Review Material

1. **Finding basis of subspaces**

Most of the subspaces that we have worked can be seen as Column spaces, Null spaces or Row spaces of matrices so here is what you need to do to calculate such spaces.

Given a matrix A, we first find the reduced row echelon form B of A, then

- **Row Space** A basis for $\mathcal{R}(A) = \mathcal{R}(B)$ is given by the non zero rows of B
- **Column Space** The column vectors of A corresponding to the column vectors of B containing the pivots (the leading 1's of B) form a basis for $\mathcal{C}(A)$.
- **Null Space** Write the solutions of the system $Bx = 0$ as a column in terms of the free variables. $x = \alpha_1 V_1 + ... + \alpha_k V_k$ where α_i's are the free variables. Then $\{V_1, ..., V_k\}$ is a basis for the null space of A. To clarify this procedure take for example

$$B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Here the free variables are y and z, so if we write the solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3y - 2z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. We obtain the vectors $\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ which are a basis for $\mathcal{N}(B)$

Problem 1: *Find bases for the row space, the column space, and the null space for the following matrix.*

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{pmatrix}$$

Solution: The R.R.E.F of A is

$$B = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

therefore a basis for $\mathcal{R}(A)$ is $(1, 3, 2)$, a basis for $\mathcal{C}(A)$ is the first column of A, and a basis for the Null space, $\mathcal{N}(A) = \mathcal{N}(B)$, is given in the above example.

Problem 2: *Find the rank of A in terms of x*

$$A = \begin{pmatrix} 2 & 2 & -6 & 8 \\ 3 & 3 & -9 & 8 \\ 1 & 1 & x & 4 \end{pmatrix}$$

Solution: Applying the operations $R_1 : (1/2)R_1$, $R_3 : R_3 - R_1$ and $R_2 : R_2 - 3R_1$, $R_1 : R_1 + R_2$ and $R_2 : (-1/4)R_2$ to A we get

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & x+3 & 0 \end{pmatrix}$$

Therefore if $x = -3$ we have 2 pivots i.e the rank is 2. Otherwise we can divide by $x + 3$ getting 3 pivots i.e rank 3.

2. **Inner Products**

Let V be a real vector space, and suppose that for each ordered pair of vectors u,v in V you have a real number (u, v). You should be able to decide whether $(,)$ is a inner product or not.

Problem 3: *Let* $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. *We define* $(u, v) = 2u_1v_1 + u_2v_1 + u_1v_2 + u_2v_2$. *Does it define an inner product?*

Solution:

- Since the definition is symmetric in u and v we have that $(u, v) = (v, u)$.
- Now $(u, v + w) = 2u_1(v_1 + w_1) + u_2(v_1 + w_1) + u_1(v_2 + w_2) + u_2(v_2 + w_2) = 2u_1v_1 + u_2v_1 + u_1v_2 + u_2v_2 + 2u_1w_1 + u_2w_1 + u_1w_2 + u_2w_2 = (u, v) + (u, w)$.
- Let c be a real number, Then $(cu, v) = 2cu_1v_1 + cu_2v_1 + cu_1v_2 + cu_2v_2 = c(2u_1v_1 + u_2v_1 + u_1v_2 + u_2v_2) = c(u, v)$.
- $(u, u) = 2u_1^2 + 2u_1u_2 + u_2^2 = u_1^2 + (u_1 + u_2)^2 \geq 0$. Moreover if $(u, u) = 0$, $u_1 = 0$ hence $u_2 = 0$

Since $(,)$ holds the four conditions to be an inner product the answer is positive.

Problem 4: *Let* $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ *and* $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. *We define* $(u, v) = u_1v_2 - u_2v_1 + u_1v_3 - u_3v_1 + u_2v_3 - u_3v_2$. *Does it define an inner product?*

Solution: Let $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Since $(u, u) = 0$ and $u \neq 0$ this is not a inner product.

3. **Orthonormal Basis:** Let $\{u_1, ..., u_n\}$ be a basis for a n-dimensional inner product space V. An orthonormal basis $\{w_1, ..., w_n\}$[1] for V can be calculate as follows :

   i) Let $v_1 = u_1$ and for $k = 2, ..., n$ define

$$v_k = u_k - \frac{(v_1, u_k)}{(v_1, v_1)}v_1 - \frac{(v_2, u_k)}{(v_2, v_2)}v_2 - ... - \frac{(v_{k-1}, u_k)}{(v_{k-1}, v_{k-1})}v_{k-1}$$

   ii) Normalize the v_k's to obtain $w_k = \frac{v_k}{\|v_k\|}$ for $k = 1, ..., n$

with this algorithm we get that

$$w_1 = \frac{u_1}{\|u_1\|}$$

$$w_k = \frac{u_k - (w_1, u_k)w_1 - (w_2, u_k)w_2 - ... - (w_{k-1}, u_k)w_{k-1}}{\|u_k - (w_1, u_k)w_1 - (w_2, u_k)w_2 - ... - (w_{k-1}, u_k)w_{k-1}\|}$$

for $k = 2, ..., n$. So for example

$$w_2 = \frac{u_2 - (w_1, u_2)w_1}{\|u_2 - (w_1, u_2)w_1\|}.$$

Problem 5: *Find an orthonormal basis for the null space* $\mathcal{N}(A)$ *of* A

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 3 & 9 & 6 \end{pmatrix}$$

Solution: In problem 1 we found that a basis for $\mathcal{N}(A)$ is $u_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$, $u_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

[1] this algortihm is known as the Gram-Schmit Orthogonalization process

Now since $(u_1, u_1) = 10$ and $(u_1, u_2) = 6$ we get that $v_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1/5 \\ -3/5 \\ 1 \end{pmatrix}$ thus $w_1 = \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} -1/\sqrt{35} \\ -3/\sqrt{35} \\ 5/\sqrt{35} \end{pmatrix}$ form an orthonormal basis for $\mathcal{N}(A)$.

Remark: Before starting to calculate an orthonormal basis be sure that you have a basis of your space, after you got it you are almost done. Now if the problem starts with a given basis it is easy you just have to orthomormalize.

4. **Orthogonal Projection**

Let V a inner product space, and let W be a non zero subspace of V. If v is a vector of V we can calculate $\mathrm{Proj}_W(v)$, *the orthogonal projection of v on W*, as follows. First calculate $\{w_1, ..., w_n\}$ an orthonormal basis for W, then

$$\mathrm{Proj}_W(v) = w_1(v, w_1) + ... + w_n(v, w_n)$$

Problem 6: *Using the standard inner product in $\mathbb{R}^3$ find the orthogonal projection of $v = \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix}$ on $\mathcal{N}(A)$ where A is the matrix of problem 5.*

Solution: Since $(v, w_1) = -\sqrt{10}$ and $(v, w_2) = 0$ we have that $\mathrm{Proj}_{\mathcal{N}(A)}(v) = -\sqrt{10}w_1 = \begin{pmatrix} 10 \\ -1 \\ 0 \end{pmatrix}$.