On the total disconnectedness of the quotient Aubry set

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Let M be a compact, connected smooth manifold without boundary, endowed with a Riemannian metric g.

A function $L : \mathrm{T}M \longrightarrow \mathbb{R}$ is called a *Tonelli Lagrangian* if:

- $L \in C^2(\mathrm{T}M)$;

- L is strictly convex in the fiber, *i.e.*, the second partial vertical derivative $\frac{\partial^2 L}{\partial v^2}(x,v)$ is positive definite, as a quadratic form, for any $(x,v) \in \mathrm{T}M$;

- L is superlinear in each fiber, *i.e.*,
$$\lim_{\|v\|_x \to +\infty} \frac{L(x,v)}{\|v\|_x} = +\infty\,.$$

Given a Lagrangian, we can define the associated *Hamiltonian*, as a function on the cotangent bundle:

$$\begin{aligned} H: \mathsf{T}^*M &\longrightarrow \mathbb{R} \\ (x,p) &\longmapsto \sup_{v\in \mathsf{T}_x M} \{\langle p,\, v\rangle_x - L(x,v)\} \end{aligned}$$

where $\langle\,\cdot,\,\cdot\,\rangle_x$ represents the canonical pairing between the tangent and cotangent space.

If L is a Tonelli Lagrangian, one can easily prove that H is finite everywhere, C^2, superlinear and strictly convex.

We say that an absolutely continuous curve $\gamma : \mathbb{R} \longrightarrow M$ is a *minimizer*, if for any interval $[a,b]$ and any other absolutely continuous curve $\gamma_1 : [a,b] \longrightarrow M$ such that $\gamma(a) = \gamma_1(a)$ and $\gamma(b) = \gamma_1(b)$, we have

$$\int_a^b L(\gamma(t),\dot\gamma(t))dt \le \int_a^b L(\gamma_1(t),\dot\gamma_1(t))dt\,.$$

Let η be a closed 1-form on M. We can define a function on the tangent space

$$\begin{aligned} \widehat{\eta} : \mathrm{T}M &\longrightarrow \mathbb{R} \\ (x,v) &\longmapsto \langle \eta(x),\, v \rangle_x \end{aligned}$$

and consider a new Tonelli Lagrangian and the associated Hamiltonian:

$$L_\eta := L - \widehat{\eta} \qquad \text{and} \qquad H_\eta(x,p) := H(x, \eta + p)\,.$$

- L and L_η have the same Euler-Lagrange flow;

- they have different minimizers, depending on the de Rham cohomology class $c = [\eta] \in H^1(M;\mathbb{R})$.

From here, the interest in considering modified Lagrangians, corresponding to different cohomology classes.

$$
\begin{array}{ccccccccc}
\dot{M}_c & \subseteq & \dot{\mathcal{A}}_c & \subseteq & \dot{N}_c & \subseteq & \dot{E}_c & \subseteq & \mathsf{T}M \\
\downarrow & & \downarrow & & & & & & \downarrow \pi \\
M_c & \subseteq & \mathcal{A}_c & & & \subseteq & & & M
\end{array}
$$

- $\dot{E}_c$ is the energy level corresponding to $\alpha(c)$ (where $\alpha : H^1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is Mather's α function);

- $\dot{N}_c$ is the *Mañé set*, *i.e.*, the union of the supports of all c-minimizers;

- $\dot{\mathcal{A}}_c$ is the *Aubry set*, *i.e.*, the union of the supports of all "*regular*" c-minimizers;

- $\dot{M}_c$ is the *Mather set*, *i.e.*, the closure of the union of the supports of all c-*minimal measures* on $\mathsf{T}M$.

For $t > 0$ and $x,\, y \in M$, consider:

$$h_{\eta,t}(x,y) = \inf \int_0^t L_\eta(\gamma(s), \dot\gamma(s))\, ds\,,$$

where the infimum is taken over all piecewise C^1 paths $\gamma : [0,t] \longrightarrow M$, such that $\gamma(0) = x$ and $\gamma(t) = y$.

We define the *Peierls barrier* as:

$$h_\eta(x,y) = \liminf_{t\to+\infty} (h_{\eta,t}(x,y) + \alpha(c)t)\,.$$

- This function is finite and Lipschitz;

- for each $x \in M$, $h_\eta(x,x) \geq 0$;

- for each $x,\, y,\, z \in M$

$$h_\eta(x,y) \leq h_\eta(x,z) + h_\eta(z,y)\,;$$

- for each $x,\, y \in M$, $h_\eta(x,y) + h_\eta(y,x) \geq 0$.

Inspired by the above properties, we can define

$$\begin{array}{rcl} \delta_c : \; M \times M & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & h_\eta(x,y) + h_\eta(y,x) \end{array}$$

- it does actually depend only the cohomology class.

- it is non-negative, symmetric and satisfies the triangle inequality;

- $\mathcal{A}_c = \{x \in M : \delta_c(x,x) = 0\}$ (*projected Aubry set*).

Therefore, δ_c is a *pseudometric* on $\mathcal{A}_c$.

The *quotient Aubry set* $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ is the metric space obtained by identifying two points in $\mathcal{A}_c$, if their δ_c-pseudodistance is zero.

We shall denote an element of this quotient by $\bar{x} = \{y \in \mathcal{A}_c : \ \delta_c(x,y) = 0\}$.

These elements (that are also called c-*static classes*), provide a partition of $\mathcal{A}_c$ into compact subsets, that can be lifted to invariant subsets of $\mathrm{T}M$.

They are really interesting, from a dynamical systems point of view, since they contain the α and ω limit sets of c-minimizing orbits.

In Mather's studies on Arnold diffusion, it turns out that understanding certain aspects of this set seems to help in understanding what orbits can be shown to exist by the method of minimization. In particular, it would be useful that it were totally disconnected (or "small" is some sense of dimension).

Mather (2003) showed the disconnectedness of $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ under the hypotheses:

- the state space has dimension ≤ 2, or
- the Lagrangian is the kinetic energy associated to a Riemannian metric, and the state space has dimension ≤ 3.

WHAT HAPPENS IN HIGHER DIMENSION?

Unfortunately, this is not true anymore!

- Mather (2004) provided several examples of mechanical Lagrangians on $\mathrm{T}\mathbb{T}^d$, with potentials $U \in C^{2d-3,1-\varepsilon}(\mathbb{T}^d)$, whose quotient Aubry sets $\bar{\mathcal{A}}_0$ were isometric to intervals.

  This construnction is closely related to Whitney's counterexample for Sard's Lemma.

- Fathi noticed that is possible to construct similar counterexamples, in which the quotient Aubry set is Lipschitz equivalent to any *doubling* metric space.

Conjecture. *If $L \in C^r(\mathrm{T}M)$, with $r \geq 2d-2$, then the quotient Aubry set $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ is totally disconnected, for any cohomology class $c \in H^1(M;\mathbb{R})$.*

Theorem (A.S., 2006). *Let M be a compact connected manifold of dimension $d \geq 1$ and assume that L is a Tonelli Lagrangian satisfying:*

- $\Lambda_L := \mathcal{L}(M \times \{0\})$ *is a Lagrangian submanifold of* T^*M *(equivalently* $\frac{\partial L}{\partial v}(x,0) \cdot dx$ *is a closed 1-form);*

- $\frac{\partial L}{\partial v}(x,0) \in C^2(M)$ *and* $L(x,0) \in C^r(M)$, *with* $r \geq 2d-2$.

Then, the quotient Aubry set $(\bar{\mathcal{A}}_{c_L}, \bar{\delta}_{c_L})$*, corresponding to the Liouville class of* Λ_L*, is totally disconnected.*

This result includes *symmetrical* (or *reversible*) Lagrangians and in particular *mechanical* ones (in these cases $c_L = 0$).

This result is clearly optimal, as Mather's counterexamples show.

Other recent related results:

- A similar (indipendent) result by Albert Fathi, Alessio Figalli and Ludovic Rifford.

- Strikingly, Patrick Bernard and Gozalo Contreras recently proved that:

  Theorem. *Generically in the sense of Mañé, for all* $c \in \mathsf{H}^1(M;\mathbb{R})$ *there are at most* $1 + \dim \mathsf{H}^1(M;\mathbb{R})$ *ergodic* c*-minimizing measures.*

  Since each static class contains at least the support of one ergodic minimizing measure, then it follows that:

  Generically, in the sense of Mañé, for all $c \in \mathsf{H}^1(M;\mathbb{R})$ *the quotient Aubry set* $\bar{\mathcal{A}}_c$ *is finite, with at most* $1 + \dim \mathsf{H}^1(M;\mathbb{R})$ *elements.*

- KEY OBSERVATION:

  There is a deep *liaison* between this problem and Sard's lemma.

- MAIN TOOLS:

  – Fathi's Weak KAM theory;

  – Morse-Sard's type lemma.

- MAIN DIFFICULTIES:

  In dimension > 1, Sard's lemma does not hold for C^1 functions!

  Therefore, one would need more regularity for the *structures* involved; but, in general, this is not so easy to retrieve.

WEAK KAM THEORY (Recall)

- A locally lipschitz function $u : M \longrightarrow \mathbb{R}$ is a *subsolution* of $H_\eta(x, d_x u) = k$, with $k \in \mathbb{R}$, if $H_\eta(x, d_x u) \leq k$ for almost every $x \in M$.

- It is possible to show that there exists $c[\eta] \in \mathbb{R}$, such that $H_\eta(x, d_x u) = k$ admits no subsolutions for $k < c[\eta]$ and has subsolutions for $k \geq c[\eta]$. The constant $c[\eta]$ is called *Mañé's critical value* and coincides with $\alpha(c)$.

- $u : M \longrightarrow \mathbb{R}$ is a η-*critical subsolution*, if $H_\eta(x, d_x u) \leq \alpha(c)$ for almost every $x \in M$.

- Fathi and Siconolfi (2004) showed that C^1 critical subsolutions are dense w.r.t. the uniform topology.

Let us denote by $\mathcal{S}^1_\eta$ the set of C^1 η - critical subsolutions and

$$\mathcal{D}^1_c := \{u - v : \ u, \ v \in \mathcal{S}^1_\eta\}\,.$$

For $x, y \in \mathcal{A}_c$:

- $h_\eta(x,y) = \sup_{u \in \mathcal{S}^1_\eta} (u(y) - u(x))\,;$
- $\delta_c(x,y) = \sup_{w \in \mathcal{D}^1_c} (w(y) - w(x))\,.$

Moreover,

- If $w \in \mathcal{D}^1_c$, then $d_x w = 0$ on $\mathcal{A}_c$. Therefore,

$$\mathcal{A}_c \subseteq \bigcap_{w \in \mathcal{D}^1_c} \text{Crit}(w)\,.$$

- If $w \in \mathcal{D}^1_c$, then it is constant on any quotient class of $\bar{\mathcal{A}}_c$.
  Namely, if $x, y \in \mathcal{A}_c$ and $\delta_c(x,y) = 0$, then $w(x) = w(y)$.

The connection with Sard's lemma becomes now more clear.

Definition. A C^1 function $f : M \longrightarrow \mathbb{R}$ is of *Morse-Sard's type* if $|f(\mathrm{Crit}(f))| = 0$.

Proposition (Key step). *Let M be a compact connected manifold of dimension $d \geq 1$, L a Tonelli Lagrangian and $c \in H^1(M;\mathbb{R})$. If each $w \in \mathcal{D}_c^1$ is of Morse-Sard's type, then the quotient Aubry set $(\bar{\mathcal{A}}_c, \bar{\delta}_c)$ is totally disconnected.*

This proposition and Sard's lemma easily imply Mather's result in dimension $d \leq 2$ (autonomous case), since one can show that w's are $C^{1,1}$.

PROOF (Key step):

Suppose by contradiction that $\bar{\mathcal{A}}_c$ is not totally disconnected; therefore it must contain a connected component $\overline{\Gamma}$ with at least two points $\bar{x}$ and $\bar{y}$.

In particular $\bar{\delta}_c(\bar{x}, \bar{y}) > 0$. Therefore, there exists $w \in \mathcal{D}_c^1$ such that $|w(y) - w(x)| > 0$, for some $x \in \bar{x}$ and $y \in \bar{y}$.

This implies that the set $w(\pi^{-1}(\overline{\Gamma}))$ is a connected set in $\mathbb{R}$ with at least two different points, hence it is a non degenerate interval and its Lebesgue measure is positive.

But

$$w(\pi^{-1}(\overline{\Gamma})) \subseteq w(\mathcal{A}_c) \subseteq w(\mathrm{Crit}(w)),$$

then

$$0 < \left| w(\pi^{-1}(\overline{\Gamma})) \right| \leq |w(\mathcal{A}_c)| \leq |w(\mathrm{Crit}(w))|$$

that contradicts our assumptions. $\square$

The main problem becomes now to understand under which conditions on L and c, these differences of subsolutions are of *Morse-Sard's type*.

Proposition. *Under the hypotheses of the main theorem, if* $w \in \mathcal{D}^1_{c_L}$*, then*

$$|w(\mathcal{A}_{c_L})| = 0\,.$$

IDEA OF THE PROOF:

Under the above assumptions, one can show that:

- Every constant function $u \equiv const$ is a η_L-critical subsolution. In particular, all η_L-critical subsolutions are such that $d_x u \equiv 0$ on $\mathcal{A}_{c_L}$.

- For every $x \in M$,

$$\frac{\partial H_{\eta_L}}{\partial p}(x,0) = \frac{\partial H}{\partial p}(x,\eta_L(x)) = 0\,.$$

In particular,

$$\mathcal{A}_{c_L} \subseteq \{L(x,0) = -\alpha(c_L)\} \quad \text{and}$$

$$\begin{aligned} \alpha(c_L) &= \sup_{x\in M}(-L(x,0)) = \\ &= -\inf_{x\in M} L(x,0) =: e_0\,. \end{aligned}$$

In general

$$e_0 \leq \min_{c\in H^1(M;\mathbb{R})} \alpha(c) = -\beta(0),$$

where $\beta : H_1(M;\mathbb{R}) \longrightarrow \mathbb{R}$ is Mather's β function (*i.e.*, the convex conjugate of α).

Moreover, $c_L \in \partial\beta(0)$, namely, it is a subgradient of β at 0.

If $u \in \mathcal{S}^1_{\eta_L}$, there exists a neighborhood W_0 of $\mathcal{A}_{c_L}$ such that, $\forall\, x \in W_0$:

$$\begin{aligned}
&\alpha(c_L) \geq H_{\eta_L}(x, d_x u) = \\
&= H_{\eta_L}(x,0) + \frac{1}{2}\frac{\partial^2 H_{\eta_L}}{\partial p^2}(x,0)(d_x u)^2 + R_2 \geq \\
&\geq H_{\eta_L}(x,0) + \frac{\tilde{\gamma}}{2}\|d_x u\|_x^2 - \frac{\tilde{\gamma}}{4}\|d_x u\|_x^2 = \\
&= H_{\eta_L}(x,0) + \frac{\tilde{\gamma}}{4}\|d_x u\|_x^2 = \\
&= -L(x,0) + \frac{\tilde{\gamma}}{4}\|d_x u\|_x^2
\end{aligned}$$

that implies

$$\|d_x u\|_x^2 \leq \frac{4}{\tilde{\gamma}}(\alpha(c_L) + L(x,0)).$$

In particular, for any $w = u - v \in \mathcal{D}^1_{c_L}$, one gets the following estimates (in an open neighborhood of the Aubry set):

$$\|d_x w\|^2 \leq C(L(x,0) + \alpha(c_L)) =: U(x)\,,$$

where $C > 0$ and $U \geq 0$ is in $C^r(M)$, with $r \geq 2d - 2$ and

$$\begin{aligned} \mathcal{A}_{c_L} &\subseteq \{x \in W_0 :\ L(x,0) = -\alpha(c_L)\} = \\ &= \{x \in W_0 :\ U(x) = 0\} \subseteq \mathrm{Crit}(U)\,. \end{aligned}$$

The result will then easily follow from the following Morse-Sard's type lemma.

Lemma. *Let $U \in C^r(M)$, with $r \geq 2d - 2$, be a non-negative function, vanishing somewhere and denote $\mathcal{A} = \{U(x) = 0\}$. If $u : M \longrightarrow \mathbb{R}$ is C^1 and satisfies $\|d_x u\|^2_x \leq U(x)$ in an open neighborhood of $\mathcal{A}$, then $|u(\mathcal{A})| = 0$.*

SKETCH OF THE PROOF:

Define, for $1 \leq s \leq r$:

$$B_s = \{x \in \mathcal{A} : \ U \text{ is } s\text{ - flat at } x\}$$

and observe that

$$\mathcal{A} = B_1 := \{x \in \mathcal{A} : DU(x) = 0\}.$$

The proof is by induction on the dimension d.

- Claim 1: If $s \geq 2d - 2$, then $|u(B_s)| = 0$. This proves the case $d = 1$.

- Observe:

  $$\mathcal{A} = (B_1 \setminus B_2) \cup \ldots \cup (B_{2d-3} \setminus B_{2d-2}) \cup B_{2d-2}.$$

  Claim 2: Every $\tilde{x} \in B_s \setminus B_{s+1}$ has a neighborhood $\tilde{V}$, such that

  $$|u((B_s \setminus B_{s+1}) \cap \tilde{V})| = 0\,.$$

Choose $\tilde{x} \in B_s \setminus B_{s+1}$; there exists a function

$$w(x) = \partial_{i_1} \partial_{i_2} \ldots \partial_{i_s} U(x)$$

such that

$$w(\tilde{x}) = 0 \qquad \text{but} \qquad \partial_1 w(\tilde{x}) \neq 0\,.$$

Define

$$\begin{array}{rcl} h : \Omega & \longrightarrow & \mathbb{R}^d \\ x & \longmapsto & (w(x),\, x_2,\, \ldots,\, x_d)\,, \end{array}$$

where $x = (x_1, x_2, \ldots, x_d)$. Clearly, $h \in C^{r-s}(\Omega)$ and $Dh(\tilde{x})$ is non-singular; hence, there is an open neighborhood V of $\tilde{x}$ such that

$$h : V \longrightarrow W$$

is a C^{r-s} isomorphism (with $W = h(V)$).
We use this fact to reduce to a $(d-1)$-dimensional situation.

Let V_1 be an open precompact set, containing $\tilde{x}$ and properly contained in V, and define $A = B_s \cap \overline{V_1}$, $A^* = h(A)$ and $g = h^{-1}$. If we consider W_1, any open set containing A^* and properly contained in W, we can prove a *rough composition theorem* à la Kneser and Glaeser and find $F : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that:

i) $F \in C^{r-1}(\mathbb{R}^d)$;

ii) $F \geq 0$;

iii) $F(x) = U(g(x)) = 0$ on A^*;

iv) F is s-flat on A^*;

v) $\{F(x) = 0\} \cap W_1 = A^*$;

vi) there exists a constant $K > 0$, such that $U(g(x)) \leq KF(x)$ on W_1.

Define

$\widehat{W} = \{(x_2, \ldots, x_d) \in \mathbb{R}^{d-1} :\ (0, x_2, \ldots, x_d) \in W_1\}$

and

$$\widehat{U}(x_2, \ldots, x_d) = C\, F(0, x_2, \ldots, x_d),$$

where C is a positive constant to be chosen sufficiently big. Observe that $\widehat{U} \in C^{r-1}(\mathbb{R}^{d-1})$ and

$$A^* = \{0\} \times \widehat{B}_1\,,$$

where $\widehat{B}_1 = \{(x_2, \ldots x_d) \in \widehat{W} :\ F(0, x_2, \ldots, x_d) = 0\}$. Denote

$$\widehat{\mathcal{A}} := \{(x_2, \ldots, x_d) \in \widehat{W} :\ \widehat{U} = 0\} = \widehat{B}_1$$

and define the following function on $\widehat{W}$:

$$\widehat{u}(x_2, \ldots, x_d) = u(g(0, x_2, \ldots, x_d)).$$

These functions satisfy the hypotheses for the $(d-1)$-dimensional case.

- $\widehat{U} \in C^{r-1}(\mathbb{R}^{d-1})$, with $r-1 \geq 2d-3 > 2(d-1)-2$;

- $\widehat{u} \in C^1(\widehat{W})$ (since g is in $C^{r-s}(W)$, where $1 \leq s \leq r-1$);

- if we denote by $\mu = \sup_{W_1} \|d_x g\| < +\infty$ (since g is C^1 on $\overline{W_1}$), then we have that for every point in $\widehat{W}$:

$$\begin{aligned}
&\|d\widehat{u}(x_2,\dots,x_d)\|^2 \leq \\
&\leq \|d_x u(g(0,x_2,\dots,x_d))\|^2 \cdot \\
&\qquad \cdot \|d_x g(0,x_2,\dots,x_d)\|^2 \leq \\
&\leq \mu^2 \|d_x u(g(0,x_2,\dots,x_d))\|^2 \leq \\
&\leq \mu^2 U(g(0,x_2,\dots,x_d)) \leq \\
&\leq \mu^2 K F(0,x_2,\dots,x_d) \leq \\
&\leq \widehat{U}(x_2,\dots,x_d),
\end{aligned}$$

  if we choose $C > \mu^2 K$.

Therefore, it follows from the inductive hypothesis, that:

$$|\widehat{u}(\widehat{\mathcal{A}})| = 0.$$

Since,

$$\begin{aligned} u(B_s \cap V_1) &\subseteq u(A) = u(g(A^*)) = \\ &= u(g(\{0\} \times \widehat{B}_1)) = \\ &= \widehat{u}(\widehat{B}_1) = \widehat{u}(\widehat{\mathcal{A}}), \end{aligned}$$

defining $\tilde{V} = V_1$, we can conclude that

$$|u(B_s \cap \tilde{V})| \leq |\widehat{u}(\widehat{\mathcal{A}})| = 0.$$

□