].

Proposition 4.2. *Consider S^2 with the standard symplectic form ω_{S^2} on it. Let H, F be smooth normalized Hamiltonians satisfying. Then the spectral invariant $\rho(\cdot;1)$ satisfies*

$$|\rho(\phi_H\phi_F;1) - (\rho(\phi_H;1) + \rho(\phi_F;1))| \leq R \tag{4.2}$$

for some constant $R = R(S^2) > 0$ depending only on ω_{S^2} but independent of H, F. In particular, the map

$$\rho(\cdot;1) : \mathcal{P}^{ham}(Symp(S^2), id) \to \mathbb{R}$$

defines a quasimorphism.

Proof. The inequality

$$\rho(\phi_H\phi_F;1) - (\rho(\phi_H;1) + \rho(\phi_F;1)) \leq 0$$

is nothing but the triangle inequality of $\rho(\cdot;1)$.

By the triangle inequality, we also have

$$\rho(\phi_H\phi_F;1) \geq \rho(\phi_H;1) - \rho(\phi_F^{-1};1).$$

On the other hand, applying Lemma 4.1 we get

$$\rho(\phi_H\phi_F;1) \geq \rho(\phi_H;1) + \rho(\phi_H;1) - R. \tag{4.3}$$

Combination of (4.2) and (4.3) finishes the proof. □

We refer to [GG2], [EP1], [Ca] for the general discussion on the basic properties of the quasimorphism.

Based on this quasimorphism $\rho(\cdot;1)$, Entov and Polterovich defined a homogeneous quasimorphism on the universal covering space $\widetilde{Ham}(S^2,\Omega)$

$$\widetilde{\mu} : \widetilde{Ham}(S^2,\Omega) \to \mathbb{R}$$

by the formula

$$\widetilde{\mu}(\widetilde{\phi}) = \mathrm{vol}(S^2) \lim_{i\to\infty} \frac{\rho(\widetilde{\phi}^m;1)}{m}. \tag{4.4}$$

Obviously this definition of homogeneous quasimorphism can be lifted to the level of Hamiltonian paths:

Definition 4.1. We define a homogeneous quasimorphism

$$\mu^{path} : \mathcal{P}^{ham}(Symp(S^2), id) \to \mathbb{R}$$

by the same formula

$$\mu^{path}(\lambda) = \mathrm{vol}(S^2) \lim_{i\to\infty} \frac{\rho(\lambda^m; 1)}{m}. \tag{4.5}$$

From its definition and from the triangle inequality of $\rho(\cdot; 1)$, we can prove

Lemma 4.3. *We have*

$$\mu^{path}(\phi_H) \;\leq\; \mathrm{vol}(S^2)\rho(\phi_H; 1) \tag{4.6}$$

$$|\mu^{path}(\phi_H) - \mu^{path}(\phi_{H'})| \;\leq\; \mathrm{vol}(S^2)(\|H' - H\| + \|\overline{H}' - \overline{H}\|) \tag{4.7}$$

Proof. The inequality (4.6) immediately follows from the triangle inequality of $\rho(\cdot; 1)$. And (4.7) follows from an examination of the proof of Proposition 3.5 [EP1], where the inequality

$$|\widetilde{\mu}(\widetilde{\psi}_H) - \widetilde{\mu}(\widetilde{\psi}_{H'})| \leq \mathrm{vol}(S^2) \int_0^1 \|H_t - H'_t\|_{C^0}\, dt$$

is proved. □

Remark 4.2. The inequalities (4.6), (4.7) can be proved by the same argument for μ^{path} defined by the same formula as (4.5) on *arbitrary closed* symplectic manifold (M, ω), whether or not it defines a quasimorphism.

The following proposition concerning the quasimorphism μ^{path} were essentially proved by Entov and Polterovich [EP1].

Proposition 4.4 (Compare with Proposition 3.3 [EP1])**.** *Suppose that $U \subset S^2$ that is displaceable, i.e., there exists $\phi \in Ham(S^2)$ such that $\phi(U) \cap \overline{U} = \emptyset$. Then we have the identity*

$$\mu^{path}(\lambda) = \mathrm{Cal}_U^{path}(\lambda)$$

for all λ with

$$\mathrm{supp}\, \lambda \subset U.$$

Entov and Polterovich called this property the *Calabi property* of a quasimorphism. We recall that $\mathrm{Cal}_U^{path}(\lambda)$ is defined by the integral

$$\mathrm{Cal}_U^{path}(\lambda) = \int_0^1 \int_U H(t, x)\, \Omega_\omega \tag{4.8}$$

on open subset U of general (M, ω) when $\lambda = \phi_H$. Here Ω_ω is the Liouville volume form normalized so that $\int_M \Omega_\omega = 1$.

For any given open subset $U \subset S^2$, we denote by

$$\mathcal{P}^{ham}(Symp_U(S^2), id)$$

the set of Hamiltonian paths supported in U.

An immediate corollary of the above two propositions is the following homomorphism property of μ restricted to $\mathcal{P}^{ham}(Symp_U(S^2), id)$.

Corollary 4.5. *Suppose that U is an open subset of S^2 such that $\overline{U}$ is displaceable on S^2 and let*

$$\lambda_1,\, \lambda_2 \in \mathcal{P}^{ham}(Symp_U(S^2), id).$$

Then we have

$$\mu^{path}(\lambda_1\lambda_2) = \mu^{path}(\lambda_1) + \mu^{path}(\lambda_2).$$

4.2. **Topological Hamiltonian case.** Now we explain how we extend μ^{path} to the group $\mathcal{P}^{ham}(Sympeo(M,\omega), id)$ of $L^{(1,\infty)}$-topological hamiltonian paths following the exposition of [Oh10] with slight modifications.

We first recall from [Oh10] the extension of spectral invariants to topological Hamiltonian paths: For any topological Hamiltonian path λ, we define

$$\rho(\lambda; 1) = \lim_{i\to\infty} \rho(\phi_{H_i}; 1) = \lim_{i\to\infty} \rho(H_i; 1)$$

for a (and so any) approximating sequence ϕ_{H_i} of λ. By the uniqueness theorem of [BS], this definition is well-defined.

Definition 4.3. Let $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$. We define

$$\overline{\mu}^{path}(\lambda) = \lim_{n\to\infty} \frac{1}{n}\rho(\lambda^n; 1).$$

The following lemma immediately follows from (4.7).

Lemma 4.6. *Let $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$. Then*

$$\overline{\mu}^{path}(\lambda) = \lim_{i\to\infty} \mu^{path}(\phi_{H_i})$$

for any approximating sequence ϕ_{H_i} of λ.

Proof. Since ϕ_{H_i} approximates $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$,

$$\|H_i - H\| + \|\overline{H}_i - \overline{H}\| \to 0$$

as $i \to \infty$, where H is the topological Hamiltonian such that $\lambda = \phi_H$. Then we apply (4.7) to H_i, H_j and taking the limit first as $j \to \infty$ and then $i \to \infty$. □

Once these are established, we are ready to state the C^0-analogue to the Calabi property of μ^{path}.

First we define the C^0 analog to $\mathcal{P}^{ham}(Symp_U(M,\omega), id)$. The following definition is taken from Definition 6.2 [OM] to which we refer readers for more detailed discussions

Definition 4.4. Let $U \subset M$ be an open subset. Define $\mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$ to be the union

$$\mathcal{P}^{ham}(Sympeo_U(M,\omega), id) := \bigcup_{K\subset U} \mathcal{P}^{ham}(Sympeo_K(M,\omega), id)$$

with the direct limit topology, where $K \subset U$ is a compact subset.

We would like to emphasize that this set is not necessarily the same as the set of $\lambda \in \mathcal{P}^{ham}(Sympeo(M,\omega), id)$ with compact $\operatorname{supp}\lambda \subset U$.

With this definition, the following is an immediate consequence of the definition of $\overline{\mu}^{path}$ and the Calabi property of μ^{path}. This proposition was stated in [Oh10] without details of a precise statement and its proof.

Proposition 4.7. *Suppose that U is displaceable. The function*

$$\overline{\mu}^{path} : \mathcal{P}^{ham}(Sympeo_U(M,\omega), id) \to \mathbb{R}$$

is a quasimorphism which satisfies

$$\overline{\mu}^{path}(\lambda) = \overline{\mathrm{Cal}}_U(\lambda) \tag{4.9}$$

for any $\lambda \in \mathcal{P}^{ham}(Sympeo_U(M,\omega), id)$.

Proof. The quasimorphism property immediately follows from that of μ^{path} by taking the limit using Lemma 4.6.

Similarly (4.9) follows from the Calabi property of μ^{path} by taking the limit using the Definition 4.4. □

It is well-known that $\pi_1(Ham(S^2)) \cong \mathbb{Z}_2$. This property is used in a crucial way in [EP1] to descend $\mu^{path} : \mathcal{P}^{ham}(Symp(S^2), id) \to \mathbb{R}$ to a quasimorphism on $Ham(S^2)$. The corresponding question for $\overline{\mu}^{path} : \mathcal{P}^{ham}(Sympeo(S^2), id) \to \mathbb{R}$ is of great importance whose detailed discussion is in order in the next section.

We strongly believe that the answer to Question 3.4 is affirmative, which would prove that $\overline{\mu}^{path}$ descends to a Calabi quasimorphism on $Hameo(S^2)$. Then the scheme of our proof in the present paper would also lead to properness of $Hameo(S^2)$ in $Homeo^\Omega(S^2)$ as well as that of $Hameo(D^2, \partial D^2) \subset Homeo^\Omega(D^2, \partial D^2)$ as pointed out in Theorem 7.8 [Oh10]. Since we do not know the answer to this question at this point, we will restrict ourselves to the disc case and take a different route of considering an extension of Calabi homomorphism on $Diff^\Omega(D^2, \partial D^2)$ to $Hameo(D^2, \partial D^2)$ instead.

5. Hamiltonian homotopy invariance of $\overline{\mathrm{Cal}}^{path}$ on D^2

Now we prove a crucial ingredient that is needed in our proof of the theorem in the next section that the Alexander isotopy on D^2 exists on the topological hamiltonian category. Here the homotopy invariance of spectral invariants $\rho(\lambda; 1)$ under the topological hamiltonian homotopy of λ which can be easily derived from Theorem 1.7 plays an essential role in this extension.

Denote by $\mathcal{P}^{ham}(Symp(D^2, \partial D^2); id)$ the group of Hamiltonian paths compactly supported in $\mathrm{Int}(D^2)$, i.e.,

$$\bigcup_{t \in [0,1]} \mathrm{supp}\, H_t \subset \mathrm{Int}(D^2).$$

We denote by $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ the $L^{(1,\infty)}$ hamiltonian completion of $\mathcal{P}^{ham}(Symp(D^2, \partial D^2); id)$.

We now fix the standard embedding of $D^2 \to S^2$ as the upper hemisphere S_+. The given identification of D^2 as S_+ we can embed

$$i^+ : \mathcal{P}^{ham}(Symp(D^2, \partial D^2); id) \hookrightarrow \mathcal{P}^{ham}(Symp(S^2); id)$$

by extending any element $\phi_H \in \mathcal{P}^{ham}(Symp(D^2, \partial D^2); id)$ to the one that is identity on the lower hemisphere by setting $H \equiv 0$ on S_-.

We recall the extended Calabi homomorphism defined in [Oh10] whose well-definedness follows from the uniqueness theorem from [BS].

Definition 5.1. Let $\lambda \in \mathcal{P}^{ham}(Symp(D^2, \partial D^2), id)$ and H be its compactly supported Hamiltonian. We define

$$\overline{\mathrm{Cal}}^{path}_{D^2}(\lambda) := \overline{\mathrm{Cal}}^{path}(H).$$

It is immediate to check that this defines a homomorphism.

The following is a crucial proposition.

Proposition 5.1. *Let $\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id) \to \mathbb{R}$ be the above extension of the Calabi homomorphism Cal^{path}. Suppose that $\lambda_0(1) = \lambda_1(1)$ and $i^+(\lambda_0)$ is hamiltonian homotopic to $i^+(\lambda_1)$ relative to the end in $\mathcal{P}^{ham}(Symp(S^2), id)$. Then we have*

$$\overline{\mathrm{Cal}}^{path}(\lambda_0) = \overline{\mathrm{Cal}}^{path}(\lambda_1) \tag{5.1}$$

Proof. By the homomorphism property of $\overline{\mathrm{Cal}}^{path}$, it is enough to prove $\overline{\mathrm{Cal}}^{path}(\lambda) = 0$ for an contractible topological hamiltonian loop $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$. Let Λ be a hamiltonian homotopy contracting λ to the identity path in

$$\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id).$$

By definition, there is some $\eta > 0$ such that

$$\mathrm{supp}\, \Lambda \subset D^2(1-\eta)$$

Then the map $i^+(\Lambda)$ provides a contractible hamiltonian homotopy of $i^+(\lambda)$ lying in $\mathcal{P}^{ham}(Sympeo_U(S^2), id)$ where $U = i^+(D^2(1-\eta))$ which is displaceable in S^2. By an abuse of notation, we just denote by $\lambda = i^+(\lambda)$ on S^2.

Then by the Calabi property of $\overline{\mu}^{path}$ we have

$$\begin{aligned} \overline{\mu}^{path}(\lambda) &= \overline{\mathrm{Cal}}^{path}(\lambda), \\ \overline{\mu}^{path}(\lambda^{-1}) &= \overline{\mathrm{Cal}}^{path}(\lambda^{-1}) = -\overline{\mathrm{Cal}}^{path}(\lambda) \end{aligned}$$

and hence $\overline{\mu}^{path}(\lambda^{-1}) = -\overline{\mu}^{path}(\lambda)$. From (4.6),

$$\begin{aligned} \overline{\mu}^{path}(\lambda) &\leq \mathrm{vol}_\omega(S^2)\rho(\lambda; 1), \\ \overline{\mu}^{path}(\lambda^{-1}) &\leq \mathrm{vol}_\omega(S^2)\rho(\lambda^{-1}; 1). \end{aligned}$$

By combining the two, we have obtained

$$-\mathrm{vol}_\omega(S^2)\rho(\lambda^{-1}; 1) \leq \overline{\mathrm{Cal}}^{path}(\lambda) \leq \mathrm{vol}_\omega(S^2)\rho(\lambda; 1).$$

Since both λ and λ^{-1} hamiltonian homotopic to the constant identity path in $\mathcal{P}^{ham}(Sympeo_{S_+}(S^2), id)$, we obtain $\rho(\lambda; 1) = 0 = \rho(\lambda^{-1}; 1)$ by Theorem 1.7. Therefore $\overline{\mu}^{path}(\lambda) = \overline{\mathrm{Cal}}^{path}_{D^2}(\lambda) = 0$ which finishes the proof. □

6. Alexander isotopy of loops in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$

For the description of Alexander isotopy, we need to consider the conjugate action of rescaling maps of D^2

$$R_a : D^2(1) \to D^2(a) \subset D^2(1)$$

for $0 < a < 1$ on $Hameo(D^2, \partial D^2)$, where $D^2(a)$ is the disc of radius a with its center at the origin. We note that R_a is a conformally symplectic map and so its conjugate action maps a symplectic map to a symplectic map whenever it is defined.

Furthermore the right composition by R_a defines a map

$$\phi \mapsto \phi \circ R_a^{-1}\,;\ Hameo(D^2(a), \partial D^2(a); R_a^*\omega_0) \to Hameo(D^2, \partial D^2)$$

and then the left composition by R_a followed by extension to the identity on $D^2 \setminus D^2(a)$ defines a map

$$Hameo(D^2, \partial D^2) \to Hameo(D^2(a), \partial D^2(a); R_a^*\omega_0) \to Hameo(D^2, \partial D^2).$$

We have the following important formula for the behavior of Calabi invariants under the Alexander isotopy.

Lemma 6.1. *Let $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ be a given a continuous Hamiltonian path on D^2. Suppose $\operatorname{supp} \lambda \subset D^2(1-\eta)$ for a sufficiently small $\eta > 0$. Consider the one-parameter family of maps λ_a defined by*

$$\lambda_a(t,x) = \begin{cases} a\lambda(t, \frac{x}{a}) & \text{for } |x| \leq a(1-\eta) \\ x & \text{otherwise} \end{cases}$$

for $0 < a \leq 1$. Then λ_a is also a topological Hamiltonian path on D^2 and satisfies

$$\overline{\mathrm{Cal}}^{path}(\lambda_a) = a^4 \overline{\mathrm{Cal}}^{path}(\lambda). \tag{6.1}$$

Proof. Let H be the unique compactly supported continuous Hamiltonian of λ. A straightforward calculation proves that λ_a is generated by the (unique) compactly supported continuous Hamiltonian H_a defined by

$$H_a(t,x) = \begin{cases} a^2 H\left(t, \frac{x}{a}\right) & \text{for } |x| \leq a(1-\eta) \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Obviously the right hand side function is the hamiltonian-limit of $\mathrm{Dev}(\lambda_{i,a})$ for a sequence λ_i of smooth hamiltonian approximation of λ where $\lambda_{i,a}$ is defined by the same formula for λ_i.

From these, we derive the formula

$$\begin{aligned} \overline{\mathrm{Cal}}_{D^2}^{path}(\lambda_a) &= \lim_{i\to\infty} \mathrm{Cal}_{D^2}^{path}(\lambda_{i,a}) = \lim_{i\to\infty} a^4\, \mathrm{Cal}_{D^2}^{path}(\lambda_i) \\ &= a^4 \lim_{i\to\infty} \int_0^1 \int_{D^2} H_i(t,y)\Omega \wedge dt \\ &= a^4 \lim_{i\to\infty} \mathrm{Cal}^{path}(\lambda_i) = a^4 \overline{\mathrm{Cal}}^{path}(\lambda). \end{aligned}$$

This proves (6.1). □

We would like to emphasize that the s-Hamiltonian F_Λ of $\Lambda(s,t) = \lambda_s^t$ does not converge in $L^{(1,\infty)}$-topology and so we cannot define its hamiltonian limit. Explanation of this relationship is now in order in the following remark.

Remark 6.1. Let $D^{2n} \subset \mathbb{R}^{2n}$ be the unit ball. Consider a smooth Hamiltonian H with $\operatorname{supp} \phi_H \subset \operatorname{Int} D^{2n} \subset \mathbb{R}^{2n}$ and its Alexander isotopy

$$\Lambda(s,t) = \phi_{H^s}^t = \lambda_s(t), \quad \lambda = \phi_H$$

Denote by H_Λ and K_Λ the t-Hamiltonian and the s-Hamiltonian respectively. Then the right hand side of (3.4) for this Λ can be re-written as

$$\frac{\partial}{\partial s}(H \circ \phi_{H^s}^t) \circ (\phi_{H^s}^t)^{-1}$$

and hence

$$\frac{\partial K}{\partial t} = \frac{\partial}{\partial s}(H \circ \phi_{H^s}^t) \circ (\phi_{H^s}^t)^{-1}. \tag{6.3}$$

But we compute

$$H_t \circ \phi_{H^s}^t(x) = s^2 H_t\left(\frac{\phi_{H^s}^t(x)}{s}\right) = s^2 H\left(t, \frac{\phi_{H^s}^t(x)}{s}\right)\left(= -s^2 \bar{H}\left(t, \frac{x}{s}\right)\right).$$

Therefore we derive

$$K(s,t,x) = 2s\int_0^t H\left(u, \frac{x}{s}\right)\, du + s\int_0^t \left\langle \left(d\overline{H}\left(u, \frac{(\phi_{H^s}^u)^{-1}(x)}{s}\right)\right), (\phi_{H^s}^u)^{-1}(x)\right\rangle du. \tag{6.4}$$

For the second summand, we use the identity $\overline{H}(t,x) = -H(t, \phi_H^t(x))$. From this expression, we note that K involves differentiating the Hamiltonian H_i and hence goes out of the $L^{(1,\infty)}$ hamiltonian category.

Theorem 6.2. *Let λ be a loop in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$. Define $\Lambda : [0,1]^2 \to Sympeo(D^2, \partial D^2)$ by*

$$\Lambda(s,t) = \lambda_s(t), \quad \lambda_0 = id.$$

Then Λ is a hamiltonian homotopy between the constant path id and λ.

Proof. We have $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ with $\lambda(0) = \lambda(1)$. Then λ_s defines a loop contained in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ for each $0 \leq s \leq 1$. Let H_i be such that ϕ_{H_i} converges to λ in $L^{(1,\infty)}$ continuous Hamiltonian topology.

We denote by Λ_s and Λ^t the paths defined by

$$\Lambda_s(t) = \Lambda(s,t),\ \Lambda^t(s) = \Lambda(s,t).$$

For a given approximating sequence Λ_i, we fix a sequence $\varepsilon_i \searrow 0$ and define

$$\Lambda_{i,\varepsilon_i} := \Lambda_i(\chi_i(s), t)$$

where $\chi_i : [0,1] \to [\varepsilon_i, 1]$ is a monotonically increasing surjective function with $\chi_i(t) = \varepsilon_i$ near $t = 0$, $\chi_i(1) = 1$ near $t = 1$, and $\chi_i \to id_{[0,1]}$ in the hamiltonian norm. We need do show that the sequence $\Lambda_{i,\varepsilon_i}$ is smooth and uniformly converges in hamiltonian topology as $i \to \infty$ over $s \in [0,1]$ and $\Lambda_{i,\varepsilon_i}^t(1) \to \lambda(t)$. Smoothness follows since the Alexander isotopy is smooth as long as $s > 0$ and by definition $\Lambda_{i,\varepsilon_i}$ involves the Alexander isotopy for $s \geq \varepsilon_i > 0$. Then the convergence immediately follows from the explicit expression of λ_a in Lemma 6.1. ∎

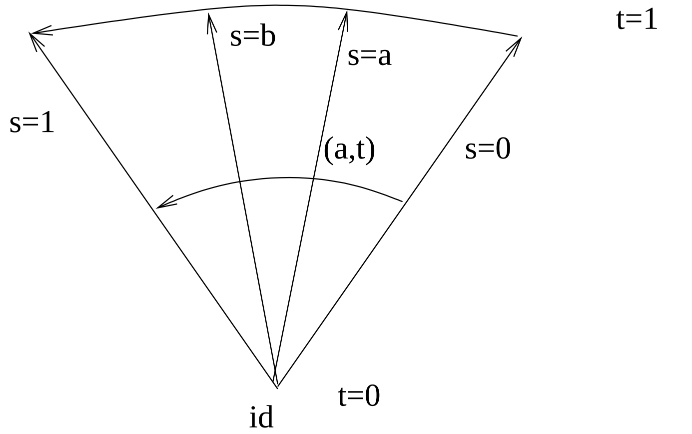

Figure 1: The hamiltonian homotopy $\Lambda(s,t)$

Corollary 6.3. *If* λ_0, $\lambda_1 \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ *and* $\lambda_0(1) = \lambda_1(1)$, *then they are hamiltonian homotopic relative to the end.*

Proof. Theorem 6.2 implies that the Alexander isotopy is a hamiltonian homotopy contracting any topological Hamiltonian loop in $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ to the identity with ends points fixed. This proves that the product loop $\lambda_0 \lambda_1^{-1}$, which is based at the identity, is contractible via a hamiltonian homotopy relative to the ends. Then this implies that λ_0 and λ_1 are hamiltonian homotopic to each other relative to the ends. □

An immediate corollary Proposition 5.1 and Corollary 6.3 is the following

Theorem 6.4. *If* λ_0, $\lambda_1 \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ *and* $\lambda_0(1) = \lambda_1(1)$, *then we have*

$$\overline{\mathrm{Cal}}^{path}(\lambda_0) = \overline{\mathrm{Cal}}^{path}(\lambda_1).$$

This theorem implies that $\overline{\mathrm{Cal}}^{path}$ restricted to $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ depends only on the final point and so gives rise to the following main theorem on the extension of Calabi homomorphism.

Theorem 6.5. *Define a map* $\overline{\mathrm{Cal}} : Hameo(D^2, \partial D^2) \to \mathbb{R}$ *by*

$$\overline{\mathrm{Cal}}(g) := \overline{\mathrm{Cal}}^{path}(\lambda)$$

for a (and so any) $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ *with* $g = \lambda(1)$. *Then this is well-defined and extends the Calabi homomorphism*

$$\overline{\mathrm{Cal}} : Hameo(D^2, \partial D^2) \to \mathbb{R}.$$

7. A wild area preserving homeomorphism on D^2

In this section, we describe an example of a compactly supported area preserving homeomorphism in $Sympeo(D^2, \partial D^2) = Homeo^{\Omega}(D^2, \partial D^2)$ not contained in $Hameo(D^2, \partial D^2)$. Then this implies that $Hameo(D^2, \partial D^2)$ is a proper normal subgroup of $Homeo^{\Omega}(D^2, \partial D^2)$.

This being said, we will focus on construction of an example of a wild area-preserving homeomorphism on D^2, which we borrow the one presented in [Oh10] with some minor correction and clarification of details for readers' convenience. We originally learned from A. Fathi [Fa2] the idea of construction of such a wild area preserving homeomorphism.

Example 7.1. With the above preparations, we consider the set of dyadic numbers $\frac{1}{2^k}$ for $k = 0, \cdots$. Let (r, θ) be polar coordinates on D^2. Then the standard area form is given by

$$\omega = r\, dr \wedge d\theta.$$

Consider maps $\phi_k : D^2 \to D^2$ of the form given by

$$\phi_k = \phi_{\rho_k} : (r, \theta) \to (r, \theta + \rho_k(r))$$

where $\rho_k : (0, 1] \to [0, \infty)$ is a smooth function supported in $(0, 1)$. It follows ϕ_{ρ_k} is an area preserving map generated by an autonomous Hamiltonian given by

$$F_{\phi_k}(r, \theta) = -\int_1^r s\rho_k(s)\, ds.$$

Therefore its Calabi invariant becomes

$$\mathrm{Cal}(\phi_k) = -\int_{D^2} \left(\int_1^r s\rho_k(s)\, ds \right) r\, dr\, d\theta = \pi \int_0^1 r^3 \rho_k(r)\, dt. \tag{7.1}$$

We now choose ρ_k inductively in the following way:

(1) $\mathrm{Cal}(\phi_1) = 1$.
(2) ρ_k has support in $\frac{1}{2^k} < r < \frac{1}{2^{k-1}}$
(3) For each $k = 1, \cdots$, we have

$$\rho_k(r) = 2^4 \rho_{k-1}(2r) \tag{7.2}$$

for $r \in (\frac{1}{2^k}, \frac{1}{2^{k-1}})$.

Since ϕ_k's have disjoint supports by construction, we can freely compose without concerning about the order of compositions. It follows that the infinite product

$$\Pi_{k=0}^{\infty} \phi_k$$

is well-defined and defines a continuous map that is smooth except at the origin at which ϕ_ρ is continuous but not differentiable : This infinite product can also be written as the homeomorphism having its values given by $\phi_\rho(0) = 0$ and

$$\phi_\rho(r, \theta) = (r, \theta + \rho(r))$$

where the smooth function $\rho : (0, 1] \to \mathbb{R}$ is defined by

$$\rho(r) = \rho_k(r) \quad \text{for } [\tfrac{1}{2^k}, \tfrac{1}{2^{k-1}}],\ k = 1,\, 2, \cdots.$$

It is easy to check that ϕ_ρ is smooth $D^2 \setminus \{0\}$ and is a continuous map, even at 0, which coincides with the above infinite product. Obviously the map $\phi_{-\rho}$ is the inverse of ϕ_ρ which shows that it is a homeomorphism. Furthermore we have

$$\phi_\rho^*(r\, dr \wedge d\theta) = r\, dr \wedge d\theta \quad \text{on } D^2 \setminus \{0\}$$

which implies that ϕ_ρ is indeed area preserving.

The following lemma motivates our construction of the above sequence ϕ_k which will play an important role in our proof of Theorem 7.3.

Lemma 7.1. *Let ϕ_k the diffeomorphisms given in Example 7.1. We have the identity*

$$R_{\frac{1}{2}} \circ \phi_{k-1}^{2^4} \circ R_{\frac{1}{2}}^{-1} = \phi_k. \tag{7.3}$$

In particular, we have

$$\mathrm{Cal}(\phi_k) = \mathrm{Cal}(\phi_{k-1}). \tag{7.4}$$

Proof. Using (7.2), we compute

$$R_{\frac{1}{2}} \circ \phi_{k-1} \circ R_{\frac{1}{2}}^{-1}(r,\theta) = (r, \theta + \rho_{k-1}(2r)) = \left(r, \theta + \frac{1}{2^4}\rho_k(r)\right)$$

where the second identity follows from (7.2). Iterating this identity 2^4 times, we obtain (7.3) from (7.2). The equality (7.4) follows from this and (6.1). □

An immediate corollary of this lemma and (7.2) is the following

Corollary 7.2. *We have*

$$\mathrm{Cal}(\phi_k) = 1.$$

for all $k = 1, \cdots$

Now we are ready to give the proof of the following theorem.

Theorem 7.3. *ϕ_ρ cannot be contained in $Hameo(D^2, \partial D^2)$.*

Proof. Suppose to the contrary that $\phi_\rho \in Hameo(D^2, \partial D^2)$, i.e., there exists a path $\lambda \in \mathcal{P}^{ham}(Sympeo(D^2, \partial D^2), id)$ with $\lambda(1) = \phi_\rho$.

Then its Calabi invariant has a finite value which we denote

$$\overline{\mathrm{Cal}}(\phi_\rho) = \overline{\mathrm{Cal}}^{path}(\lambda) = C_1 \tag{7.5}$$

for some $C_1 \in \mathbb{R}$.

Writing $\phi_\rho = \psi_N \widetilde{\psi}_N$ where

$$\begin{aligned} \psi_N &= \Pi_{i=1}^N \phi_i \\ \widetilde{\psi}_N &= \Pi_{i=N+1}^\infty \phi_i, \end{aligned}$$

for $N = 1$, we derive

$$C_1 = \overline{\mathrm{Cal}}(\psi_1) + \overline{\mathrm{Cal}}(\widetilde{\psi}_1)$$

from the homomorphism property of $\overline{\mathrm{Cal}}$. Here we note that ψ_N is smooth and so obviously lies in $Hameo(D^2, \partial D^2)$. Therefore it follows from the group property of $Hameo(D^2, \partial D^2)$ that $\widetilde{\psi}_N$ lies in $Hameo(D^2, \partial D^2)$ if ϕ_ρ does so. We also derive

$$\overline{\mathrm{Cal}}(\psi_1) = \overline{\mathrm{Cal}}(\phi_1) = 1$$

from Corollary 7.2, and hence

$$\overline{\mathrm{Cal}}(\widetilde{\psi}_1) = C_1 - 1. \tag{7.6}$$

On the other hand, applying (7.3) iteratively to the infinite product

$$\widetilde{\psi}_1 = \prod_{i=2}^\infty \phi_i,$$

we show that $\widetilde{\psi}_1$ satisfies the identity

$$\widetilde{\psi}_1(r,\theta) = \begin{cases} R_{\frac{1}{2}} \circ \phi_\rho^{2^4} \circ R_{\frac{1}{2}}^{-1}(r,\theta) & \text{for } 0 < r \le \frac{1}{2} \\ (r,\theta) & \text{for } \frac{1}{2} \le r \le 1. \end{cases} \tag{7.7}$$

In particular, we have

$$\widetilde{\psi}_1 = ev_1((\lambda^{2^4})_{1/2}). \tag{7.8}$$

This property, Lemma 6.1 applied for $a = 1/2$ and the homomorphism property of $\overline{\text{Cal}}^{path}$ give rise to

$$\begin{aligned} \overline{\text{Cal}}(\widetilde{\psi}_1) &= \overline{\text{Cal}}^{path}((\lambda^{2^4})_{1/2}) = \left(\frac{1}{2}\right)^4 \overline{\text{Cal}}^{path}(\lambda^{2^4}) \\ &= \left(\frac{1}{2}\right)^4 2^4 \overline{\text{Cal}}^{path}(\lambda) = \overline{\text{Cal}}^{path}(\lambda) = C_1 \end{aligned} \tag{7.9}$$

It is manifest that (7.6) and (7.9) contradict to each other. This finishes the proof. □

8. Appendix

8.1. **Proof of Lemma 4.1.** Next we prove Lemma 4.1 which was postponed until this section. Let H be a smooth Hamiltonian. Since $\rho(\phi_H; 1)$ depends only on the homotopy class of ϕ_H, we will work with its homotopy class $[\phi_H] \in \widetilde{Ham}(M, \omega)$. We denote an element of $\widetilde{Ham}(M, \omega)$ by f, g, ... and so on.

We follow the argument used by Entov and Polterovich [EP1] in their construction of a spectral quasimorphism.

Consider the subspace $QH^0(M) \subset QH(M)$ of degree zero elements. This is a commutative Frobenius algebra with unit $1 = PD[M]$ over the field $K = \Lambda_\omega^{(0)}$. Here $\Lambda_\omega^{(0)}$ is the degree 0 part of Λ_ω.

We consider the spectral function

$$\rho_1 : \widetilde{Ham}(M, \omega) \to \mathbb{R}$$

defined by $\rho_1(f) = \rho(f; 1)$. Obviously it satisfies

$$\rho_1(fg) \leq \rho_1(f) + \rho_1(g).$$

We now ask under which condition, ρ_1 satisfies the lower inequality

$$\rho_1(fg) \geq \rho_1(f) + \rho_1(g) - R \tag{8.1}$$

for some constant R independent of f, g. The following lemma is a slight variation of Entov-Polterovich [EP1]. (Entov-Polterovich proved this lemma for the monotone (M, ω), which is all we need to consider the case S^2. See [U] for the proof of arbitrary compact symplectic manifolds.)

Lemma 8.1 (Lemma 2.2 [EP1]). *Let $b \in QH^*(M) \setminus \{0\}$ and denote by $\Upsilon(b)$ the set of $a \in QH^*(M)$ with $\Pi(b, a) \neq 0$. Then we have*

$$\rho(\phi_H^{-1}; b) = -\inf_{a \in \Upsilon(b)} \rho(\phi_H; a). \tag{8.2}$$

We note that S^2 and the quantum cohomology class $1 \in QH^0(S^2)$ satisfies the hypothesis required in [EP1]. Furthermore the right hand side of (8.2) becomes

$$-\inf_{a \in QH^0(S^2) \setminus \{0\}} \rho(g; a).$$

$QH^0(S^2)$ is factorized into

$$QH^0(S^2) = \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}\{[\omega] q^{-1}\}$$

where $\mathbf{1} = PD[S^2]$ and q is the formal parameter with its valuation $\nu(q) = 4\pi$, and any nonzero element in $QH^0(S^2)$ is invertible.

Now for any $a \in QH^0(S^2) \setminus \{0\}$ which is invertible, we have

$$\rho(g; a) \geq \rho(g; a \cdot a^{-1}) - \rho(1; a^{-1}) = \rho(g; 1) + \nu(a^{-1})$$

by the Normalization axiom of the spectral invariants (see [Oh5]) which gives rise to $\rho(1; a^{-1}) := -\nu(a^{-1})$. But we have

$$\inf_{b \in QH^0(S^2)\setminus\{0\}} \nu(b^{-1}) = -4\pi.$$

This implies $\rho(g; a) \geq \rho(g; 1) - 4\pi$ for all $a \in QH^0(S^2) \setminus \{0\}$.

Let f, $g \in \widetilde{Ham}(M, \omega)$. By the triangle inequality, we have

$$\rho(fg; 1) \geq \rho(f; 1) - \rho(g^{-1}; 1).$$

Applying Lemma 8.1 we get

$$\rho(fg; 1) \geq \rho(f; 1) + \rho(g; 1) - 4\pi$$

which finishes the proof with $R = 4\pi$ by substituting $f = g^{-1}$. □

One interesting consequence of this proof gives rise to the following bound for the spectral diameter of $Ham(S^2)$.

Theorem 8.2. *Denote by* Diam_ρ *the spectral diameter defined by*

$$\mathrm{Diam}_\rho(Ham(S^2)) := \sup_H \{\gamma(\phi_H) \mid H \in C^\infty([0, 1] \times M, \mathbb{R})\}.$$

Then we have $\mathrm{Diam}_\rho(Ham(S^2)) \leq 4\pi$ *or equivalently,*

$$\gamma(\phi_H) \leq 4\pi (= \mathrm{Area}(S^2)) \tag{8.3}$$

for all H*.*

This is quite a contrast to Polterovich's theorem in [P1] which states that the diameter of $Ham(S^2)$ with respect to the Hofer distance is infinite. In fact a similar bound holds whenever the quantum cohomology $QH^*(M)$ becomes a field, e.g., such as $\mathbb{C}P^n$. It would be an interesting theme of future research to find out the precise criterion for (M, ω) to have finite (or infinite) spectral diameter and to understand its implication in the point of Hamiltonian dynamics.

8.2. **High dimensional ball** $B^{2n}(1)$**.** In this subsection, we prove the following high dimensional analog to the main properness result.

Theorem 8.3. *Equip* $D^{2n} = D^{2n}(1) \subset \mathbb{C}^n$ *with the standard symplectic form* ω*. Then* $Hameo(D^{2n}, \partial D^{2n})$ *is a non-trivial proper normal subgroup of* $Sympeo(D^{2n}, \partial D^{2n})$*.*

Proof. Firstly, we note that we can symplectically embed D^{2n} into $(\mathcal{P}^n(\mu), \omega)$ *as a displaceable domain* for a suitable choice of sufficiently large $\mu > 0$. (For example, any $\mu \geq 8\pi$ will be more than enough.) Here ω is the Fubini-Study form with $\omega(\alpha) = \mu$ for the generator $\alpha \in H_2(\mathbb{C}P^n) \cong \mathbb{Z}$.

Secondly, the quantum cohomology of $QH^*(\mathbb{C}P^n)$ is a field and so $\phi_H \to \rho(\phi_H; 1)$ defines a Calabi quasimorphism on $\mathcal{P}^{ham}(Symp(\mathbb{C}P^n), id)$ as shown essentially by Entov-Polterovich [EP1].

These two combined with Buhovsky-Seyfaddini's uniqueness result [BS] enable us to extend the homogeneous quasimorphism

$$\mu^{path} : \mathcal{P}^{ham}(Symp(\mathbb{C}P^n), id) \to \mathbb{R}$$

to a homogeneous quasimorphism

$$\overline{\mu}^{path} : \mathcal{P}^{ham}(Sympeo(\mathbb{C}P^n), id) \to \mathbb{R}$$

and Calabi homomorphism $\mathrm{Cal}^{path} : \mathcal{P}^{ham}(Symp(D^{2n}, \partial D^{2n}) \to \mathbb{R}$ to

$$\overline{\mathrm{Cal}}^{path} : \mathcal{P}^{ham}(Sympeo(D^{2n}, \partial D^{2n}), id) \to \mathbb{R}.$$

We denote

$$\iota : \mathcal{P}(Sympeo(D^{2n}, \partial D^{2n}), id) \to \mathcal{P}(Sympeo(\mathbb{C}P^n), id)$$

the embedding induced by the above fixed symplectic embedding of $D^{2n} \to \mathbb{C}P^n(\mu)$. This one restricts to an embedding

$$\iota : \mathcal{P}^{ham}(Sympeo(D^{2n}, \partial D^{2n}) \to \mathcal{P}^{ham}(Sympeo(\mathbb{C}P^n), id).$$

Then the Calabi property of $\overline{\mu}^{path}$ implies

$$\overline{\mu}^{path}(\iota(\lambda)) = \overline{\mathrm{Cal}}^{path}(\lambda)$$

for all $\lambda \in \mathcal{P}(Sympeo(D^{2n}, \partial D^{2n}), id)$.

Thirdly, we recall that the homotopy invariance of $\rho(\lambda; 1)$, Theorem 1.1 [Oh11] applies to $\mathbb{C}P^n$ since $\mathbb{C}P^n$ is rational. Then by the same proof as in section 6 the Alexander isotopy is a hamiltonian homotopy and so $\overline{\mathrm{Cal}}^{path}$ descends to a nontrivial homomorphism

$$\overline{\mathrm{Cal}} : Hameo(D^{2n}, \partial D^{2n}) \to \mathbb{R}$$

which extends the standard Calabi homomorphism $\mathrm{Cal} : Ham(D^{2n}, \partial D^{2n}) \to \mathbb{R}$.

Finally the same construction of wild homeomorphism as in section 7 can be applied to construct a symplectic homeomorphism $\rho_\rho \in Sympeo(D^{2n}, \partial D^{2n})$ that is not contained in $Hameo(D^{2n}, \partial D^{2n})$. The only difference will be the followings:

(1) The equality (6.1) is replaced by $\overline{\mathrm{Cal}}^{path}(\lambda_a) = a^{n+2}\overline{\mathrm{Cal}}^{path}(\lambda)$.
(2) So we replace (7.2) by

$$\rho_k(r) = 2^{n+2}\rho_{k-1}(2r)$$

in the construction of the sequence ϕ_k in section 7.
(3) Then we consider the autonomous Hamiltonian

$$F(z) = -\int_1^r s\rho_k(s)\, ds,$$

where $z = (z_1, \cdots, z_n)$ and $r = \sqrt{\sum_{i=1} |z_i|^2}$. The corresponding Hamiltonian diffeomorphism is nothing but the complex multiplication of $e^{\rho_k(r)\sqrt{-1}}$

$$\phi_k(z) = e^{\rho_k(r)\sqrt{-1}} \cdot z.$$

Combining these steps, we have finished the proof. □

References

[AD] Aubry, S., Le Daeron, P. Y., *The discrete Frenkel-Kontorova model and its extensions. I. Exact results for the ground-states*, Phys. D 8 (1983), no. 3, 381–422.

[B] Banyaga, A., *Sur la structure du groupe des difféomorphismses qui préservent une forme symplectique*, Comm. Math. Helv. 53 (1978), 174-227.

[BS] Buhovsky, L, Seyfaddini, S., *Uniqueness of generating Hamiltonians for continuous Hamiltonian flows*, J. Symp. Geom. (to appear), arXiv:1003.2612.

[C] Calabi, E., *On the group of autormorphisms of a symplectic manifold*, Problems in Analysis (symposium in honor of Bochner), Princeton University Press (1970), 1- 26.

[Ca] Calegari, D. scl (Stable Commutator Length), MSJ Memoirs vol 20, Math. Soc. of Japan, 2009.

[El] Eliashberg, Y., *A theorem on the structure of wave fronts and applications in symplectic topology*, Funct. Anal. and its Appl. 21 (1987), 227-232.

[EP1] Entov, M., Polterovich, L., *Calabi quasimorphism and quantum homology*, Int. Math. Res. Not. no 30 (2003), 1635–1676.

[EP2] Entov, M., Polterovich, L., *Quasi-states and symplectic intersections,* Comment. Math. Helv. 81 (2006), 75-99.

[Fa1] Fathi, A. *Structure of the group of homeomorphisms preserving a good measure on a compact manifold*, Ann. Scient. Èc. Norm. Sup. 13 (1980), 45-93.

[Fa2] Fathi, A., private communication, 2005.

[Fl2] Floer, A., *Symplectic fixed points and holomorphic spheres,* Commun. Math. Phys. 120 (1989), 575-611.

[GG1] Gambaudo, J.-M., Ghys, É., *Enlacements asymptotiques*, Topology 36 (1997), 1355 - 1379.

[GG2] Gambaudo, J.-M., Ghys, É., *Commutators and diffeomorphisms of surfaces*, Ergod. Th. & Dynam. Sys. 24 (2004), 1591-1617.

[Gh] Ghys, É., *Knots and dynamics*, Proceedings of ICM-2006 vol 1, pp 247-277, Madrid, EMS, 2006.

[Gr] Gromov, M., *Soft and hard symplectic geometry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 81–98, Amer. Math. Soc., Providence, RI, 1987.

[H] Hofer, H. *On the topological properties of symplectic maps*, Proc. Royal Soc. Edinburgh 115 (1990), 25-38.

[leR] Le Roux, F., *Simplicity of the group of compactly supported area preserving homeomorphisms of the open disc and fragmentation of symplectic diffeomorphisms*, preprint 2009, arXiv:0901.2428.

[M1] Mather, J., *Existence of quasiperiodic orbits for twist homeomorphisms of the annulus*, Topology 21 (1982), no. 4, 457–467.

[M2] Mather, J., *Dynamics of area preserving maps*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 1190–1194, Amer. Math. Soc., Providence, RI, 1987.

[M3] Mather, J., private communication via A. Fathi.

[Maz] Mazur, B., *The method of infinite repetition in pure topology: I*, Ann. of Math., 80 (1964), 190-199.

[Mu] Müller, S., *The group of Hamiltonian homeomorphisms in the L^∞-norm*, J. Korean Math. Soc. 45 (2008), No. 6, 1769–1784.

[Oh1] Oh, Y.-G., *Symplectic topology as the geometry of action functional, I*, J. Differ. Geom. 46 (1997), 499-577.

[Oh2] Oh, Y.-G., *Symplectic topology as the geometry of action functional, II*, Commun. Anal. Geom. 7 (1999), 1-55.

[Oh3] Oh, Y.-G., *Chain level Floer theory and Hofer's geometry of the Hamiltonian diffeomorphism group*, Asian J. Math. 6 (2002), 579-624 ; *Erratum* 7 (2003), 447-448.

[Oh4] Oh, Y.-G., *Normalization of the Hamiltonian and the action spectrum*, J. Korean Math. Soc. 42 (2005), 65-83.

[Oh5] Oh, Y.-G., *Construction of spectral invariants of Hamiltonian paths on closed symplectic manifolds*, in "The Breadth of Symplectic and Poisson Geometry", Prog. Math. 232, 525 - 570, Birkhäuser, Boston, 2005.

[Oh6] Oh, Y.-G., *Spectral invariants, analysis of the Floer moduli space and geometry of Hamiltonian diffeomorphisms*, Duke Math. J. 130 (2005), 199 - 295; math.SG/0403083

[Oh7] Oh, Y.-G., *Lectures on Floer theory and spectral invariants of Hamiltonian flows,* "Morse Theoretic Methods in Non-linear Analysis and Symplectic Topology", Seminaire de Mathematique Superieure-Summer School, University of Montreal, June 21 - July 2, 2004, Nato Science Series, II/vol 217, pp. 321 - 416, Springer, 2005.

[Oh8] Oh, Y.-G., *Floer mini-max theory, the Cerf diagram and spectral invariants*, J. Korean Math. Soc. 46 (2009), 363-447.

[Oh9] Oh, Y.-G., *C^0-coerciveness of Moser's problem and smoothing area preserving homeomorphisms,* preprint; arXiv:math/0601183.

[Oh10] Oh, Y.-G., *The group of Hamiltonian homeomorphisms and continuous Hamiltonian flows,* pp 149-177, Contemp. Math., 512, Amer. Math. Soc., Providence, RI, 2010.

[Oh11] Oh, Y.-G., *Homotopy invariance of spectral invariants of topological Hamiltonian flows and its Lagrangian analog*, preprint, 2011.

[OM] Oh, Y.-G., Müller, S., *The group of Hamiltonian homeomorphisms and C^0 symplectic topology*, J. Symp. Geom. 5 (2007), 167 - 219.

[P1] Polterovich, L., *Hofer's diameter and Lagrangian intersections*, Intern. Math. Res. Notices, 1998 No. 4, 217-223.

[P2] Polterovich, L., The Geometry of the Group of Symplectic Diffeomorphisms, Lectures in Mathematics ETH Zürich. Birkhuser Verlag, Basel, 2001.

[Sc] Schwarz, M., *On the action spactrum for closed symplectically aspherical manifolds*, Pacific, J. Math. 193 (2000), 419-461.

[Si] Sikorav,J.-C., *Approximation of a volume-preserving homeomorphism by a volume-preserving diffeomorphism*, preprint, September 2007; available from http://www.umpa.ens-lyon.fr/ symplexe.

[T] Thurston, W., *On the structure of the group of volume preserving diffeomorphisms,* unpublished.

[U] Usher, M., *Duality in filtered Floer-Novikov complexes,* J. Topol. Anal. 2 (2010), no. 2, 233-258.

[V] Viterbo, C., *On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonian flows*, Internat. Math. Res. Notices, vol 2006, Article ID 34028, 9 pages; Erratum, ibid, vol 2006, Article ID 38784, 4 pages.

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