Foliations, Homology and Dynamics

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3. **Analytical or Lyapunov growth**

Geometric growth rate

$$\chi(x,r) = \limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{Vol}(f^n(W_r(x)))$$

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χ is independent of r and χ is also independent of the Riemannian metric on M.

One dimensional foliations and asymptotical cycles

For any segment on the unstable foliation $W_r(x)$ and any integer i, let l_i be the loop obtained by joining two end points of $f^i(W_r(x))$ by the shortest oriented curve. Let

$$[l_i] \in H_1(M, \mathbb{Z})$$

and $|l_i|$ be the length of l_i.

$$[l_i]/|l_i| \in H_1(M, \mathbb{R})$$

Definition: We say that the invariant foliation W *carries a non-trivial homology* $h_W \in H_1(M,\mathbb{R})$, $h_W \neq 0$ if there are $x \in M$, $r > 0$ and a subsequence $n_i \to \infty$ such that

$$\lim_{i\to\infty} \frac{[l_{n_i}]}{|l_{n_i}|} = h_W \in H_1(M,\mathbb{R}).$$

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We say that the invariant foliation W *carries a unique non-trivial homology* $h_W \in H_1(M, \mathbb{R})$, $h_W \neq 0$ if h_W defined above is unique up to rescaling.

Proposition: If W carries a unique non-trivial homology $h_W \in H_1(M, \mathbb{R})$, then h_W is an eigenvector of the induced linear map

$$f_* : H_1(M, \mathbb{R}) \to H_1(M, \mathbb{R}).$$

Theorem: (Saghin & X, 2006) If an invariant foliation W carries a unique nontrivial homology h_W, then its geometric expansion $\chi(x, r)$ on W is independent of the point $x \in M$ and independent of $r > 0$.

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Moreover, let λ_W be the corresponding eigenvalue for eigenvector h_W for the linear map $f_* : H_1(M,\mathbb{R}) \longrightarrow H_1(M,\mathbb{R})$, then

$$\begin{aligned}\chi &= \chi(x,r) = \limsup_{i\to\infty} \frac{1}{i} \ln |f^i(W_r(x))| \\ &= \lim_{i\to\infty} \frac{1}{i} \ln |f^i(W_r(x))| = \ln \lambda_W\end{aligned}$$

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The currents are uniformly bounded and there are weak limits.

$$\lim_{i \to \infty} C_{n_i}(\omega) = C(\omega)$$

A current C is said to be *closed* if for any exact k-form $\omega = d\alpha$, we have $C(\omega) = C(d\alpha) = 0$.

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From Stokes' Theorem, we have:

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Every limit current is closed for $k = 1$. High dimensions?

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It is conjectured that all Anosov diffeomorphisms are transitive.

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We say that a k-dimensional invariant foliation W carries a unique non-trivial homology (up to rescale) if all subsequential limits of the currents C_i are closed and the homologies it carries are unique up to scalar multiplication.

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Lyapunov expansion:

Let μ be an invariant probablity measure on M. Let E be an invariant sub-bundle of TM, e.g., $E = TW$. Let $\Lambda_E(x)$ be the sum (counting multiplicity n_i) of the Lyapunov exponents corresponding to the splittings in E_x.

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$$\Lambda_E = \int_M \log(\|\Lambda^k D_x f|_{\Lambda^k E_x}\|) d\mu.$$

When μ is ergodic, $\Lambda_E(x) = \Lambda_E$ a.e. (μ).

We call Λ_E the *Lyapunov growth*.

Proposition: Let $f \in \mathrm{Diff}^r_\mu(M)$ be a diffeomorphism on M, preserving a smooth volume μ. Let W be a k dimensional foliation of M, invariant under f. If the foliation W is absolutely continuous, then

$$\Lambda_W \leq \chi_W.$$

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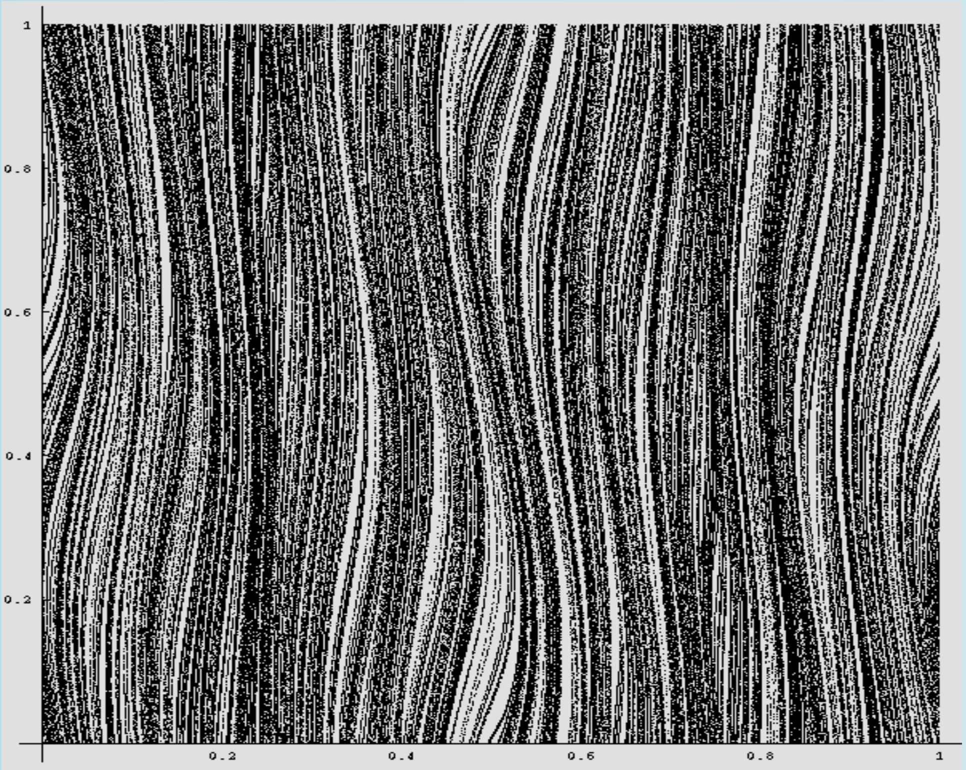
Shub and Wilkinson give the first examples of stably pathological foliations with compact circles, where a full measure set intersects the center foliation in finite number of points.

A pathological foliation



Corollary: Let $f \in \mathrm{Diff}^r_\mu(M)$ be diffeomorphism on M, preserving a smooth volume μ. Let W be a k dimensional foliation of M, invariant under f. If $\chi < \Lambda_W$, then the foliation W is not absolutely continuous.

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Moreover, if μ is ergodic, then there is a full measure set $A \in M$ such that every leaf $W(x)$ of the foliation W intersect A in a zero measure set,

$$\mu_W(W(x) \cap A) = 0,$$

for all $x \in M$, where μ_W is the conditional measure of μ on the leaves of W.

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i.e., W is a $pathological$ foliation where Fubini's theorem fails miserably.

Perturbations and examples

Let $A : \mathbb{T}^n \to \mathbb{T}^n$ be a linear toral automorphism such that $\det(A) = 1$ and A has at least two distinct nonegative (or nonpositive eigenvalues).

Theorem: (Saghin & X 2006) For any such automorphism A of the torus $\mathbb{T}^n$ there exist an open set of volume preserving diffeomorphisms U, C^1 arbitrarily close to A, such that, for any $f \in U$, the weaker foliation is non-absolutely continuous and moreover it is pathological.

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- Katok: For $C^{1+\alpha}$ diffeomorphisms on compact surfaces, topological entropy is lower semi-continuous. (Pesin Theory for Hyperbolic measures)

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- The dimension of the center foliation is 2 or less;
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Then the topological entropy h_{top} : $\mathrm{Diff}^\infty(M) \to \mathbb{R}$ is continuous at f.

Furthermore, if the center foliation has dimension 1, then h_{top} is a constant in a small neighborhood of $f \in \mathrm{Diff}^1(M)$.

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Theorem: (Hua, Saghin & X 2006) Let f be a partially hyperbolic diffeomorphism on a compact manifold M. Let ν be an ergodic measure and let λ_i^c be the Lyapunov exponents corresponding to the center distribution E^c. Then the following estimate holds

$$h_\nu(f) \leq \sum_{\lambda_i^c > 0} \lambda_i^c + \chi_u(f).$$

- R. Saghin, Z. Xia, *Geometric expansion, Lyapunov exponents and foliations*, preprint 2006, math.DS/0611324

- Y. Hua, R. Saghin, Z. Xia, *Topological Entropy and Partially Hyperbolic Diffeomorphisms*, preprint 2006, math.DS/0608720