C^0-rigidity of Poisson brackets

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December 10, 2007

Abstract

Consider a functional associating to a pair of smooth functions on a closed symplectic manifold the maximum of their Poisson bracket. We show that this functional is lower semi-continuous with respect to the product uniform norm on the space of pairs of smooth functions. This extends previous results of Cardin-Viterbo and Zapolsky. The proof involves theory of geodesics of the Hofer metric on the group of Hamiltonian diffeomorphisms. We also discuss a failure of a similar semi-continuity phenomenon for multiple Poisson brackets of three or more functions.

1 Statement of results

The subject of this note is function theory on symplectic manifolds. Let (M, ω) be a closed symplectic manifold. Denote by $C^\infty(M)$ the space of smooth functions on M and by $\|\cdot\|$ the standard *uniform norm* (also called the C^0-*norm*) on it: $\|F\| := \max_{x \in M} |F(x)|$.

The definition of the Poisson bracket $\{F, G\}$ of two smooth functions $F, G \in C^\infty(M)$ involves first derivatives of the functions. Thus *a priori* there is no restriction on possible changes of $\{F, G\}$ when F and G are slightly perturbed in the uniform norm. Amazingly such restrictions do exist: this was first pointed out by F.Cardin and C.Viterbo [2] who showed that

$$\{F, G\} \not\equiv 0 \Longrightarrow \liminf_{F', G' \xrightarrow{C^0} F, G} \|\{F', G'\}\| > 0.$$

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[a] Partially supported by E. and J. Bishop Research Fund.

[b] Partially supported by the Israel Science Foundation grant # 509/07.

Our main result is as follows:

Theorem 1.1.

$$\max\{F,G\} = \liminf_{F',G' \xrightarrow{C^0} F,G} \max\{F',G'\}$$

for any closed symplectic manifold M and any pair of smooth functions $F, G \in C^\infty(M)$.

Replacing F by $-F$, we get a similar result for $-\min\{F,G\}$. In particular, this yields

$$\|\{F,G\}\| = \liminf_{F',G' \xrightarrow{C^0} F,G} \|\{F',G'\}\| \ , \tag{1}$$

which should be considered as a quantitative version of Cardin-Viterbo theorem, and which answers in positive a question posed in [4]. In the case $\dim M = 2$ formula (1) has been first proved by F.Zapolsky [14] by methods of two-dimensional topology. A generalization of the Cardin-Viterbo result in a different direction has been found by V.Humilière [6].

Remark 1.2. Note that (1) does *not* imply that $\{F',G'\} \xrightarrow{C^0} \{F,G\}$ when $F',G' \xrightarrow{C^0} F,G$ – see e.g. [6] for counterexamples.

Remark 1.3. Clearly $\liminf$ cannot be replaced in the theorem by $\lim$: the maximum of the Poisson bracket of two functions can be arbitrarily increased by arbitrarily C^0-small perturbations of the functions.

In the proof of Theorem 1.1 we use the following ingredient from "hard" symplectic topology: Sufficiently small segments of one-parameter subgroups of the group *Ham*(M) of Hamiltonian diffeomorphisms of M minimize the "positive part of the Hofer length" among all paths on the group in their homotopy class with fixed end points. This was proved by D.McDuff in [10, Proposition 1.5], see also papers [1], [8], [3], [9], [7] [11] for related results in this direction.

Remark 1.4. Theorem 1.1 should extend to compactly supported functions on arbitrary, not necessarily closed, symplectic manifolds. Clearly, it extends to those open symplectic manifolds which can be exhausted by open subsets which admit a compactification by a closed manifold, for instance to the standard linear space $\mathbb{R}^{2n}$. The only place in the proof of Theorem 1.1 where we use that M is closed is the above-mentioned geodesic property of one-parameter subgroups of Hamiltonian diffeomorphisms. It sounds likely that this property holds true on open manifolds as well, though to the best of our knowledge such a result does not appear in the literature in full generality.

The next result gives an evidence for a failure of C^0-rigidity for multiple Poisson brackets. Given an arbitrary (i.e. not necessarily closed) symplectic M, denote by $C_c^\infty(M)$ the space of smooth functions on M with compact support. The uniform norm on $C_c^\infty(M)$ is defined in the same way as in the closed case.

Theorem 1.5. *Let M be an arbitrary symplectic manifold. There exists a constant $N \in \mathbb{N}$, depending only on the dimension of M, such that* **for any** *smooth functions $F_1, \ldots, F_N \in C_c^\infty(M)$ there exist $F_1', \ldots, F_N' \in C_c^\infty(M)$ arbitrarily close in the uniform norm, respectively, to $F_1, \ldots, F_N$ which satisfy the following relation:*

$$\{F_1', \{F_2', \ldots \{F_{N-1}', F_N'\}\} \ldots\} \equiv 0.$$

We shall see in Section 2.3 below that in the case $\dim M = 2$ the result above holds for $N = 3$.

Question 1.6. Does the theorem above remain valid with $N = 3$ on an arbitrary symplectic manifold?

The following claim, though it does not answer Question 1.6, shows that Theorem 1.1 cannot be formally extended to the triple Poisson bracket.

Theorem 1.7. *For any symplectic manifold M* **one can find** *3 functions $F, G, H \in C_c^\infty(M)$ satisfying $\{F, \{G, H\}\} \not\equiv 0$ such that there exist smooth functions $F', G', H' \in C_c^\infty(M)$ arbitrarily close in the uniform norm, respectively, to F, G, H and satisfying the condition*

$$\{F', \{G', H'\}\} \equiv 0.$$

The theorem will be proved in Section 2.3. The proof shows that the phenomenon is local: we just implant a 2-dimensional example (see the remark after Theorem 1.5) in a Darboux chart.

Surprisingly, the next problem is open even in dimension 2:

Problem 1.8. Compare

$$\liminf_{F',G' \xrightarrow{C^0} F,G} \max\{\{F',G'\},G'\}$$

with $\max\{\{F,G\},G\}$ for some/all pairs of functions F,G on some/all symplectic manifolds.

2 Proofs

2.1 Preliminaries

Given a (time-dependent) Hamiltonian $H : M \times [0,1] \to \mathbb{R}$, denote by X_H the (time-dependent) Hamiltonian vector field generated by H. The Poisson bracket of two functions $F, G \in C^\infty(M)$ is defined by $\{F,G\} = dF(X_G)$.

Write $\widetilde{Ham}(M)$ for the universal cover of the group $Ham(M)$ of Hamiltonian diffeomorphisms of (M,ω), where the base point is chosen to be the identity map $\mathbb{1}$. Denote by ψ_H^t, $t \in \mathbb{R}$, the Hamiltonian flow generated by H (i.e. the flow of X_H). Let $\psi_H := \psi_H^1$ and let $\widetilde{\psi}_H \in \widetilde{Ham}(M)$ be the lift of ψ_H associated to the path $\{\psi_H^t\}, t \in [0;1]$. We will say that ψ_H and $\widetilde{\psi}_H$ *are generated by* H. We will also denote $\|H\| = \max_{M\times[0,1]} |H(x,t)|$ (for time-independent Hamiltonians this norm coincides with the uniform norm on $C^\infty(M)$ introduced above). Set $H_t = H(\cdot,t)$.

Recall that the flow $\psi_H^t\psi_K^t$ is generated by the Hamiltonian $H\sharp K(x,t) = H(x,t) + K((\psi_H^t)^{-1}x,t)$ and the flow $\psi_H\psi_K^t(\psi_H)^{-1}$ by $K\big((\psi_H)^{-1}x,t\big)$.

Denote by $\mathcal{F}$ the set of all the Hamiltonians H on M such that H_t has zero mean for all t (for a time-independent Hamiltonian H the condition $H \in \mathcal{F}$ means just that H has zero mean). Note that if $H, K \in \mathcal{F}$ then both $H\sharp K$ and $K\big((\psi_H)^{-1}x,t\big)$ also belong to $\mathcal{F}$.

For $a, b \in \widetilde{Ham}(M,\omega)$ write $[a,b]$ for the commutator $aba^{-1}b^{-1}$.

Lemma 2.1. *Assume $H, K \in C^\infty(M)$ are time-independent Hamiltonians. Then $[\widetilde{\psi}_H, \widetilde{\psi}_K]$ can be generated by $L(x,t) = H(x) - H\big(\psi_K^{-1}\psi_H^{-t}x\big)$.*

Proof. It is easy to see that the element $[\widetilde{\psi}_H, \widetilde{\psi}_K] \in \widetilde{Ham}(M)$ can be represented by the path $\{\psi_H^t\psi_K\psi_H^{-t}\psi_K^{-1}\}$ where $t \in [0;1]$. The flow ψ_H^{-t} is generated by $-H$ and therefore the flow $\psi_K\psi_H^{-t}\psi_K^{-1}$ is generated by the Hamiltonian $-H\circ\psi_K^{-1}$. Thus the flow $\psi_H^t\psi_K\psi_H^{-t}\psi_K^{-1}$ is generated by $H\sharp\big(-H\circ\psi_K^{-1}\big)(x,t) = H(x) - H\big(\psi_K^{-1}\psi_H^{-t}x\big)$. □

The group $\widetilde{Ham}(M)$ carries conjugation-invariant functionals ρ^+ and ρ defined by

$$\rho^+(\widetilde{\psi}) := \inf_H \int_0^1 \max_{x\in M} H(x,t)\, dt$$

and

$$\rho(\widetilde{\psi}) := \inf_H \int_0^1 (\max_{x\in M} H(x,t) - \min_{x\in M} H(x,t))\, dt\ ,$$

where the infimum is taken over all (time-dependent) Hamiltonians $H \in \mathcal{F}$ generating $\widetilde{\psi}$. The functional ρ is the Hofer (semi)-norm [5] (see e.g. [12] for an introduction to Hofer's geometry). It gives rise to the bi-invariant Hofer (pseudo-)metric on $\widetilde{Ham}(M)$ by $d(\widetilde{\phi},\widetilde{\psi}) = \rho(\widetilde{\phi}^{-1}\widetilde{\psi})$. The functional ρ^+, which is sometimes called the "positive part of the Hofer norm", satisfies the triangle inequality but is not symmetric. Note also that $\rho^+ \le \rho$. We shall use the following properties of these functionals. By the triangle inequality for ρ^+

$$|\rho^+(\widetilde{\phi}) - \rho^+(\widetilde{\psi})| \le \max(\rho^+(\widetilde{\phi}^{-1}\widetilde{\psi}), \rho^+(\widetilde{\psi}^{-1}\widetilde{\phi})) \le d(\widetilde{\phi},\widetilde{\psi})\ . \tag{2}$$

This readily yields

$$|\rho^+(\widetilde{\psi}_H) - \rho^+(\widetilde{\psi}_K)| \le d(\widetilde{\psi}_H,\widetilde{\psi}_K) \le 2||H-K|| \tag{3}$$

for any $H, K \in \mathcal{F}$. McDuff showed [10, Proposition1.5] that for every *time-independent* function $H \in \mathcal{F}$ there exists $\delta > 0$ so that

$$\rho^+(\widetilde{\psi}_{tH}) = t \cdot \max H \ \ \forall t \in (0;\delta)\ . \tag{4}$$

Lemma 2.2. *Assume $H, K \in C^\infty(M)$ are time-independent Hamiltonians with zero mean. Then $\rho^+([\widetilde{\psi}_H,\widetilde{\psi}_K]) \le \max\{H,K\}$.*

Proof. By Lemma 2.1 $[\widetilde{\psi}_H,\widetilde{\psi}_K]$ can be generated by

$$L(x,t) = H(x) - H\big(\psi_K^{-1}\psi_H^{-t}x\big).$$

Note that

$$\int_0^1 \max L(x,t)\, dt = \int_0^1 \max(H - H\circ\psi_K^{-1}\circ\psi_H^{-t})\, dt$$

$$= \int_0^1 \max(H\circ\psi_H^t - H\circ\psi_K^{-1})\, dt$$

$$= \int_0^1 \max(H - H \circ \psi_K^{-1})\, dt = \int_0^1 \max(H \circ \psi_K - H)\, dt$$

since H is constant on the orbits of the flow ψ_H^t. Taking into account that

$$H(\psi_K x) - H(x) = \int_0^1 \frac{d}{dt} H(\psi_K^t x) dt = \int_0^1 \{H, K\}(\psi_K^t x) dt,$$

we get that

$$\rho^+([\widetilde{\psi}_H, \widetilde{\psi}_K]) \leq \int_0^1 \max L(x,t)\, dt \leq \max\{H, K\},$$

which yields the lemma. $\square$

2.2 Proof of Theorem 1.1

As it is easy to see, we can assume without loss of generality that all the functions F_i, G_i, F, G have zero mean.

Denote by f_s, g_t, $s,t \in [0,1]$, the Hamiltonian flows generated by F and G, and by $\widetilde{f}_s, \widetilde{f}_t$ their respective lifts to $\widetilde{Ham}(M)$. Note that for fixed s and t the elements $\widetilde{f}_s$ and $\widetilde{g}_t$ are generated, respectively, by the Hamiltonians sF and tG.

By Lemma 2.1 for fixed s,t the commutator $[\widetilde{f}_s, \widetilde{g}_t]$ can be generated by the Hamiltonian $L_{s,t}(x,\tau) = sF(x) - sF(g_t^{-1} f_{\tau s}^{-1} x)$ (use Lemma 2.1 with $H = sF$, $K = tG$ and note that $\psi_{sF}^\tau = f_{\tau s}$). Clearly $L_{s,t} \in \mathcal{F}$ since $F, G \in \mathcal{F}$.

Lemma 2.3. *$L_{s,t} = st\{F,G\} + K_{s,t}$, where $\|K_{s,t}\|/st \to 0$ as $s,t \to 0$.*

Proof. We need to compute the relevant terms in the expansion of $L_{s,t}$ with respect to s,t at $s = 0, t = 0$.

Clearly, $L_{0,0} \equiv 0$.

The first order terms are as follows:

$$\frac{\partial L_{s,t}}{\partial s}(x,\tau) = \partial\big(sF(x) - sF(g_t^{-1} f_{\tau s}^{-1} x)\big)/\partial s =$$
$$= F(x) - F(g_t^{-1} f_{\tau s}^{-1} x) - s dF \circ dg_t^{-1}(X_{-sF}(x)) =$$
$$= F(x) - F(g_t^{-1} f_{\tau s}^{-1} x) + s^2 dF \circ dg_t^{-1}(X_F(x)),$$

and

$$\frac{\partial L_{s,t}}{\partial t}(x,\tau) = \partial\big(sF(x) - sF(g_t^{-1} f_{\tau s}^{-1} x)\big)/\partial t =$$

$$= -sdF\big(X_{-G}(f_{\tau s}^{-1}x)\big) = s\{F,G\}(f_{\tau s}^{-1}x).$$

Evaluating $\partial L_{s,t}/\partial s(x,\tau)$ and $\partial L_{s,t}/\partial t(x,\tau)$, respectively, at the points $(s,0)$ and $(0,t)$ (for a fixed (x,τ)) we see that

$$\frac{\partial L_{s,0}}{\partial s}(x,\tau) \equiv 0$$

(since F is constant on the orbits of the flow f_s, $s \in \mathbb{R}$) and

$$\frac{\partial L_{0,t}}{\partial t}(x,\tau) \equiv 0.$$

Thus

$$\left.\frac{\partial^k}{\partial s^k}\right|_{(s,t)=(0,0)} L_{s,t}(x,\tau) = 0 = \left.\frac{\partial^k}{\partial t^k}\right|_{(s,t)=(0,0)} L_{s,t}(x,\tau), \text{ for any } k \geq 1.$$

Finally, let us compute $\left.\frac{\partial^2}{\partial s \partial t}\right|_{(s,t)=(0,0)} L_{s,t}(x,\tau)$:

$$\left.\frac{\partial^2}{\partial s \partial t}\right|_{(s,t)=(0,0)} L_{s,t}(x,\tau) = \left.\frac{\partial}{\partial s}\right|_{s=0} \frac{\partial L_{s,0}}{\partial t}(x,\tau) =$$

$$= \left.\frac{\partial}{\partial s}\right|_{s=0} s\{F,G\}(f_{\tau s}^{-1}x) = \{F,G\}(x).$$

This finishes the proof of the lemma. □

Now we are ready to complete the proof of Theorem 1.1. The inequality

$$\max\{F,G\} \geq \liminf_{F',G' \xrightarrow{C^0} F,G} \max\{F',G'\}$$

is trivial so we only need to prove the opposite one. Let F_i, G_i be sequences of smooth functions such that

$$F_i, G_i \xrightarrow{C^0} F, G, \ i \to +\infty,$$

and

$$\max\{F_i, G_i\} \to A, \ i \to +\infty.$$

We need to show that $\max\{F,G\} \leq A$.

Assume on the contrary that $\max\{F,G\} > A$. Pick B such that $A < B < \max\{F,G\}$. Then for any sufficiently large i

$$\max\{F_i, G_i\} \leq B.$$

Denote by $f_{s,i}, g_{t,i}$, respectively, the time-s and time-t maps of the flows generated by F_i and G_i. Their lifts to $\widetilde{Ham}(M)$ will be decorated by tildes. The right inequality in (3) easily implies that the sequences $\widetilde{f}_{s,i}$ and $\widetilde{g}_{t,i}$ converge, respectively, to $\widetilde{f}_s$ and $\widetilde{g}_s$ in the Hofer (pseudo-)metric. Since by (2) the functional ρ^+ is continuous in the Hofer (pseudo-)metric,

$$\rho^+([\widetilde{f}_{i,s}, \widetilde{g}_{i,t}]) \to \rho^+([\widetilde{f}_s, \widetilde{g}_t]) \text{ as } i \to \infty \; .$$

By Lemma 2.2,

$$\rho^+([\widetilde{f}_{i,s}, \widetilde{g}_{i,t}]) \leq st \cdot \max\{F_i, G_i\} \leq stB$$

for any sufficiently large i. Hence, taking the limit in the left-hand side as $i \to +\infty$, we get

$$\rho^+([\widetilde{f}_s, \widetilde{g}_t]) \leq stB. \tag{5}$$

Choose $\epsilon > 0$ such that $B + 2\epsilon < \max\{F,G\}$. Take sufficiently small $s, t > 0$ so that the function $K_{s,t}$ from Lemma 2.3 admits a bound

$$\|K_{s,t}\| \leq \epsilon st \tag{6}$$

and so that the Hamiltonian $st\{F,G\}$ is sufficiently small and satisfies

$$\rho^+(\widetilde{\psi}_{st\{F,G\}}) = st \cdot \max\{F,G\} \; , \tag{7}$$

see formula (4). Lemma 2.3 and inequalities (6), (3) yield

$$|\rho^+([\widetilde{f}_s, \widetilde{g}_t]) - \rho^+(\widetilde{\psi}_{st\{F,G\}})| \leq 2\epsilon st \; .$$

Hence,

$$\rho^+([\widetilde{f}_s, \widetilde{g}_t] \geq \rho^+(\widetilde{\psi}_{st\{F,G\}}) - 2\epsilon st = st(\max\{F,G\} - 2\epsilon).$$

Combining this with (5), we get

$$st(\max\{F,G\} - 2\epsilon) \leq \rho^+([\widetilde{f}_s, \widetilde{g}_t]) \leq stB,$$

and hence

$$\max\{F,G\} - 2\epsilon \leq B$$

which contradicts our choice of B and ϵ. We have obtained a contradiction. Hence $\max\{F,G\} \leq A$ and the theorem is proven. □

2.3 Proofs of Theorems 1.5, 1.7

Proof of Theorem 1.5.

For simplicity we will prove the result in the case $\dim M = \mathbb{T}^2$ with $N = 3$. The general case can be done in a similar way using [13].

Define a *thick grid* T with mesh c in M as a union of pair-wise disjoint squares on M such that each square has a side $2c$ and the centers of the squares form a rectangular grid with the mesh $3c$. A T*-tamed function* is a smooth function which is constant in a small neighborhood of each square of the thick grid T (but its values may vary from square to square).

One can easily construct a sequence $c_i \to 0$ and $N = 3$ thick grids U_i, V_i, W_i with mesh c_i so that $U_i \cup V_i \cup W_i = M$ for all i. (See [13] on how to construct a similar covering of an arbitrary M by a number of thick grids depending only on $\dim M$).

Now for every $\epsilon > 0$ there exists i large enough so that every triple of functions $F_1, F_2, F_3 \in C_c^\infty(M)$ can be ϵ-approximated, respectively, by U_i, V_i, W_i-tamed functions $F_1', F_2', F_3' \in C_c^\infty(M)$. Take any point $x \in M$. Then at least one of the functions F_1', F_2', F_3' is constant near x. Thus $\{F_1', \{F_2', F_3'\}\} \equiv 0$, and the claim follows. □

Proof of Theorem 1.7.

Assume $\dim M = 2n > 2$ (the case $\dim M = 2$ has been dealt with in the proof of Theorem 1.5). In a local Darboux chart with coordinates $p_1, q_1, \ldots, p_n, q_n$ on M choose an open cube

$$P = K^{2n-2} \times K^2,$$

where K^{2n-2} is an open cube in the $(p_1, q_1, \ldots, p_{n-1}, q_{n-1})$-coordinate plane and K^2 is a open square in the (p_n, q_n)-coordinate plane. Fix a smooth compactly supported non-zero function χ on K^{2n-2}. Given a smooth compactly supported function L on K^2, define the function $\chi L \in C_c^\infty(M)$ as

$$\chi L(p_1, q_1, \ldots, p_n, q_n) := \chi(p_1, q_1, \ldots, p_{n-1}, q_{n-1}) L(p_n, q_n)$$

on P and as zero outside P.

Now pick any functions $F_1, G_1, H_1 \in C_c^\infty(K^2)$ such that

$$\{F_1, \{G_1, H_1\}\} \not\equiv 0.$$

Set

$$F := \chi F_1, G := \chi G_1, H := \chi H_1 \in C_c^\infty(M).$$

As in the proof of Theorem 1.5 (note that in the case of the two-dimensional square the construction of the thick grids is as easy as in the case of $\mathbb{T}^2$), choose C^0-small perturbations $F_1', G_1', H_1' \in C_c^\infty(K^2)$ of F_1, G_1, H_1 so that

$$\{F_1', \{G_1', H_1'\}\} \equiv 0.$$

Then $F' := \chi F_1', G' := \chi G_1', H' := \chi H_1' \in C_c^\infty(M)$ satisfy

$$\{F', \{G', H'\}\} = \{\chi F_1', \{\chi G_1', \chi H_1'\}\} = \chi^3 \{F_1', \{G_1', H_1'\}\} \equiv 0,$$

because of the Leibniz rule for Poisson brackets and because the Poisson bracket of χ and any function of p_n, q_n vanishes identically. For the same reason

$$\{F, \{G, H\}\} = \{\chi F_1, \{\chi G_1, \chi H_1\}\} = \chi^3 \{F_1, \{G_1, H_1\}\} \not\equiv 0.$$

Clearly, by choosing F_1', G_1', H_1' arbitrarily C^0-close to F_1, G_1, H_1 in $C_c^\infty(K^2)$ we can turn F', G', H' into arbitrarily C^0-small perturbations of F, G, H in $C_c^\infty(M)$. Thus we have constructed F, G, H, F', G', H' satisfying the required conditions. □

Acknowledgement. We thank Dima Burago for a useful discussion.

References

[1] Bialy, M., Polterovich, L., *Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms,* Duke Math. J. **76** (1994), 273-292.

[2] Cardin, F., Viterbo, C., *Commuting Hamiltonians and multi-time Hamilton-Jacobi equations,* preprint, math/0507418, 2005.

[3] Entov, M., *K-area, Hofer metric and geometry of conjugacy classes in Lie groups,* Invent. Math **146** (2001), 93-141.

[4] Entov, M., Polterovich, L., Zapolsky, F., *Quasi-morphisms and the Poisson bracket*, Pure and Applied Math. Quarterly, **3**:4 (2007), 1037-1055.

[5] Hofer, H., *On the topological properties of symplectic maps*, Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), 25–38.

[6] Humilière, V., *Hamiltonian pseudo-representations*, preprint, math/0703335, 2007.

[7] Kerman, E., Lalonde, F., *Length minimizing Hamiltonian paths for symplectically aspherical manifolds*, Ann. Inst. Fourier (Grenoble) **53** (2003), 1503–1526.

[8] Lalonde, F., McDuff, D., *Hofer's L^∞-geometry: energy and stability of flows II,* Invent. Math. **122** (1995), 35-69.

[9] McDuff, D., Slimowitz, J., *Hofer-Zehnder capacity and length minimizing Hamiltonian paths,* Geom. and Topology **5** (2001), 799-830.

[10] McDuff, D., *Geometric variants of the Hofer norm*, J. Symplectic Geom. **1** (2002),197–252.

[11] Oh, Y.-G., *Spectral invariants and the length minimizing property of Hamiltonian paths*, Asian J. Math. **9** (2005), 1–18.

[12] Polterovich, L., *The geometry of the group of symplectic diffeomorphisms*, Lectures in Mathematics ETH Zürich, Birkhäuser, 2001.

[13] Rudyak, Y., Schlenk, F., *Minimal atlases of closed symplectic manifolds*, preprint, arXiv:math/0605350, 2006.

[14] Zapolsky, F., *Quasi-states and the Poisson bracket on surfaces,* J. of Modern Dynamics, **1**:3, 2007, 465-475.

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