# A HOFER-LIKE METRIC ON THE GROUP OF SYMPLECTIC DIFFEOMORPHISMS

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Abstract. Using a "Hodge decomposition" of symplectic isotopies on a compact symplectic manifold $(M,\omega)$ , we construct a norm on the identity component in the group of all symplectic diffeomorphisms of $(M,\omega)$ whose restriction to the group $Ham(M,\omega)$ of hamiltonian diffeomorphisms is bounded from above by the Hofer norm. Moreover, $Ham(M,\omega)$ is closed in $Symp(M,\omega)$ equipped with the topology induced by the extended norm. We give an application to the $C^0$ symplectic topology. We also discuss extensions of Oh's spectral distance.

## 1. Introduction and statement of the main results

Let $Symp(M,\omega)$ denote the group of all symplectic diffeomorphisms of a compact symplectic manifold $(M,\omega)$, endowed with the $C^\infty$ compact-open topology, and $Symp(M,\omega)_0 = G_\omega(M)$ the identity component in $Symp(M,\omega)$. $Symp(M,\omega)_0$ consists of symplectic diffeomorphisms $h$ such that there is a symplectic isotopy $h_t$ from the identity to $h$. By definition $h_t$ is a symplectic isotopy if the map $(x,t)\mapsto h_t(x)$ is smooth and for all t , $h_t^*\omega=\omega$. We denote by $Iso(M)$ the set of all symplectic isotopies, and by $Iso(\phi)$ the set of all symplectic isotopies from the identity to $\phi\in Symp(M,\omega)_0$.

Let $Ham(M,\omega)\subset Symp(M,\omega)_0$ be the subgroup of Hamiltonian diffeomorphisms. A diffeomorphism $\psi$ is Hamiltonian iff it is the time 1 map of a smooth family of diffeomorphisms $\psi_t$ such that if

$$\dot\psi_t(x)=\frac{d\psi_t}{dt}(\psi_t^{-1}(x)),\ \psi_0(x)=x \tag{1}$$

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there exists a smooth family of functions $u_t$ such that

$$i_{(\dot{\psi}_t)}\omega = du_t. \tag{2}$$

The family of diffeomorphisms $\psi_t$ above is called a hamiltonian isotopy.
We denote by $HIso(\phi)$ the set of all hamiltonian isotopies from $\phi \in Ham(M, \omega)$ to the identity, and by $HIso(M)$ the set of all hamiltonian isotopies.

In equation (2), $i_{(.)}$ denotes the interior product: $i_X\omega$ is the 1-form such that $i_X\omega(Y) = \omega(X, Y)$ . Recall that a symplectic form is a closed 2-form $\omega$ such that the map assigning to a vector field $X$ the 1-form $i_X\omega$ is an isomorphism $\tilde{\omega}$. For any 1-form $\alpha$, we denote by $\alpha^{\#}$ the vector field $\tilde{\omega}^{-1}(\alpha)$.

The *Hofer length* of a hamiltonian isotopy $\psi_t$ is defined as:

$$l_H(\psi_t) = \int_0^1 (max_x u_t(x) - min_x u_t(x))dt \tag{3}$$

One also denotes

$$max_x u_t(x) - min_x u_t(x)) = osc(u_t(x))$$

and call it the oscillation of $u_t$.

Hence the Hofer length is the mean oscillation of the hamiltonian $u_t$ of the hamiltonian isotopy $\Phi = (\phi_t)$.

For $\psi \in Ham(M, \omega)$, the *Hofer norm* is defined as:

$$||\psi||_H = inf(l_H(\psi_t)) \tag{4}$$

where the infimum is taken over all hamiltonian isotopies $\psi_t \in HIso(\psi)$ and $u_t$ is the function in equation (2).

The Hofer distance between two hamiltonian diffeomorphisms $\phi$ and $\psi$ is:

$$d_H(\phi, \psi) = ||\phi\psi^{-1}||_H$$

It is easy to see that the formula above defines a bi-invariant pseudo-metric but it is very challenging to show that it is not degenerate and hence it is a genuine distance [5],[7],[12], [13].

In this paper we propose a formula for the length of a symplectic isotopy $\Phi = (\phi_t)$ (5), which generalizes the length of a hamiltonian isotopy (3).

Fix a riemannian metric on $M$ and consider the Hodge decomposition of $i_{(\dot{\phi}_t)}\omega$

$$i_{(\dot{\phi}_t)}\omega = \mathcal{H}_t^\Phi + du_t^\Phi$$

where $\mathcal{H}_t^\Phi$ and $u_t^\Phi$ are smooth family of harmonic 1-forms and functions respectively.

We define the length $l(\Phi)$ of the isotopy $\Phi$ by:

$$l(\Phi) = \int_0^1 (|\mathcal{H}_t^\Phi| + (max_x(u_t^\Phi) - min_x(u_t^\Phi))dt \qquad (5)$$

Here $|\mathcal{H}_t^\Phi|$ is the "Euclidean" norm of the harmonic 1-form $\mathcal{H}_t^\Phi$ ( see (13), (14)).

This formula reduces to (3) for hamiltonian isotopies.

For any $\phi \in Symp(M,\omega)_0$, we define the energy $e_0(\phi)$ of $\phi$ as:

$$e_0(\phi) = inf_{\Phi\in Iso(\phi)}(l(\Phi))$$

Our main result is the following

**Theorem 1.**

*Let $(M,\omega)$ be a closed symplectic manifold. Consider the map $e : Symp(M,\omega)_0 \to \mathbb{R}\cup\{\infty\}$*

$$e(\phi) = 1/2(e_0(\phi) + e_0(\phi^{-1})).$$

*Then $e$ is a norm on $Symp(M,\omega)_0$ whose restriction to $Ham(M,\omega)$ is bounded from above by the Hofer metric.*

*Moreover the subgroup $Ham(M,\omega)$ is closed in $Symp(M,\omega)$ endowed with the metric topology defined by $e$.*

We define a distance on $Symp(M,\omega)$ by:

$$d(\phi,\psi) = e(\phi\psi^{-1})$$

This distance is obviously right invariant, but not left invariant.

**Remark**

The fact that (5) reduces to (3) when $\Phi$ is a hamiltonian isotopy implies that

$$e(\phi) \leq ||\phi||_H$$

for all $\phi \in Ham(M,\omega)$.

**Conjecture** The restriction of the norm $e$ to $Ham(M,\omega)$ is equivalent to the Hofer norm.

What is the interest of our construction? From the Hofer norm, there are easy ways of constructing bi-invariant norms on $Symp(M,\omega)$. One is given by Han [4]:

fix a positive number $K$ and define

$$||\phi||_K = \begin{cases} min(||\phi||_H, K), & \text{if } \phi \in Ham(M,\omega) \\ K & \text{otherwise.} \end{cases}$$

Another is given by Lalonde-Polterovich [8]:

fix a real number $\alpha$ and define

$$||\phi||_\alpha = sup\{||\phi f \phi^{-1} f^{-1}||_H | f \in Ham(M,\omega), ||f||_H \leq \alpha\}.$$

In both cases the restriction of these metrics back to $Ham(M,\omega)$ gives different topologies on $Ham(M,\omega)$. In particular $Ham(M,\omega)$ in these topology has always a finite diameter which is known to be untrue for the Hofer norm in several cases.

Hence the advantage of our construction is that its restriction to $Ham(M,\omega)$ gives a "better" topology, which may be the same if the conjecture is true.

Moreover the metric $e$ comes from a Finsler metric. This is useful for the definition of the *symplectic topology* on $Iso(M)$. We apply this to the $C^0$ symplectic topology by defining a group $SSympeo(M,\omega)$ of strong symplectic homeomorphisms, analoguous to the group $Hameo(M,\omega)$ of hamiltonian homeomorphisms [10].

## 2. Hamiltonian and harmonic diffeomorphisms

For each symplectic isotopy $\Phi = (\phi_t)$, consider the following 1-form:

$$\Sigma(\Phi) = \int_o^1 (i_{\dot{\phi}_t}\omega)dt \tag{6}$$

It is shown in [1], ( see also [2]) that the cohomology class $[\Sigma(\Phi] \in H^1(M,\mathbb{R})$ of the form $\Sigma(\Phi)$ depends only on the class $[\Phi]$ of $\Phi$ in the universal covering $\tilde{G}(M,\omega)$ of $Symp(M,\omega)_0 = G(M,\omega)$ and that the map $[\Phi] \mapsto [\Sigma(\Phi)]$ is a surjective homomorphism

$$\tilde{S} : \tilde{G}(M,\omega) \to H^1(M,\mathbb{R}) \tag{7}$$

The group

$$\Gamma = \tilde{S}(\pi_1(G(M,\omega))) \subset H^1(M,\mathbb{R})$$

is called the **flux group**.

In [1], it was observed that $\Gamma$ was discrete in several examples and the author wrote " I do not know any flux group which is not discrete". The statement that " $\Gamma$ is discrete" became known as the "Flux conjecture". This conjecture has been recently proved by Ono [11] using Floer-Novikov homology.

**Theorem (Ono).**

*Let $(M,\omega)$ be a compact symplectic manifold, then the flux group is discrete.*

Consider the induced homomorphism :

$$S : G(M,\omega) \to H^1(M,\mathbb{R})/\Gamma \tag{8}$$

In [1], [2], it is shown that the Kernel of S coincides with the group $Ham(M,\omega)$ of Hamiltonian diffeomorphisms, and it is a simple group, which coincides with the commutator subgroup $[G(M,\omega), G(M,\omega)]$ of $G(M,\omega)$. We summarize:

$$Ham(M,\omega) = KerS = [G(M,\omega), G(M,\omega)] \tag{9}$$

for all closed symplectic manifolds $(M,\omega)$.

We will need to represent in a unique way cohomology classes ; this is achieved by Hodge theory on compact riemannian manifolds.

The Hodge decomposition theorem ( see for instance [14]) asserts that any smooth family of p-forms $\theta_t$ on a compact oriented riemannian manifold $(M,g)$ can be decomposed in a unique way as

$$\theta_t = \mathcal{H}_t + d\alpha_t + \delta\beta_t \tag{10}$$

where $\mathcal{H}_t$ is harmonic, i.e $d\mathcal{H}_t = \delta\mathcal{H}_t = 0$ . Here $\delta$ denotes the codifferential.
If $d\theta_t = 0$ , then $\delta\beta_t = 0$ . The forms $\mathcal{H}_t$, $\alpha_t$ and $\beta_t$ depend smoothly on $t$.

The harmonic form $\mathcal{H}_t$ is a unique representative of the cohomology class $[\theta_t] \in H^1(M,\mathbb{R})$ of $\theta_t$.

**Definition 1.**

*Let $(M,\omega)$ be a compact symplectic manifold, equipped with some riemannian metric. A vector field $X$ on $M$ is said to be a harmonic vector field if $i_X\omega$ is a harmonic form.*

*A diffeomorphism $\phi$ of $M$ is said to be a harmonic diffeomorphism if if there exists a smooth family $\mathcal{H}_t$ of harmonic 1-forms such that $\phi$ is the time 1 map of the symplectic isotopy $\phi_t$ such that*

$$\dot{\phi}_t = (\mathcal{H}_t)^{\#}. \tag{11}$$

*We say that $\phi_t$ is a harmonic isotopy.*

Let $symp(M,\omega)$ be the set of symplectic vector fields, $harm(M)$ the set of harmonic vector fields and $ham(M,\omega)$ the space of hamiltonian vector fields. If

$X \in symp(M, \omega)$ then $i_X\omega$ is closed. The decomposition $i_X\omega = \mathcal{H} + du$ expresses $X$ as

$$X = H + X_u \tag{12}$$

where $H = (\mathcal{H})^{\#}$ is harmonic and $X_u$ is the hamiltonian vector field with hamiltonian $u$.

Hence $symp(M, \omega)$ is the Cartesian product of $harm(M)$ and $ham(M, \omega)$. We give $symp(M, \omega)$ the product metric :

$$|X| = |H| + max_x u(x) - min_x u(x) \tag{13}$$

where $|H|$ is the norm on $harm(M)$ given below:

the space $harm(M)$, which is isomorphic to the space of harmonic 1-forms is a finite dimensional vector space whose dimension is the first Betti number of $M$.

In this paper , we fix a basis $h_1, ..., h_r$ of harmonic 1-forms and consider $(H_i) = (h_i^{\#})$ the corresponding basis of $harm(M)$. We give these 2 vector spaces the following Euclidean metric : if $h = \sum_i \lambda_i h_i$, $H = \sum_i \lambda_i H_i$, then

$$|h| = |H| = \sum_i |\lambda_i| \tag{14}$$

In view of (13), the length formula (5) gives a Finsler metric on $Symp(M, \omega)$.

**Remark**

The function $u$ in the Hodge decomposition $i_X\omega = \mathcal{H} + du$ is not necessary normalized. However if in (13) $|X| = 0$, then $|H| = 0$, i.e $i_X\omega = du$ and $osc(u) = 0$ implies that $u$ is constant, and hence $du = 0$. Therefore $X = 0$.

**Lemma 1.**

*Any symplectic isotopy* $\Phi = (\phi_t)$ *on a compact symplectic manifold* $(M, \omega)$ *can be decomposed in a unique way as*

$$\phi_t = \rho_t . \psi_t$$

*where $\rho_t$ is a harmonic isotopy and $\psi_t$ is a hamiltonian isotopy. In particular, if $\phi_t$ is a hamiltonian isotopy, then $\phi_t = \psi_t$ and $\rho_t = id_M$.*

*Proof.*

By Hodge decomposition theorem $i_{(\dot{\phi}_t)}\omega$ can be decomposed in a unique way as

$$i_{(\dot{\phi}_t)}\omega = \mathcal{H}_t^\Phi + du_t^\Phi$$

where $\mathcal{H}_t^\Phi$ and $u_t^\Phi$ are smooth family of harmonic 1-forms and functions respectively. Let $\rho_t$ be the harmonic isotopy such that $\dot{\rho}_t = (\mathcal{H}_t)^\#$. Set now $\psi_t = (\rho_t)^{-1}.\phi_t$. From $\phi_t = \rho_t.\psi_t$ , we get:

$$\dot{\phi}_t = \dot{\rho}_t + (\rho_t)_*\dot{\psi}_t \tag{15}$$

Since $i_{(\dot{\phi}_t - \dot{\rho}_t)}\omega = du_t = i_{(X_{(u_t)})}\omega$ where $X_{u_t}$ is the hamiltonnian vector field of $u_t$, we see that

$$\dot{\phi}_t = \dot{\rho}_t + X_{u_t} = \dot{\rho}_t + (\rho_t)_*((\rho_t)^{-1})_*(X_{u_t})$$

Hence $\dot{\psi}_t = (\rho_t)^{-1})_*(X_{u_t}) = X_{(u_t \circ \rho_t)}$. This shows that $\psi_t$ is a hamiltonnian isotopy.

□

In formula (5), $\int_0^1 osc(u_t^\Phi)dt$ is nothing else than $l_H(\psi_t)$ and formula (5) can be written

$$l(\Phi) = \int_0^1 |i(\dot{\rho}_t)\omega|dt + l_H(\psi_t))dt. \tag{5'}$$

## 3. Prooof of theorem 1

Clearly , $e(\phi) \geq 0$ for all $\phi$ and by definition $e(\phi) = e(\phi^{-1})$.

To see that the triangular inequality holds, fix a small positive number $\epsilon \leq 1/8$ and a smooth increasing function $a : [0,1] \to [0,1]$ such that $a_{|[0,\epsilon)} = 0$ and a $a_{|(1-\epsilon),1]} = 1$ and let $\lambda(t) = a(2t)$ for $0 \leq t \leq 1/2$ and $\mu(t) = a(2t-1)$ for $1/2 \leq t \leq 1$.

If $\Phi \in Iso(\phi)$ and $\Psi \in Iso(\psi)$ , we get an isotopy $\Phi * \Psi = (\sigma_t) \in Iso(\phi\psi)$ defined as:

$$\sigma_t = \begin{cases} \phi_{\lambda(t)}, & \text{for } 0 \leq t \leq 1/2 \\ \phi_1 \psi_{\mu(t)}, & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Let $c(\Phi, \Psi)$ be the set of all isotopies from $\phi\psi$ to the identity obtained as above.

Clearly :

$$e_0(\phi\psi) \leq inf_{\mathcal{R}}(l(\mathcal{R}))$$

where $\mathcal{R} \in c(\Phi, \Psi)$.

Since

$$\dot{\sigma}_t = \begin{cases} \lambda' \dot{\phi}_{\lambda(t)}, & \text{for } 0 \leq t \leq 1/2 \\ \mu' \dot{\psi}_{\mu(t)}, & \text{for } 1/2 \leq y \leq 1, \end{cases}$$

we have:

$$i(\dot{\sigma}_t)\omega = \begin{cases} \lambda' \mathcal{H}^{\Phi}_{\lambda(t)} + d(\lambda' u^{\Phi}_{\lambda(t)}) & \text{for } 0 \leq t \leq 1/2 \\ \mu' \mathcal{H}^{\Phi'}_{\mu(t)} + d(\mu' u^{\Phi'}_{\mu(t)}), & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

Therefore

$$l(\Phi * \Psi) = \int_0^{1/2} (|\lambda' \mathcal{H}^{\Phi}_{\lambda(t)}| + osc(\lambda' u^{\Phi}_{\lambda(t)})|)dt + \int_{1/2}^{1} (|\mu' \mathcal{H}^{\Phi'}_{\mu(t)}| + osc(\mu' u^{\Phi'}_{\mu(t)})dt$$

By the change of variable formula , we get:

$$l(\Phi * \Psi) = l(\Phi) + l(\Psi)$$

Finally,

$$e_0(\phi\psi) \leq inf_{\mathcal{R}}(l(\mathcal{R})) \leq inf_{\Phi} l(\Phi) + inf_{\Psi} l(\Psi) = e_0(\phi) + e_0(\psi).$$

Therefore the triangular inequality holds true for $e_0$, and hence for $e$ as well.

Showing that $e$ is non-degenerate is more delicate. Suppose that $e_0(\phi) = 0$.

**Step 1**

The statement $e_0(\phi) = inf(l(\Phi)) = 0$ means that for every $N$, there exists an isotopy $\Phi^N$ from $\phi$ to the identity such that $l(\Phi^N) \leq 1/N$.

Thus:

$$\int_0^1 |\mathcal{H}_t^{\Phi^N}|dt \leq 1/N \tag{16}$$

and

$$\int_0^1 osc(u^{\Phi^N})dt \leq 1/N$$

Hence

$$|\mathcal{H}(\Phi^N)| = |\int_0^1 \mathcal{H}_t^{\Phi^N} dt| \leq \int_0^1 |\mathcal{H}_t^{\Phi^N}|dt \leq 1/N$$

For any symplectic isotopy from $\phi$ to the identity $\Phi = (\phi_t)$, the 1-form

$$\mathcal{H}(\Phi) = \int_0^1 \mathcal{H}_t^{\Phi} dt$$

is the harmonic representative of the cohomology class $\tilde{S}([\phi_t])$.

For any symplectic isotopy $\Phi = (\phi_t)$ from $\phi$ to the identity

$$\mathcal{H}(\Phi^N) - \mathcal{H}(\Phi) = \gamma(\Phi) \in \Gamma \tag{17}$$

since $\mathcal{H}(\Phi^N) - \mathcal{H}(\Phi)$ is the harmonic representative of the image by $\tilde{S}$ of the class $[\phi_t^N * \phi_{(1-t)}]$ of the loop $\phi_t^N * \phi_{(1-t)}$.

By (16) and (17), the distance $d(\mathcal{H}(\Phi), \Gamma)$ from $\mathcal{H}(\Phi)$ to $\Gamma$ satisfies:

$$d(\mathcal{H}(\Phi), \Gamma) \leq |\mathcal{H}(\Phi) - (-\gamma(\Phi^N)| = |\mathcal{H}(\Phi^N)| \leq 1/N$$

This says that $(\mathcal{H}(\Phi))$ is arbitrarly close to $\Gamma$. Hence $(\mathcal{H}(\Phi)) \in \Gamma$. This means that $\phi \in KerS = Ham(M, \omega)$.

The facts that $\mathcal{H}(\Phi^N) \in \Gamma$ and $|\mathcal{H}(\Phi^N)| \leq 1/N$ imply that $\mathcal{H}(\Phi^N) = 0$ for $N$ large enough since $\Gamma$ is discrete (Ono's theorem).

Fix now an isotopy $\Phi^N$ such that $\mathcal{H}(\Phi^N) = 0$. To simplify the notations, we denote by $\Phi = (\phi_t)$ the isotopy $\Phi^N = (\phi_t^N)$.

The Hodge decomposition of the isotopy $\phi_t$ gives:

$$\phi_t = \rho_t \mu_t$$

where $\rho_t$ is harmonic and $\mu_t$ is hamiltonian. We have:

$$i(\dot{\phi}_t)\omega = \mathcal{H}_t + du_t$$

$$\dot{\rho}_t = (\mathcal{H}_t)^{\#} = H_t,$$

$$\int_0^1 \mathcal{H}_t dt = 0$$

and

$$\int_0^1 (|\mathcal{H}_t| + osc(u_t))dt \leq 1/N$$

Hence

$$\int_0^1 |\mathcal{H}_t| dt \leq 1/N; \int_0^1 osc(u_t)dt \leq 1/N \tag{18}$$

**Step 2**

We are now going to deform the isotopy $\rho_t$ fixing the extremities to a hamiltonian isotopy following [1], proposition II.3.1.

Let $Z_{(s,t)}$ be the family of symplectic vector fields:

$$Z_{(s,t)} = t\dot{\rho}_{(s.t)} - 2s(\int_0^t (i(\dot{\rho}_u)\omega)du))\# \tag{19}$$

Clearly, $Z_{(0,t)} = 0$ and we have:

$$\int_0^1 i(Z_{(s,t)})\omega ds = 0 \tag{20}$$

Let $G_{(s,t)}$ be the 2-parameter family of diffeomorphisms obtained by integrating $Z_{(s,t)}$ with $t$ fixed, i.e. $G_{(s,t)}$ is defined by the following equations:

$$\frac{d}{ds} G_{(s,t)}(x) = Z_{(s,t)}(G^{-1}_{(s,t)}(x)), G_{(0,t)}(x) = x \tag{21}$$

By (20), $G_{(1,t)}$ is a hamiltonnian diffeomorphism for all $t$. Since $Z_{(s,1)} = \dot{\rho}_s - 2s((\int_0^1 (i(\dot{\rho}_u)\omega)du) = \dot{\rho}_s$ , $s \mapsto G_{(s,1)}$ is an isotopy from the identity to $G(1,1) = \rho_1$. Hence the $g_t = G_{(1,t)}$ is an isotopy in $Ham(M, \omega)$ from $\rho_1$ to the identity.

Consider the 2-parameter family of vector fields $V_{(s,t)}$ defined by:

$$V_{(s,t)}(x) = \frac{d}{dt} G_{(s,t)}((G^{-1}_{(s,t)}(x))$$

Clearly $\dot{g}_t = V_{(1,t)}$.

We have ( see [1], proposition I.1.1):

$$\frac{\partial}{\partial s}V_{(s,t)} = \frac{\partial}{\partial t}Z_{(s,t)} + [V_{(s,t)}, Z_{(s,t)}] \tag{22}$$

We will need the following

**Proposition.**

$$i(V_{(1,t)})\omega = du_t$$

*where* $u_t = \int_0^1 \omega(Z_{(s,t)}, V_{(s,t)})ds$

*Proof.*

From equation 22

$$0 = \frac{\partial}{\partial t}[\int_0^1 i(Z_{(s,t)})\omega ds] = \int_0^1 i(\frac{\partial}{\partial t}(Z_{(s,t)}))\omega ds$$

$$= \int_0^1 i(\frac{\partial}{\partial s}(V_{(s,t)}))\omega ds - \int_0^1 i([Z_{(s,t)}, V_{(s,t)}])\omega ds = \int_0^1 (\frac{\partial}{\partial s} i(V_{(s,t)})\omega)ds - \int_0^1 i([Z_{(s,t)}, V_{(s,t)}])\omega ds$$

$$= i(V_{(1,t)})\omega - i(V_{(0,t)})\omega - \int_0^1 i([Z_{(s,t)}, V_{(s,t)}])\omega ds = i(V_{(1,t)})\omega - d(\int_0^1 \omega(Z_{(s,t)}, V_{(s,t)})ds)$$

We used the facts that $V_{(0,t)} = 0$, $i([Z, V]\omega = L_Z i_V \omega - i_V L_Z \omega$ and $L_V \omega = L_V \omega = 0$.

□

**Step 3: Norm estimates**

The harmonic vector fields $\dot{\rho}_t$ can be written as $\dot{\rho}_t = \sum_0^k \lambda_i(t) H_i$, where $H_i = h_i^{\#}$ and $(h_i)$ is a basis of harmonic 1-forms. Formula (19) just says:

$$Z_{(s,t)} = \sum_i (t\lambda_i(st) - 2s\int_0^t \lambda_i(u)du)H_i = \sum_i \mu_i(s,t)H_i \tag{23}$$

Hence:

$$|Z_{(s,t)}| = \sum_i |\mu_i(s,t)| \leq t|\dot{\rho}_{st}| + 2s\int_0^t |\mathcal{H}_t|dt$$

$$\leq t|\dot{\rho}_{st}| + 2s\int_0^1 |\mathcal{H}_t|dt \leq t|\dot{\rho}_{st}| + 2s/N.$$

On the other hand, we have:

$$\omega(Z_{(s,t)}, V_{(s,t)}) = (i_{(Z_{(s,t)}}\omega)(V_{(s,t)}) = \sum_i \mu_i(s,t)h_i(V_{(s,t)})$$

Consequently:

$$|\omega(Z_{(s,t)}, V_{(s,t)})| \leq \sum_i |\mu_i(s,t)h_i(V_{(s,t)})|$$

Let $||h_i||$ be the sup norm of the 1-forms $h_i$, i.e $||h_i|| = sup_{x\in M}|||h_i(x)|||$ and $|||h_i(x)|||$ is the norm of the linear map $h_i(x)$ on the tangent space $T_xM$.

We have:

$$\sum_i |\mu_i(s,t)h_i(V_{(s,t)})| \leq (\sum_i |\mu_i(s,t)|)|V_{(s,t)}|E = |Z_{(s,t)}||V_{(s,t)}|E$$

where $E = max\{||h_i||\}$.

Hence

$$|w_t| = |\int_0^1 \omega(Z_{(s,t)}, V_{(s,t)})ds| \leq \int_0^1 |\omega(Z_{(s,t)}, V_{(s,t)})|ds$$
$$\leq E(\int_0^1 (t|\dot{\rho}_{st}| + 2s/N)|V_{s,t})|ds.$$

Let $A = sup_{s,t}|V_{s,t}|$, then

$$|w_t| \leq AE\int_0^1 (t|\dot{\rho}_{st}| + 2s/N)ds = AE(\int_0^t (|\dot{\rho}_u|du) + 1/N)$$
$$\leq AE(\int_0^1 (|\dot{\rho}_u|du + 1/N) \leq 2AE/N.$$

Therefore $osc(w_t) \leq 4AE/N$, hence the length of the isotopy $\rho_t$ is less or equal to $4AE/N$, and therefore the Hofer norm of $\rho$ : $||\rho||_H \leq 4AE/N$, where $\rho = \rho_1$.

**Step 4**

Let $\mathcal{M}$ denote the space of smooth maps $c : I = [0,1] \to W$, where $W$ is the space of symplectic vector fields on $(M,\omega)$ such that $c(0) = 0$ and $c(1)$ is a hamiltonian vector field with the Hofer norm

$$||c|| = \int_0^1 |c(t)|dt$$

Here $|c(t)|$ is the norm given by formulas 13 and 14.

On the space $\mathcal{M} \times I$ we define the distance $d(c,s),(c',s')) = ((||c-c'||^2 + (s-s')^2)^{1/2}$

Let $\mathcal{N}$ be the space of smooth functions $u : I \times I \to U$, where $U$ is the space of symplectic vector fields with the metric $||u|| = sup_{s,t}|u(s,t)|$.

The family of vector fields $V_{s,t}$ above is the image of $\dot{\rho}_t$ by the following map:

$$\mathcal{R} : \mathcal{M} \times I \to \mathcal{N}$$

where $\mathcal{R} = \partial_t \circ I_s \circ a_s$ with

$a_s : c(t) \mapsto tc(st) - 2s(\int_0^t i(c(u)\omega du)^{\#}$

$I_s : U_{s,t} \mapsto G_{s,t} : M \to M$ where the family of diffeomorphisms $G_{s,t}$ is obtained by integrating in $s$ like in formula 21.

and finally $\partial t : g_{s,t} \mapsto \partial/\partial t(g_{s,t})$ ( formula 22).

The mapping $\mathcal{R}$ is a smooth map since all its components are smooth, consequently it is Lipschitz. Therefore there is a constant $K$ such that $d(\mathcal{R}(\dot{\rho}_t, s),(0,0)) = sup_{s,t}|V_{s,t}| \leq K(||\dot{\rho}_t||^2 + s^2)^{1/2}$ (Observe that $\mathcal{R}(0,0) = 0$).

Therefore

$$A = sup_{s,t}|V_{s,t}| \leq K((1/N)^2 + s^2)^{1/2} \leq K((1/N)^2 + 1)^{1/2}$$

Finally, we get:

$$||\rho||_H \leq (4E(K((1/N)^2 + 1)^{1/2}))/N$$

Remember now that $\phi = \rho\mu$ and $||\mu||_H \leq 1/N$. Hence $||\phi||_H \leq ((4EK((1/N)^2 + 1)^{1/2}))) + 1))/N$ for all $N$. Hence $||\phi||_H = 0$ and consequently $\phi = id$. □

$Ham(M,\omega)$ **is closed in** $Symp(M,\omega)$

Let $(h_n) \in Ham(M,\omega)$ be a sequence converging to $g \in Symp(M,\omega)$. There exists $N_0$ such that for all $N \geq N_0$, there exists an isotopy $\Phi^N \in Iso(g^{-1}h_N)$ with length $l(\Phi^N) \leq 1/N$. By step 1, $g^{-1}h_N$ is hamiltonian for N large. Hence $g$ is also hamiltonian. □

## 4. Applications to the $C^0$ symplectic topology

In [10], Oh and Muller defined the group of symplectic homeomorphisms, $Sympeo(M, \omega)$ as the closure of the group $Symp(M, \omega)$ of $C^\infty$ symplectic diffeomorphisms of $(M, \omega)$ in the group $Homeo(M)$ of homeomorphisms of $M$ with the $C^0$ topology, and the group $Hameo(M, \omega)$ of hamiltonian homeomorphisms. The group $Sympeo(M\omega)$ has only $C^0$ topology induced from $Homeo(M)$, but $Hameo(M, \omega)$ has a more involved topology, called the *hamiltonian topology*, which combines the $C^0$ topology and the Hofer topology.

Using our construction, we define a *symplectic topology* on the space $Iso(M)$ of symplectic isotopies of $(M, \omega)$ as follows:
Fix a distance $d_0$ on $M$ (coming from some riemannian metric) and define the distance $\overline{d}$ on the space $Homeo(M)$ of homeomorphismes of $M$ as

$$\overline{d}(\phi, \psi) = max\{d(\phi, \psi), d(\phi^{-1}, \psi^{-1})$$

where

$$d(h, g) = max_x(d_0(h(x), g(x))$$

for all $h, g \in Homeo(M)$.

Then $(Homeo(M), \overline{d})$ is a complete metric space and its metric topology is just the $C^0$ topology. On the space $\mathcal{P}Homeo(M)$ of continuous paths $\lambda : [0,1] \to Homeo(M)$, we put the metric topology from the distance

$$\overline{d}(\lambda, \mu) = sup_{t\in[0,1]}\overline{d}(\lambda(t), \mu(t)).$$

We define the *symplectic distance* on $Iso(M)$ by:

$$d_{symp}(\Phi, \Psi) = \overline{d}(\Phi, \Psi) + l(\Phi.\Psi^{-1})$$

We call the *symplectic topology* on $Iso(M)$ the metric topology defined by the above distance. This is the equivalent of the "hamiltonian topology" of [10]

We now define a set (which we conjecture to be a group) $SSympeo(M,\omega)$ as follows: $h \in SSympeo(M)$ iff there exists a continuous path $\lambda : [0,1] \to Homeo(M)$ such that $\lambda(0) = id; \lambda(1) = h$ and a sequence $\phi_n^t$ of symplectic isotopies, which converge to $\lambda$ in the $C^0$ topology ( induced by the norm $\overline{d}$) and such that $l(\phi_j^t(\phi_j^t)^{-1})$ tends to zero when $i$ and $j$ go to infinity. Here the length $l(\Phi)$ of a symplectic isotopy $\Phi = (\phi_t)$ is defined by formula (5). This "group" will play a major role in the $C^0$ symplectic topology.

## 5. Final Remarks

The metric $e$ obtained here is not an " extension" of the Hofer metric since we do not know if $e(\phi) = ||\phi||_H$ when $\phi \in Ham(M,\omega)$. We only know that $e(\phi) \leq ||\phi||_H$. The problem of extending the Hofer norm was considered in [3]. Here we would like to make some remarks about the results of [3].

**Extension of Oh's spectral norm**. It is obvious that formulas of the extensions of the Hofer metric given in [3] give in fact extensions for any bi-invariant metric on $Ham(M,\omega)$. Theorem 2 in [3] uses only the properties of bi-invariance and not the Hofer norm. Then theorem 2 of [3] can be rephrased as

**Theorem 2.**

*Let $(M,\omega)$ be a symplectic manifold such that the homomorphism $S$ admits a continuous homomorphic right inverse, then any bi-invariant metric on $Ham(M,\omega)$ extends to a right invariant metric on $Symp(M,\omega)$.*

Under the hypothesis of the theorem above, the spectral norm $||.||_{\mathcal{O}}$ of Oh extends to all of $Symp(M,\omega)_0$. For the definition of Oh's spectral norm, we refer to [9]. An example where this hypothesis holds is $T^{2n}$ with its natural symplectic form.

**Theorem 3.**

*If $\Gamma = 0$, Oh's spectral distance extends to $Symp(M,\omega)_0$.*

*Proof.*

Let $\phi_i, i = 1, 2$ two symplectomorphisms and $\Phi_i = (\phi_t^i) \in Iso(\phi_i)$. The harmonic 1-forms $\mathcal{H}(\Phi_i)$ depend only on $\phi_i$. Let $\rho_i$ be the time one of the 1-parameter group generated by $\mathcal{H}(\Phi_i)^{\#}$, then $\psi_i = \phi_i \rho_i^{-1} \in Ham(M, \omega)$. We define the Oh distance $d_{\mathcal{O}}$ of $\phi_1$ and $\phi_2$ by:

$$d_{\mathcal{O}}(\phi_1, \phi_2) = |\mathcal{H}(\Phi_1) - \mathcal{H}(\Phi_2)| + ||\psi_1 \psi_2^{-1}||_{\mathcal{O}}.$$

The cases where $\Gamma = 0$ include oriented compact surfaces of genus bigger than one. More recently, Kedra, Kotschick and Morita [6] found a longer list of compact symplectic manifolds with vanishing flux group.

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