Review Material

1. **Diagonalizable Matrices**

Recall that an $n \times n$ matrix M is diagonalizable if there are matrices P invertible and D diagonal such that $P^{-1}MP = D$. What you need to know about this is how to decide when a given matrix M is diagonalizable, and when this happen how to find such P and D.

- **Step 1** Find the eigenvalues $\{\lambda_1, ..., \lambda_k\}$ of M. i.e find the roots of the polynomial $p(x) = det(xI - A)$
- **Step 2** For each eigenvalue λ_i calculate $n_i :=$ the dimension of the null space of $\lambda_i I - M$. Recall that this is pretty easy you just reduce the matrix and then count the number of free variables.
- **Step 3**
  Now add all the numbers that you found in step 2. If this number is n, the number of columns of M, your matrix is diagonalizable otherwise it is not.

Remark: Sometimes we don't even have to go over step 2. If we have that M has n different eigenvalues automatically M is diagonalizable.

Examples: Decide which of the following matrices is diagonalizable

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$$

Solution:

i) The characteristic polynomial of A is $x^2 - 5x$, so A has two different eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = 5$. Since A is a 2×2 matrix, A diagonalizable.

ii) The characteristic polynomial of B is $(x-1)^2$, so B has $\lambda_1 = 1$ as its unique eigenvalue.

$$\lambda_1 I - B = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

The null space of $\lambda_1 I - B$ has dimension 1 thus the sum of all the null space dimensions is $1 \neq 2$ hence B is not diagonalizable.

iii) The characteristic polynomial of C is $(x-3)(x+3)^2$, so C has two different eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -3$
The row reduced echelon form of $\lambda_1 I - C$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

which has null space of dimension 1. (z is the only free variable)
The row reduced echelon form of $\lambda_2 I - C$ is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which has null space of dimension 2. (y and z are the free variables)

How to find P and D ?

Suppose M is diagonalizable with eigenvalues $\{\lambda_1, ..., \lambda_k\}$. For each $Null(\lambda_i I - M)$ calculate a basis B_i. Then P is the matrix whose columns are all the vectors of the basis that we just calculated. Then $P^{-1}MP = D$ is the diagonal matrix whose diagonal entries are the eigenvalues of M that correspond to the columns of P. All this process is known as *diagonalization* of M.

Examples: In the previous example we found out that the matrices A and C are diagonalizable. Calculate one diagonalization for each one of them.

Solution:

i) The characteristic polynomial of A is $x^2 - 5x$, so A has two different eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = 5$.

$$\lambda_1 I - A = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}$$

Then a basis for $Null(\lambda_1 I - B)$ is given by

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\lambda_2 I - A = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

Then a basis for $Null(\lambda_2 I - B)$ is given by

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

thus

$$P = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

ii) C has two different eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -3$. From the row reduced echelon form of $\lambda_1 I - C$, which we found in the previous example, we see that a basis for $Null(\lambda_1 I - C)$ is given by

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Also from the row reduced echelon form of $\lambda_2 I - C$, which we found in the previous example, we see that a basis for $Null(\lambda_2 I - C)$ is given by

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

thus

$$P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

2. **Diagonalizing symmetric matrices**

Recall that a matrix Q is call *orthogonal* if $Q^t = Q^{-1}$. This condition is also equivalent to the columns of Q form an orthonormal basis. For symmetric matrices we can find P and D as before but with the extra condition that P is orthogonal. The procedure is the following:

Suppose M is symmetric with eigenvalues $\{\lambda_1, ..., \lambda_k\}$. For each $Null(\lambda_i I - M)$ calculate an orthonormal basis B_i(via Gram-Schmidt). Then P is the matrix whose columns are all the vectors of the basis that we just calculated. Then $P^{-1}MP = D$ is the diagonal matrix whose diagonal entries are the eigenvalues of M that correspond to the columns of P. This process is known as *Orthogonal diagonalization* of M.

Example: Find an orthogonal diagonalization for the matrices A and C of the previous example.

Solution:

i) A has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$. Since $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ form a basis for $Null(\lambda_1 I - B)$, to get an orthonormal basis we just have to divide it by its length. Then an orthonormal basis for $Null(\lambda_1 I - B)$ is given by $\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$. Doing the same with $Null(\lambda_2 I - B) = \text{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we get that the matrix

$$P = \begin{pmatrix} -2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix}$$

Let's check that in fact P is an orthogonal matrix,

$$P^t P = \begin{pmatrix} -2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix} \begin{pmatrix} -2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then we have that

$$P^t A P = \begin{pmatrix} -2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} -2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 5\sqrt{5} & 10\sqrt{5} \end{pmatrix} \begin{pmatrix} -2\sqrt{5} & 1\sqrt{5} \\ 1\sqrt{5} & 2\sqrt{5} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

ii) C has two different eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -3$. The vector space associated to λ_1 has basis $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ thus an orthonormal basis for this space is given by $\begin{pmatrix} 1/\sqrt{3} \\ 1\sqrt{3} \\ 1\sqrt{3} \end{pmatrix}$. The vector space associated to λ_2 has basis $u_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $u_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Now Gram-Schmidt tells us that an orthonormal basis for this space is given by

$$w_1 = \frac{u_1}{\|u_1\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \qquad w_2 = \frac{u_2 - (w_1, u_2)w_1}{\|u_2 - (w_1, u_2)w_1\|} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$$

Then we get the matrices

$$P = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

3. **Problems:**

(a) Which of the following matrices are diagonalizable?

$$i) \begin{pmatrix} -2 & 2 \\ 5 & 1 \end{pmatrix} \qquad ii) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

(b) Diagonalize each given matrix

$$i) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \qquad ii) \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) For each given matrix find an orthogonal matrix P such that $P^{-1}MP$ is diagonal.

$$M = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$