

### Solution for section 5.5

5. We have  $V = P_3$  with the inner product given by

$$(p(t), q(t)) = \int_0^1 p(t)q(t) dt.$$

Denote a general element  $p$  of  $P_3$  as

$$p(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$

If  $p$  is orthogonal to the subspace  $\text{span}\{t - 1, t^2\}$ , we have

$$0 = (p(t), t - 1) = \int_0^1 p(t)(t - 1) dt$$

$$0 = (p(t), t^2) = \int_0^1 p(t)t^2 dt.$$

But we have

$$p(t)(t - 1) = a_3t^4 + (a_2 - a_3)t^3 + (a_1 - a_2)t^2 + (a_0 - a_1)t - a_0$$

and

$$p(t)t^2 = a_3t^5 + a_2t^4 + a_1t^3 + a_0t^2.$$

Hence we obtain the equations

$$0 = \frac{a_3}{5} + \frac{a_2 - a_3}{4} + \frac{a_1 - a_2}{3} + \frac{a_0 - a_1}{2} - a_0$$

$$0 = \frac{a_3}{6} + \frac{a_2}{5} + \frac{a_1}{4} + \frac{a_0}{3}.$$

Simplifying the equations, we have

$$0 = \frac{a_3}{10} + \frac{a_2}{6} + \frac{a_1}{3} + a_0$$

$$0 = \frac{a_3}{6} + \frac{a_2}{5} + \frac{a_1}{4} + \frac{a_0}{3}.$$

Solving the equation, we obtain

$$a_2 = r, a_3 = s, a_1 = -\frac{52}{65}r - \frac{52}{25}s, a_0 = -\frac{1}{30}r + \frac{89}{150}s.$$

Therefore a basis for  $W^\perp$  is given by

$$\left\{ t^3 - \frac{52}{25}t + \frac{89}{150}, t^2 - \frac{52}{65}t - \frac{1}{30} \right\}$$

**Please check this answer!**