Final Exam Solutions

– Partially based on the solutions of D. Holcomb and Jaeho Lee –

1. *Use the universal coefficient theorem to show tht if* $H_*(X;\mathbb{Z})$ *is finitely generated, so the Euler characteristic* $\chi(X) = \sum_n$ *rank* $H_n(X;\mathbb{Z})$ *is defined, then for any coefficient field* F *we have* $\chi(X) = \sum_n$ *rank* $H_n(X;F)$*.*

Solution: If we have that $H_*(X;\mathbb{Z})$ is finitely generated, then it can be written as the finite sum

$$H_*(X;\mathbb{Z}) = \oplus_{n=0}^{N} H_n(X;\mathbb{Z})$$

for some $N \in \mathbb{Z}_+$ and each $H_n(X;\mathbb{Z})$ is finitely generated. By the Fundamental Theorem of Algebra, we can write

$$H_n(X;\mathbb{Z}) = \mathbb{Z}^d \oplus \left(\oplus_i \mathbb{Z}_{m_i}^{a_i}\right)$$

for some d and a_i's and $m_i = p_i^{k_i} \in \mathbb{N}$ for some k_i with p_i primes. If we have that $a = \sum_i a_i$ then a free resolution of $H_n(X;\mathbb{Z})$ is

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^d \arrow[r] & \mathbb{Z}^{d+a} \arrow[r] & \mathbb{Z}^d \oplus \left(\oplus_i \mathbb{Z}_{m_i}^{a_i}\right) \arrow[r] & 0\,. \end{tikzcd}$$

Therefore we have that

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(H_n(X;\mathbb{Z}),F) \arrow[r] & \mathbb{Z}^d \otimes F \arrow[r] & \mathbb{Z}^{d+a} \otimes F \arrow[r] & \mathbb{Z}^d \oplus \left(\oplus_i \mathbb{Z}_{m_i}^{a_i}\right) \otimes F \arrow[r] & 0 \end{tikzcd}$$

is exact.

Now we split the proof into two cases, one the case with $\mathrm{char}(F) = 0$ and the other with $\mathrm{char}(F) = p$ non-zero prime.

First assume $\mathrm{char}(F) = 0$. Then by properties of tensor products any elements of finite order when tensored with F are 0 giving us that

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(H_n(X;\mathbb{Z}),F) \arrow[r] & F^d \arrow[r, "\varphi"] & F^{a+d} \arrow[r] & F^d \arrow[r] & 0 \end{tikzcd}$$

is exact. Therefore since we have that

$$\mathrm{Tor}(H_n(X;\mathbb{Z}),F) = \ker\varphi$$

and $\ker\varphi = 0$, we have that $\mathrm{Tor}(H_n(X;\mathbb{Z}),F) = 0$. Now by applying the universal coefficient theorem for homology we get that

$$H_n(X;F) = H_n(X;\mathbb{Z}) \otimes F.$$

This gives us that

$$\begin{aligned} \text{rank } H_n(X;F) &= \text{rank } (\mathbb{Z}^d \oplus_i \mathbb{Z}_{m_i}^{a_i}) \otimes F \\ &= \text{rank } F^d \\ &= d = \text{rank } H_n(X;\mathbb{Z}). \end{aligned}$$

Therefore $\chi(X) = \sum_n (-1)^n \text{rank } H_n(X;\mathbb{Z}) = \sum_n (-1)^n \text{rank } H_n(X;F)$ since the ranks are equal at each level.

Next we consider the case $\text{char}(F) = p \neq 0$. In this case, we need to calculate $\text{Tor}(H_n(X;\mathbb{Z});F)$. We have

$$\begin{aligned} \text{Tor}(H_n(X;\mathbb{Z});F) &= \text{Tor}(\mathbb{Z}^d \oplus \left(\mathbb{Z}^d \oplus \left(\oplus_i \mathbb{Z}_{m_i}^{a_i}\right);F\right) \\ &= \text{Tor}(\mathbb{Z}^d;F) \oplus \left(\oplus_i \text{Tor}(\mathbb{Z}_{m_i}^{a_i};F)\right) \\ &= \oplus_i \text{Tor}(\mathbb{Z}_{m_i}^{a_i};F) = \oplus_i \text{Tor}(\mathbb{Z}_{m_i};F)^{a_i}. \end{aligned}$$

But we compute $\text{Tor}(\mathbb{Z}_{m_i};F)$. But we have $\text{Tor}(\mathbb{Z}_{m_i};F) = \ker(F \overset{m_i}{\to} F)$. Therefore if $p_i = p$, $\text{Tor}(\mathbb{Z}_{m_i};F) = F$, and $\mathbb{Z}_{m_i} \otimes F = F$. If $p_i \neq p$, $\text{Tor}(\mathbb{Z}_{m_i};F) = 0$ and $\mathbb{Z}_{m_i} \otimes F = 0$. (Recall $\mathbb{Z}_a \otimes \mathbb{Z}_b = \mathbb{Z}_{(a,b)}$.)

Therefore if $p_i \neq p$, the torsion group does not contribute.

On the other hand if $p_i = p$, each such summand in $\text{Tor}(H_{n-1}(X;\mathbb{Z});F)$ contributes to rank $H_n(X;F)$. In summary, we obtain

$$\dim_F(H_n(X;F)) = \dim_F(H_n(X;\mathbb{Z}) \otimes F \oplus \text{Tor}(H_{n-1}(X;\mathbb{Z});F)) = b_n + c_n + c_{n-1}$$

where $b_n = \text{rank } H_n(X;\mathbb{Z})$ and $c_n = \#(p\text{-multiple-torsion-summands of } H_n(X;\mathbb{Z})$. We sum

$$\begin{aligned} \sum_{n=0}^{N+1} (-1)^n \dim_F H_n(X;F) &= \sum_{n=1}^{N+1} (-1)^n (b_n + c_n + c_{n-1}) \\ &= \sum_{n=1}^{N+1} (-1)^n b_n + \sum_{n=1}^{N+1} (-1)^n c_n - \sum_{n=0}^{N} (-1)^n c_n \\ &= \chi(X) + (-1)c_{N+1} - c_0. \end{aligned}$$

We recall that H_0 has no torsion part and so $c_0 = 0$. We also have $c_{N+1} = 0$ since $H_{N+1}(X;\mathbb{Z}) = 0$ by the choice of N. This finishes the proof.

2. *Show that Tor($A, \mathbb{Q}/\mathbb{Z}$) is isomorphic to the torsion subgroup of A. Deduce that A is torsion free iff Tor(A, B) $= 0$ for all B.*

Solution: First note that $\text{Tor}(A, \mathbb{Q}/\mathbb{Z}) = \text{Tor}(\mathbb{Q}/\mathbb{Z}, A)$, then consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\iota"] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Then properties of Tor give us that

$$\begin{tikzcd} 0 \arrow[r] & \text{Tor}(\mathbb{Z}, A) \arrow[r] & \text{Tor}(\mathbb{Q}, A) \arrow[r] & \text{Tor}(\mathbb{Q}/\mathbb{Z}, A) \arrow[r] & \mathbb{Z} \otimes A \arrow[r, "\iota"] & \mathbb{Q} \otimes A \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes A \arrow[r] & 0 \end{tikzcd}$$

is an exact sequence. Since $\mathbb{Z}$ and $\mathbb{Q}$ are torsion free this leaves us with

$$\begin{tikzcd} 0 \arrow[r] & \text{Tor}(\mathbb{Q}/\mathbb{Z}, A) \arrow[r] & \mathbb{Z} \otimes A \arrow[r, "\iota"] & \mathbb{Q} \otimes A \arrow[r] & \mathbb{Q}/\mathbb{Z} \otimes A \arrow[r] & 0 \end{tikzcd}$$

Now consider the inclusion map in the above exact sequence it can be interpreted as a map $A \to \mathbb{Q} \otimes A$ by $a \mapsto 1 \otimes a$. Since we are mapping into $\mathbb{Q} \otimes A$ then any element of finite order n in A has the property that $1 \otimes a \sim 1/n \otimes na \sim 0$, so the kernel of the map is exactly the element of A with finite order, or the torsion subgroup of A. Since the sequence is exact this gives us that $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$ is isomorphic to the kernel of the inclusion map, and so to the torsion subgroup of A.

Now this means that if $\mathrm{Tor}(A, B) = 0$ for all B, then $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) = 0$ meaning that A has no torsion subgroup and so is torsion free. And by properties of Tor we have that if A is torsion free then $\mathrm{Tor}(A, B) = 0$, therefore $\mathrm{Tor}(A, B) = 0$ for all B iff A is torsion free.

3. *Show that if $\tilde{H}^n(X;\mathbb{Q})$ and $\tilde{H}^n(X;\mathbb{Z}_p)$ are zero for all n and all primes p, then $\tilde{H}_n(X;\mathbb{Z}) = 0$ for all n, and hence $\tilde{H}^n(X;G) = 0$ for all G and n.*

It seems to me that this problem should assume that $H_*(X;\mathbb{Z})$ is finitely generated or at least X is the direct limit of X_α whose homology are finitely generated. :

Solution: The universal coefficient theorem gives us that

$$\tilde{H}^n(X;\mathbb{Q}) \quad = \quad \mathrm{hom}(\tilde{H}_n(X), \mathbb{Q}) \oplus \mathrm{Ext}\ (H_{n-1}(X), \mathbb{Q})$$

and

$$\tilde{H}^n(X;\mathbb{Z}_p) \quad = \quad \mathrm{hom}(\tilde{H}_n(X), \mathbb{Z}_p) \oplus \mathrm{Ext}\ (H_{n-1}(X), \mathbb{Z}_p).$$

Therefore we have

$$\mathrm{hom}(\tilde{H}_n(X), \mathbb{Q}) = 0 = \mathrm{Ext}\ (H_{n-1}(X), \mathbb{Q}), \quad \mathrm{hom}(\tilde{H}_n(X), \mathbb{Z}_p) = 0 = \mathrm{Ext}\ (H_{n-1}(X), \mathbb{Z}_p)$$

for all n and for all prime p. Set $A_n = \widetilde{H}_n(X;\mathbb{Z})$.

First consider the case where A_n is finitely generated. Then we can write $A_n = F_n \oplus T_n$ with F_n free and T_n torsion by FTA. But then we have

$$0 = \mathrm{Hom}(A_n, \mathbb{Q}) \cong \mathrm{Hom}(F_n;\mathbb{Q}) \oplus \mathrm{Hom}(T_n;\mathbb{Q}) \cong \mathrm{Hom}(F_n;\mathbb{Q}).$$

Since we have $F_n = \mathbb{Z}^{d_n}$ for some integer $d_n \geq 0$ and so $\mathrm{Hom}(F_n;\mathbb{Q}) \cong \mathbb{Q}^{d_n}$, we find $d_n = 0$ and so $F_n = 0$.

On the other hand, we recall $F_n = \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_m^{r_m}}$ for some prime $p_i \geq 2$ and $r_i \geq 0$, where p_i are not necessarily different. By taking $\mathrm{Ext}(F_n;\mathbb{Z}_p)$ for various p's, we have

$$0 \cong \mathrm{Ext}(F_n;\mathbb{Z}_p) \cong \oplus_{i=1}^m \mathrm{Ext}(\mathbb{Z}_{p_i^{r_i}}, \mathbb{Z}_p) \cong \oplus_{i=1}^m \mathbb{Z}_p / p_i^{r_i} \mathbb{Z}_p.$$

(Here we recall $\mathrm{Ext}(\mathbb{Z}_m, G) \cong G/mG$.) In other words, we have

$$\mathbb{Z}_p = p_i^{r_i} \mathbb{Z}_p.$$

for all prime p and in particular, we have

$$p_i^{r_i}\mathbb{Z}_{p_i} = \mathbb{Z}_{p_i}.$$

But this is possible only if $r_i = 0$. Since this holds for all $i = 1, \cdots, m$, we have also proved $T_n = 0$ and hence $A_n = 0$ for all n. This finished proof.

4. *Show that* $\otimes$ *and Tor commute with direct limits:* $(\varinjlim A_\alpha) \otimes B = \varinjlim(A_\alpha \otimes B)$ *and* $Tor(\varinjlim A_\alpha, B) = \varinjlim Tor(A_\alpha, B)$.

Solution: First we show that the tensor product commutes with direct limits in the following way. Let $\{f_{\alpha\beta} : A_\alpha \to A_\beta\}$ be the given directed system and denote $A = \varinjlim A_\alpha$. Denote by $\nu_\alpha : A_\alpha \to A$ the canonical map induced by the directed system.

If we consider the groups $A_\alpha \otimes B$ this is a directed system with maps given by $g_{\alpha\beta} = f_{\alpha\beta} \otimes id$. Then we consider the maps

$$\nu_\alpha \otimes id : A_\alpha \otimes B \to A \otimes B$$

which are compatible with the directed system $\{g_{\alpha\beta} : A_\alpha \otimes B \to A_\beta \otimes B\}$ and hence induces a map

$$\nu : \varinjlim(A_\alpha \otimes B) \to A \otimes B.$$

On the other hand, we also have the canonical map

$$\mu_\alpha : A_\alpha \otimes B \to \varinjlim(A_\alpha \otimes B)$$

associated to the direct system $g_{\alpha\beta}$ such that

$$\mu_\alpha = \mu_\beta \circ g_{\alpha\beta}.$$

We define a map $\mu : A \otimes B \to \varinjlim(A_\alpha \otimes B)$ as follows. Let $a \otimes b \in A \otimes B$. By definition, we have $a = [a_\alpha]$ for some $a_\alpha \in A_\alpha$. Then we define

$$\mu(a \otimes b) = \mu_\alpha(a_\alpha \otimes b).$$

One checks that this is well-defined as μ_α satisfies the compatibility condition. We claim $\nu \circ \mu = id$. But

$$\nu \circ \mu(a \otimes b) = \nu \circ \mu([a_\alpha] \otimes b) = \nu(\mu_\alpha(a_\alpha \otimes b) = \nu([a_\alpha \otimes b]).$$

By the definition of the canonical map ν which is the limit of $\nu_\alpha \otimes id$, we have

$$\nu([a_\alpha \otimes b]) = [a_\alpha] \otimes b = a \otimes b.$$

This proves $\nu \circ \mu = id$. On the other hand, we have

$$\mu \circ \nu([a_\alpha \otimes b]) = \mu([a_\alpha] \otimes b) = \mu_\alpha(a_\alpha \otimes b) = [a_\alpha \otimes b]$$

where we use the definition of ν and μ above for the first and the second identity respectively. Hence $\mu \circ \nu = id$. This finishes the proof.

Now to show that Tor commutes with direct limits, we recall the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A, B_\alpha) \arrow[r] & F_1 \otimes B_\alpha \arrow[r, "\varphi"] & F_0 \otimes B_\alpha \arrow[r] & A \otimes B_\alpha \arrow[r] & 0 .
\end{tikzcd}$$

We also have the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A, B_\alpha) \arrow[r] \arrow[d, "\mathrm{Tor}(id, f_{\alpha\beta})"] & F_1 \otimes B_\alpha \arrow[r, "\varphi"] \arrow[d, "id \otimes f_{\alpha\beta}"] & F_0 \otimes B_\alpha \arrow[r] \arrow[d, "id \otimes f_{\alpha\beta}"] & A \otimes B_\alpha \arrow[r] \arrow[d, "id \otimes f_{\alpha\beta}"] & 0 . \\
0 \arrow[r] & \mathrm{Tor}(A, B_\beta) \arrow[r] & F_1 \otimes B_\beta \arrow[r] & F_0 \otimes B_\beta \arrow[r] & A \otimes B_\alpha \arrow[r] & 0
\end{tikzcd}$$

By taking the direct limit of the horizontal exact sequences, we obtain the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A, B_\alpha) \arrow[r] \arrow[d, "\mathrm{Tor}(id, \nu_\alpha)"] & F_1 \otimes B_\alpha \arrow[r, "\varphi"] \arrow[d, "\nu_{1,\alpha}"] & F_0 \otimes B_\alpha \arrow[r] \arrow[d, "\nu_{0,\alpha}"] & A \otimes B_\alpha \arrow[r] \arrow[d, "\nu_{A,\alpha}"] & 0 \\
0 \arrow[r] & \varinjlim \mathrm{Tor}(A, B_\beta) \arrow[r] & \varinjlim (F_1 \otimes B_\beta) \arrow[r] & \varinjlim (F_0 \otimes B_\beta) \arrow[r] & \varinjlim (A \otimes B_\beta) \arrow[r] & 0
\end{tikzcd}$$

where $\nu_{i,\alpha} : F_i \otimes B_\alpha \to \varinjlim (F_i \otimes B_\alpha)$ and $\nu_{A,\alpha} : A \otimes B_\alpha \to \varinjlim (A \otimes B_\alpha)$ are the canonical maps. From this, it follows that the sequence

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \mathrm{Tor}(A, B_\beta) \arrow[r] & \varinjlim (F_1 \otimes B_\beta) \arrow[r] & \varinjlim (F_0 \otimes B_\beta) \arrow[r] & \varinjlim (A \otimes B_\beta) \arrow[r] & 0
\end{tikzcd}$$

is also exact. One more time, by composing with the canonical vertical maps, the above commuting diagram also induces the commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}(A, B_\alpha) \arrow[r] \arrow[d, "\mathrm{Tor}(id, \nu_\alpha)"] & F_1 \otimes B_\alpha \arrow[r, "\varphi"] \arrow[d, "id \otimes \nu_\alpha"] & F_0 \otimes B_\alpha \arrow[r] \arrow[d, "id \otimes \nu_\alpha"] & A \otimes B_\alpha \arrow[r] \arrow[d, "id \otimes \nu_\alpha"] & 0 \\
0 \arrow[r] & \mathrm{Tor}(A, B) \arrow[r] & F_1 \otimes B \arrow[r] & F_0 \otimes B \arrow[r] & A \otimes B \arrow[r] & 0
\end{tikzcd}$$

where $\nu_\alpha : B_\alpha \to B$ is the canonical map. This in turn induces the diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \mathrm{Tor}(A, B_\beta) \arrow[r] \arrow[d, "\mu"] & \varinjlim (F_1 \otimes B_\beta) \arrow[r] \arrow[d, "\cong"] & \varinjlim (F_0 \otimes B_\beta) \arrow[r] \arrow[d, "\cong"] & \varinjlim (A \otimes B_\beta) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Tor}(A, B) \arrow[r] & F_1 \otimes B \arrow[r] & F_0 \otimes B \arrow[r] & A \otimes B \arrow[r] & 0
\end{tikzcd}$$

where $B = \varinjlim B_\beta$ and $\mu : \varinjlim \mathrm{Tor}(A, B_\alpha) \to \mathrm{Tor}(A, B)$ is the limit map of $\mathrm{Tor}(id, \nu_\alpha) : \mathrm{Tor}(A, B_\alpha) \to \mathrm{Tor}(A, B)$. Here the isomorphism property of the 2nd-4th vertical maps follows since we have proved direct limit commutes with tensoring. Now by the Five Lemma, it follows that μ is an isomorphism. In other words, we have proved

$$\varinjlim \mathrm{Tor}(A, B_\alpha) \cong \mathrm{Tor}(A, \varinjlim B_\alpha).$$

5. *From the fact that Tor$(A, B) = 0$ if A is free, deduce that Tor$(A, B) = 0$ if A is torsionfree by applying the previous problem to the directed system of finitely generated subgroups A_α of A.*

Solution: First note that $\varinjlim A_\alpha = A$ because $\varinjlim A_\alpha$ is defined to be the direct sum $\oplus_\alpha A_\alpha$ quotiented by the subgroup generated by all elements of the form $a - f_{\alpha\beta}(a)$ for $a \in A_\alpha$, that is if we have $x \in A_\alpha \subset A$ and $x \in A_\beta \subset A$, then we identify x with itself. In this way what remains is exactly A.

Now if we consider any finitely generated subgroup A_α of A, then we have that $\mathrm{Tor}(A_\alpha, B) = 0$ since A_α is free. Now since direct limits commute with Tor we have that

$$\begin{aligned} \varinjlim \mathrm{Tor}(A_\alpha, B) &= \mathrm{Tor}(\varinjlim A_\alpha, B) \\ \varinjlim 0 &= \mathrm{Tor}(A, B) \\ 0 &= \mathrm{Tor}(A, B) \end{aligned}$$

And so $\mathrm{Tor}(A, B) = 0$ if A is torsion free.

6. *Show that Tor(A, B) is always a torsion group, ant that Tor(A, B) contains an element of order n iff both A and B contains elements of order n.*

Solution: First we assume that B is finitely generated. Then we can decompose

$$B = F(B) \oplus T(B)$$

(noncanonically) into the free and the torsion parts by FTA. Therefore we have

$$\mathrm{Tor}(A, B) = \mathrm{Tor}(A, F(B)) \oplus \mathrm{Tor}(A, T(B)) = \mathrm{Tor}(A, T(B))$$

as $\mathrm{Tor}(A, F(B)) = 0$ since $F(B)$ is free. Therefore let us assume that B is a torsion group. Consider the free resolution

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "\vartheta"] & F_0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is a free resolution of A, then we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(A, B) \arrow[r] & F_1 \otimes B \arrow[r, "\varphi"] & F_0 \otimes B \arrow[r] & A \otimes B \arrow[r] & 0 \end{tikzcd}$$

Now let x be any element of the kernel of φ and let $x = \sum_{i=1}^m f_i \otimes b_i$ with $f_i \in F_1$ and $b_i \in B$. Then since B is a torsion group each $f_i \otimes b_i$ has finite order and so does x. Since $\mathrm{Tor}(A, B) \simeq \ker \varphi$ we also have that any element of $\mathrm{Tor}(A, B)$ has finite order and so $\mathrm{Tor}(A, B)$ is a torsion group.

Since any B an be written as the direct limit of a finitely generated subgroups and the Tor functor commutes with the direct limit, we have

$$\varinjlim \mathrm{Tor}(A, B_\alpha) = \mathrm{Tor}(A, B).$$

We know that each $\mathrm{Tor}(A, B_\alpha)$ is a torsion group. It is enough to prove
Claim: A direct limit of torsion groups is again a torsion group.
Proof of the claim: Let T be the limit of $(T_\alpha, f_{\alpha\beta})$ for torsion groups T_α. Denote by $f_\alpha : T_\alpha \to T$ the canonical map. Since any element $x \in T$ has the form $x = [x_\alpha] = f_\alpha(x_\alpha)$, x must have order less than that of x_α whose order we know is finite by the assumption. Therefore x has a finite order and hence T is a torsion group.

Now to see that $\mathrm{Tor}(A, B)$ has an element of order n if and only if both A and B do, first assume that $\mathrm{Tor}(A, B)$ has an element $x \in \ker \varphi \subset F_1 \otimes B$ of order n. Let $x = \sum_{i=1}^m f_i \otimes b_i$ for $f_i \in F_1$ and $b_i \in B$. By rearranging f_i's, we may assume that f_i's are linearly independent. We have

$$x = \sum_{i=1}^{m} f_i \otimes (nb_i) \cong \oplus nb_i$$

where the last congruence follows from the hypothesis that F_i is free and f_i's are linearly independent. Therefore it follows that

$$\mathrm{ord}(x) = \max_i \mathrm{ord}(b_i).$$

Therefore if $\mathrm{Tor}(A, B)$ has order n, b must have an element of order n. Since Tor is symmetric, the same conclusion holds for A.

Now assume that both A and B have elements of order n. Again consider the free resolution

$$0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{j} A \to 0.$$

Let $0 \neq a \in A$ and $0 \neq b \in B$ have order n. Since j is surjective, $j(f_0) = a$ for some $f_0 \in F_0$. Then $0 = na = nj(f_0) = j(nf_0)$ and so $nf_0 = i(f_1)$ for some $f_1 \in F_1$ by the exactness of the resolution. Now consider

$$0 \to \mathrm{Tor}(A, B) \xrightarrow{k} F_1 \otimes B \xrightarrow{i \otimes id} F_0 \otimes B \xrightarrow{j \otimes id} A \otimes B \to 0.$$

Since b has order n, we have

$$0 = f_0 \otimes nb == nf_0 \otimes b = i(f_1) \otimes b = (i \otimes id)(f_1 \otimes b).$$

Therefore by the exactness, there is some $t \in \mathrm{Tor}(A, B)$ such that

$$k(t) = f_1 \otimes b.$$

We have $f_1 \otimes b \neq 0$ since F_1 is free and $b \neq 0$. Since k is a homomorphism, $t \neq 0$. Finally since b has order n, we have $f_1 \otimes nb = 0$ and so $k(nt) = nk(t) = n(f_1 \otimes b) = f_1 \otimes nb = 0$. Therefore the injectivity of k implies $nt = 0$ and so t has order n. This finishes the proof.