

# BIRTH-DEATH PROCESSES AND $q$ -CONTINUED FRACTIONS

TONY FENG, RACHEL KIRSCH, ELISE MCCALL, AND MATT WAGE

ABSTRACT. In the 1997 paper of Lenin, Parthasarathy, Schoutens, and Van Assche [15], the authors study a birth-death process related to the Rogers-Ramanujan continued fraction  $r(q)$ . We generalize their results to establish a correspondence between birth-death processes and a larger family of  $q$ -continued fractions. It turns out that many of these continued fractions, including  $r(q)$ , play important roles in number theory: specifically, in the theory of modular forms and  $q$ -series. We draw upon the number theoretic properties of modular forms to give identities between the transition probabilities of different birth-death processes.

## 1. INTRODUCTION

A *birth-death process* is a continuous-time Markov chain in which a population of size  $n$  can transition to either size  $n + 1$  or size  $n - 1$ . Birth-death processes are an important object of study in probability, stochastic processes, and queuing theory, as they can be used to model population growth, epidemics, and queue length [17]. In [15], Lenin et al. establish a correspondence between a particular birth-death model and the celebrated Rogers-Ramanujan continued fraction

$$r(q) := \frac{1}{1 + \frac{q}{1 + \frac{q^2}{\ddots}}} = 1 - q + q^2 - q^4 + \dots \quad (1.1)$$

The Rogers-Ramanujan continued fraction has deep number-theoretic and analytic properties. It is this correspondence between birth-death models and  $q$ -continued fractions that we examine more generally here.

For each state  $n$  in a birth-death process there is a birth rate  $\lambda_n$  and a death rate  $\mu_n$  (for a formal definition of birth-death processes, see §2.1). Let  $p_0(t)$  be the probability that the population has size 0 at time  $t$ , given that it has size 0 at time 0. Define

$$f_0(0) := \int_0^\infty p_0(t) dt. \quad (1.2)$$

As we shall see in §2.1, it turns out that  $f_0(0)$  has the following suggestive representation

$$f_0(0) = \frac{1}{\lambda_0 - \frac{\lambda_0 \mu_1}{\lambda_1 + \mu_1 - \frac{\lambda_1 \mu_2}{\ddots}}}. \quad (1.3)$$

Since we are interested in finding birth-death processes corresponding to certain  $q$ -continued fractions, we let  $\{a_n\} := \{a_n(q)\}$  and  $\{b_n\} := \{b_n(q)\}$  be sequences with all positive terms

for  $q \in (0, 1)$  and we define  $\{\lambda_n\} := \{\lambda_n(q)\}$ ,  $\{\mu_n\} := \{\mu_n(q)\}$  by  $\lambda_0 = b_0$  and

$$\begin{cases} \lambda_{n-1} \mu_n = a_n, & n \geq 1, \\ \lambda_n + \mu_n = b_n, & n \geq 1. \end{cases} \quad (1.4)$$

Notice that  $\{a_n\}$  (resp.  $\{b_n\}$ ) will be the numerators (resp. denominators) of the continued fraction (1.3). By (1.4) we have

$$\lambda_n = b_n - \frac{a_n}{\lambda_{n-1}}, \quad n \geq 1. \quad (1.5)$$

Call a birth-death model *good* if its birth rates and death rates are positive. Observe that if  $\lambda_n > 0$ , then also  $\mu_{n+1} > 0$  since  $\lambda_n \mu_{n+1} = a_{n+1} > 0$ . Therefore to show that a process is good it suffices to show that  $\lambda_n > 0$  for  $n \geq 0$ . In §2.4 we establish conditions for a sequence of recursively defined birth rates  $\{\lambda_n(q)\}$  to give rise to a good birth-death process. Specifically, with  $\{a_n\}$  and  $\{b_n\}$  as above we prove the following two theorems for  $q \in (0, 1)$ :

**Theorem 1.1.** *Suppose  $b_n$  is bounded above by some constant  $0 < k \in \mathbb{R}$  for  $q \in (0, 1)$ , and suppose that there is a constant  $C_q$ , depending on  $q$ , such that*

$$0 \leq \frac{d}{dq} \sqrt{a_n} < C_q \quad \text{and} \quad 0 \leq -\frac{d}{dq} b_n < C_q.$$

*Then there exists a  $q_1 \in [0, 1]$  such that  $\lambda_n > 0$  for all  $n$  if and only if  $q \leq q_1$ .*

*Remark.* The main result of [15] is the special case of Theorem 1.1 where  $b_n = 1$  and  $a_n = q^n$ .

**Theorem 1.2.** *For  $\alpha > 0$ , if  $\lambda_k \geq \frac{1}{\alpha} \sqrt{a_{k+1}}$  for some  $k \geq 0$  and  $b_n - \alpha \sqrt{a_n} \geq \frac{1}{\alpha} \sqrt{a_{n+1}}$  for all  $n \geq k$ , then for all  $n > k$*

$$\lambda_n \geq b_n - \alpha \sqrt{a_n} > 0.$$

A process is *recurrent* if it returns to some state with probability 1 and *transient* otherwise. It is *positive recurrent* if the expected time of return is finite. (For a more formal definition, see §2.5.) It turns out that recurrence of a birth-death process is related to the convergence of (1.3). In §2.5 we prove that a birth-death process satisfying the hypotheses of Theorem 1.1 is recurrent for  $q < q_1$  and transient for  $q = q_1$ , and we show that a birth-death process satisfying the hypotheses of Theorem 1.2 for  $q \in (0, 1)$  is recurrent on that range. We also examine the asymptotic behavior of the birth and death rates.

In §2.3 we analyze the continued fractions from a  $q$ -series perspective to compute formulas for polynomials associated with the birth-death processes. We then discuss a method of recovering transition probabilities such as  $p_0(t)$  from the corresponding continued fraction. These results end up requiring important constructions in the theory of  $q$ -series. For example, the following result expresses the convergents in terms of the Pochhammer symbol and the  $q$ -binomial coefficient (see §2.3).

**Corollary.** *If  $A_n(s)/B_n(s)$  be the  $n$ th convergent of the continued fraction*

$$\frac{1}{s+1 + \frac{aq}{s+1 + bq + \frac{aq^2}{s+1 + bq^2 \dots}}}, \quad (1.6)$$

then

$$\frac{A_n(s)}{B_n(s)} = \frac{q^{1/2} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(s+1)(-bq^{1/2}/(s+1); q)_j}{(-bq^{1/2}/(s+1); q)}}{\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b/(s+1); q)_j}{(-b/(s+1); q)}}. \quad (1.7)$$

The above equation allows us to approximate the transition probabilities of a birth-death process from the corresponding continued fraction via (1.3).

*Remark.* The Rogers-Ramanujan continued fraction (1.1) is the case  $a = 1, b = s = 0$  of the continued fraction (1.6).

In §4 we give several examples of famous number-theoretic  $q$ -continued fractions which correspond to good birth-death processes. One of them is

$$\frac{1}{1 - \frac{q^3}{1 + q^5 - \frac{q^5}{\ddots}}} \quad (1.8)$$

whose power series expansion is essentially the classical Jacobi theta function

$$\vartheta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2}. \quad (1.9)$$

Given a good birth-death model  $\mathcal{B}$ , denote by  $f_{\mathcal{B}}(q)$  the normalized Laplace transform evaluated at 0 associated with  $\mathcal{B}$  (in other words, just the  $f_0(0)$  from the (1.2) for the particular birth-death model  $\mathcal{B}$ ). In the last section we prove some surprising relations between the transition probabilities  $p_0(t)$  for different birth-death processes, including one for  $r(q)$ . For example, let the birth-death process corresponding to (1.8) be  $\mathcal{B}_3$ ; we will show that if  $y > 0$ ,

$$1 + 2e^{-2\pi y} f_{\mathcal{B}_3}(e^{-2\pi y}) = \sqrt{2/y} \left( 1 + e^{-\frac{\pi}{2y}} f_{\mathcal{B}_3} \left( e^{-\frac{\pi}{2y}} \right) \right). \quad (1.10)$$

The above equation is a consequence of the fact that  $\vartheta_3(q)$  essentially transforms to itself under the mapping  $\tau \rightarrow -1/\tau$  on the upper half of the complex plane, which is in turn a consequence of the theory of modular forms. (Recall that a *modular form* is a type of meromorphic function that satisfies a special transformation property. For a formal definition, see [14].)

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## 2. THEORY OF $q$ -CONTINUED FRACTIONS AND BIRTH-DEATH PROCESSES

Here we recall the formal definition of a birth-death process and important facts about orthogonal polynomials and  $q$ -continued fractions. We also prove Theorems 1.1 and 1.2 and study the asymptotic behavior of the birth-death processes.

**2.1. Birth-death processes.** Here we closely follow the treatment of the subject as presented in [17]. Given that a population has size  $m$  initially, let  $p_{m,n}(t)$  be the probability that the population is at size  $n$  after time  $t$ . By definition, a birth-death process has the *transition probabilities* for  $m \geq 0$

$$p_{m,m+1}(t) := \lambda_m t + o(t)$$

$$p_{m,m-1}(t) := \mu_m t + o(t),$$

where  $\lim_{t \rightarrow 0} o(t)/t = 0$ . We call  $\lambda_n$  the birth rates and  $\mu_n$  the death rates of this process. These probabilities  $p_{m,n}$  satisfy the *Kolmogorov forward equations for birth-death processes* [17]:

$$p'_{m,0}(t) = -\lambda_0 p_{m,0}(t) + \mu_1 p_{m,1}(t), \quad (2.1)$$

$$p'_{m,n}(t) = \lambda_{n-1} p_{m,n-1}(t) - (\lambda_n + \mu_n) p_{m,n}(t) + \mu_{n+1} p_{m,n+1}(t), \quad (2.2)$$

where the derivative is taken with respect to  $t$ . We are interested in the case where the initial population is zero, so we will denote  $p_n(t) := p_{0,n}(t)$ .

Following [11, p. 366], let  $f_n(s)$  be the normalized Laplace transform of  $p_n(t)$ :

$$f_n(s) := (-1)^n \left( \prod_{i=1}^n \mu_i \right) \int_0^\infty e^{-st} p_n(t) dt,$$

$$f_0(s) := \int_0^\infty e^{-st} p_0(t) dt.$$

Taking the normalized Laplace transforms of (2.1) and (2.2) yields the recurrence relations

$$\frac{1}{f_0(s)} = s + \lambda_0 + \frac{f_1(s)}{f_0(s)}, \quad (2.3)$$

$$\frac{f_n(s)}{f_{n-1}(s)} = \frac{-\lambda_{n-1} \mu_n}{s + \lambda_n + \mu_n + \frac{f_{n+1}(s)}{f_n(s)}}. \quad (2.4)$$

These recurrence relations yield the following continued fraction with  $a_n := \lambda_{n-1} \mu_n$  and  $b_n := \mu_n + \lambda_n$ :

$$f_0(s) = \frac{1}{s + b_0 -} \frac{a_1}{s + b_1 -} \frac{a_2}{s + b_2 +} \dots \quad (2.5)$$

(Henceforth we adopt the notation for continued fractions as in (2.5).) We will be examining  $f_0(0) = \int_0^\infty p_0(t) dt$ , which is well-defined for  $q < q_1$  (see §2.5).

**2.2. Orthogonal polynomials.** Central to the study of birth-death processes and continued fractions are orthogonal polynomials. A sequence of polynomials is called *orthogonal* if there exists a distribution function  $\psi$  such that

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) d\psi(x) = 0 \text{ for } m \neq n, \text{ and}$$

$$\int_{-\infty}^{\infty} P_n^2(x) d\psi(x) \neq 0,$$

By Favard's Theorem [6, p. 21], polynomials defined by the following recurrence relation are orthogonal:

$$P_{n+1}(x) = (x - b_n) P_n(x) - a_n P_{n-1}(x), \quad (2.6)$$

where  $P_{-1}(x) = 0$ ,  $P_0(x) = 1$ ,  $b_n \in \mathbb{R}$  and  $a_n > 0$ .

In [13], Karlin and McGregor exhibit deep connections between birth-death processes and orthogonal polynomials, using the polynomials to compute approximations of the transition probabilities. Orthogonal polynomials will play a key role in our methods as well, as they have many useful and interesting properties. We now list some properties of orthogonal polynomials that will be used in later sections.

- (F1) The zeros of  $P_n$  are real and distinct. The zeros of  $P_n$  and  $P_{n+1}$  interlace. [6, p. 28]  
 (F2) The *spectrum* of  $\psi$ , defined by

$$S := \{x : \psi(x + \delta) - \psi(x - \delta) > 0 \text{ for all } \delta > 0\}$$

is countably infinite, and for each  $s \in S$ , every neighborhood of  $s$  contains a zero of infinitely many polynomials  $P_n(x)$ . [6, p. 51, p. 60]

- (F3) The orthonormal polynomials corresponding to  $P_n(x)$  are [6, p. 23]

$$p_n(x) := \frac{P_n(x)}{(a_1 a_2 \cdots a_n)^{1/2}}.$$

- (F4) If  $s \in S$ , then [6, p. 63]

$$0 < \psi(s+) - \psi(s-) \leq \left( \sum_{k=0}^{\infty} p_k^2(s) \right)^{-1}.$$

**2.3. Explicit formulas through generating functions.** In this section we analyze generating functions for sequences of orthogonal polynomials associated with birth-death probability models. We derive explicit formulas for the orthogonal polynomials (and consequently the birth and death rates) and the convergents of their corresponding continued fractions, which allows us to approximate the normalized Laplace transforms of the transition probabilities.

Let  $\{\lambda_n\}$  be a sequence of real numbers satisfying  $\lambda_0 = 1$  and  $\lambda_{n-1}(1 + bq^n - \lambda_n) = aq^n$  for  $n \geq 1$ . Define  $R_n$  by  $R_0 = 1$  and  $\lambda_n = R_{n+1}/R_n$ . Then we have

$$R_{n+1} = (1 + bq^n)R_n - aq^n R_{n-1}.$$

Let  $\{P_n(x)\}$  be a sequence of polynomials satisfying the conditions  $P_{-1}(x) = 0$ ,  $P_0(x) = 1$ , and

$$P_{n+1}(x) = (x + bq^n)P_n(x) - aq^n P_{n-1}(x) \quad (2.7)$$

so that  $P_n(1) = R_n$ . Observe that the  $P_n(x)$  are orthogonal polynomials with respect to some distribution  $\psi$  (see (2.6)).

**Theorem 2.1.** *With notation as above,*

$$\frac{1}{x^n} P_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2} / x^{2j} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b/x; q)_{n-j}}{(-b/x; q)_j}. \quad (2.8)$$

Above and throughout  $(\alpha; q)_n = \prod_{i=0}^{n-1} (1 - \alpha q^i)$  is the  $q$ -Pochhammer symbol and

$$\begin{bmatrix} a \\ b \end{bmatrix} = (q; q)_a / ((q; q)_b (q; q)_{b-a})$$

is the  $q$ -binomial coefficient.

*Proof.* We follow the approach of Ismail in [9]. First we construct the generating function  $F(z) := \sum_{n=0}^{\infty} z^n P_n(x)$ . Multiplying the recurrence relation (2.7) by  $z^{n+1}$  and summing from  $n = 0$  to  $\infty$  yields

$$F(z) = \frac{1}{1 - xz} (1 + bz(1 - azq/b)F(qz)).$$

Iterating, we have

$$F(z) = \sum_{k=0}^{\infty} z^k b^k q^{k(k-1)/2} \frac{(azq/b; q)_k}{(xz; q)_{k+1}}. \quad (2.9)$$

Now we apply Gauss's  $q$ -binomial formula (see [12, p. 15])

$$(z; q)_k = \sum_{j=0}^k \frac{(q; q)_k}{(q; q)_j (q; q)_{k-j}} (-z)^j q^{j(j+1)/2} \quad (2.10)$$

and Heine's  $q$ -binomial formula [5, p. 8]

$$\frac{(\alpha z; q)_{\infty}}{(z; q)_{\infty}} = \sum_{i=0}^{\infty} \frac{(\alpha; q)_i}{(q; q)_i} z^i \quad (2.11)$$

to obtain

$$F(z) = \sum_{k=0}^{\infty} z^k b^k q^{k(k-1)/2} \sum_{i=0}^{\infty} \frac{(q^{k+1}; q)_i}{(q; q)_i} (xz)^i \sum_{j=0}^k \frac{(q; q)_k (-az/b)^j}{(q; q)_j (q; q)_{k-j}} q^{j(j+1)/2}.$$

Collecting terms, switching the order of summation (see [19]), and substituting  $n = i + j + k$ , we have

$$F(z) = \sum_{n=0}^{\infty} z^n \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j q^{j(j+1)/2} a^j b^{-j} x^{n-j} \frac{(q; q)_{n-j}}{(q; q)_j} \sum_{k=j}^{n-j} x^{-k} b^k \frac{q^{k(k-1)/2}}{(q; q)_{k-j} (q; q)_{n-j-k}}.$$

Now equating coefficients of  $z^n$  and rewriting  $(\alpha; q)_{n-j-k}$  as  $\frac{(\alpha; q)_{n-j} q^{k(k+1)/2} q^{-k(n-j)}}{(q^{k+j-n}/\alpha; q)_k (-\alpha)^k}$  yields

$$P_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j q^{j(j+1)/2} a^j b^{-j} x^{n-j}}{(q; q)_j} \sum_{k=j}^{n-j} x^{-k} b^k q^{k(n-j)} \frac{(q^{j-n}; q)_k}{(q; q)_{k-j}}.$$

Next, set  $m = k - j$  and apply Heine's binomial formula (2.11) to obtain

$$P_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j q^{n-j(j-1)/2} a^j x^{n-2j} \frac{(q^{j-n}; q)_j (-q^j b/x; q)_{\infty}}{(q; q)_j (-q^{n-j} b/x; q)_{\infty}}.$$

Equation (2.8) then follows after straightforward manipulations.  $\square$

**Corollary 2.2.** *With notation as above, we have*

$$R_n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b; q)_{n-j}}{(-b; q)_j}. \quad (2.12)$$

*Remark.* This affords us an explicit formula for  $\lambda_n$ .

**Corollary 2.3.** *If*

$$f(z) = (-bz; q)_\infty \sum_{j=0}^{\infty} \frac{(-1)^j a^j q^{j^2} z^{2j}}{(q; q)_j (-bz; q)_j}, \quad (2.13)$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{x^n} P_n(x) = f(1/x). \quad (2.14)$$

Furthermore, let the positive elements of the support of the distribution function  $\psi$  with respect to which the polynomials  $P_n(x)$  are orthogonal be  $\{x_1, x_2, \dots\}$  where  $x_1 > x_2 > x_3 > \dots > 0$  and let the zeros of (2.13) be  $0 < z_1 < z_2 < z_3 \dots$ . Then we have  $z_k = 1/x_k$ .

*Proof.* Equation (2.14) follows immediately by letting  $n \rightarrow \infty$  in (2.8) and setting  $z = 1/x$ . Let the zeros of  $P_n(x)$  be  $x_{n,k}$ ,  $k = 1, 2, \dots, n$  where  $x_{n,1} > x_{n,2} > x_{n,3} \dots x_{n,n}$ . It is a well known property of orthogonal polynomials (see §2.2 (F2)) that  $\lim_{n \rightarrow \infty} x_{n,k} = x_k$ . By (2.14),  $\lim_{n \rightarrow \infty} f(1/x_{n,k}) = f(1/x_k) = 0$ .  $\square$

*Remark.* If we set  $b = 0, a = 1$ , then the polynomials are given by  $P_0 = 1, P_{-1} = 0$ , and  $P_{n+1}(x) = xP_n(x) - q^n P_{n-1}(x), n \geq 1$  and we recover (2.3) in [15]:  $F(z; q) = \sum_{k=0}^{\infty} (-1)^k z^{2k} q^{k^2} / (q; q)_k$ .

**Corollary 2.4.** *Let  $A_n(s)/B_n(s)$  be the  $n$ th convergent of the continued fraction*

$$\frac{1}{s+1} + \frac{aq}{s+1+aq} + \frac{aq^2}{s+1+aq^2} + \dots \quad (2.15)$$

Then

$$\frac{A_n(s)}{B_n(s)} = \frac{q^{1/2} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2+j} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(s+1)(-bq^{1/2}/(s+1); q)_j}{(-bq^{1/2}/(s+1); q)_j}}{\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j a^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} \frac{(-b/(s+1); q)_j}{(-b/(s+1); q)_j}}. \quad (2.16)$$

Moreover, as  $n \rightarrow \infty$  we find

$$\frac{1}{s+1} + \frac{aq}{s+1+aq} + \frac{aq^2}{s+1+aq^2} + \dots = \frac{(-b/(s+1); q)_\infty q^{1/2} \sum_{j=0}^{\infty} \frac{a^j q^{j^2+j}}{(q; q)_j (-bq^{1/2}/(s+1); q)_j}}{(s+1)(-bq^{1/2}/(s+1); q)_\infty \sum_{j=0}^{\infty} \frac{a^j q^{j^2}}{(q; q)_j (-b/(s+1); q)_j}}$$

*Remark.* Ramanujan mentions the case  $s = 0$  of (2.15) in his first letter to Hardy [3, p. 22].

*Proof.* We have the recursive relations for convergents of a continued fraction

$$\begin{cases} A_0(s) = 0 \\ A_1(s) = 1 \\ A_{n+1}(s) = (s+1+b^n)A_n(s) - aq^n A_{n-1}(s), \quad n \geq 1 \end{cases}$$

and

$$\begin{cases} B_{-1}(s) = 0 \\ B_0(s) = 1 \\ B_{n+1}(s) = (s+1+b^n)A_n(x) - aq^n B_{n-1}(s), \quad n \geq 0. \end{cases}$$

Observe  $A_n(s)$  and  $B_n(s)$  satisfy similar recurrences to those for  $P_n(s)$ . In fact, it is straightforward to verify that  $B_n(s) = P_n(s + 1)$  and  $A_n(s) = q^{(n-1)/2} P_{n-1}((s + 1)/q^{1/2})$ . The corollary follows by applying (2.8).  $\square$

As we saw in §2.1, the fraction in (2.15) is the normalized Laplace transform of  $p_0(t)$  when  $\lambda_n + \mu_n = bq^n$  and  $\lambda_{n-1}\mu_n = aq^n$ . We now outline a well-known method (see [15] for details) to compute the inverse Laplace transform of the convergents of the fraction, which allows us to approximate the transition probabilities  $p_m(t)$ . We know that  $A_n(s)/B_n(s)$  is the  $n$ th convergent of (2.15). Since the polynomials  $B_n(s)$  are orthogonal, their zeros are real and distinct. Using (2.16), we can then write  $A_n(s)/B_n(s)$  as a sum of partial fractions and easily find the inverse Laplace transform using the formula

$$\mathcal{L}^{-1}\left(\frac{A_n(s)}{B_n(s)}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{st} \frac{A_n(s)}{B_n(s)} d\omega \quad (2.17)$$

where  $s = c + i\omega$ ,  $c > 0$  ([11, p. 369]).

**2.4. Conditions for good birth-death processes.** In this section, we prove Theorems 1.1 and 1.2. Recall that we establish these theorems to give conditions under which a  $q$ -continued fraction gives rise to a good birth-death process.

*Proof of Theorem 1.1.* Since  $b_n$  is bounded above by  $k$ , we may write  $b_n = k - \beta_n$  where  $0 \leq \beta_n < k$ . Define  $R_n$  by  $R_0 = 1$  and

$$\frac{R_{n+1}}{R_n} = \lambda_n, \quad n \geq 1.$$

By (1.5), we know  $R_{n+1} = (k - \beta_n)R_n - a_n R_{n-1}$ . Next, define the polynomials  $P_n(x) = P_n(x, q)$  by  $P_{-1}(x) = 0$ ,  $P_0(x) = 1$ , and

$$P_{n+1}(x) = (x - \beta_n)P_n(x) - a_n P_{n-1}(x), \quad n \geq 0. \quad (2.18)$$

Notice that  $P_n(k) = R_n$ . We will study the zeros of the polynomials  $P_n(x)$  in order to determine the signs of  $P_n(k)$  and hence the signs of  $\lambda_n$ . By Favard's Theorem (2.6), the polynomials  $P_i(x)$  are orthogonal with respect to some distribution  $\psi$ , and therefore all the roots of  $P_n(x)$  are distinct and real (see §2.2 (F1)). Define  $x_n = x_n(q)$  to be the largest zero of  $P_n(x)$ . Also, define

$$X(q) := \lim_{n \rightarrow \infty} x_n(q).$$

By orthogonality, all roots of the polynomials  $P_i(x)$  are bounded and the zeros of  $P_i(x)$  and  $P_{i+1}(x)$  interlace (see §2.2 (F1)). Therefore,  $X$  exists and is in fact an element of the spectrum of  $\psi$  as it is a limit point of  $x_n$  (see §2.2 (F2)).

Next, notice that the zeros of  $P_n$  are the eigenvalues of the tridiagonal matrix

$$M_n(q) := \begin{pmatrix} \beta_0 & \sqrt{a_1} & 0 & \cdots & 0 \\ \sqrt{a_1} & \beta_1 & \sqrt{a_2} & \ddots & \vdots \\ 0 & \sqrt{a_2} & \beta_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sqrt{a_{n-1}} \\ 0 & \cdots & 0 & \sqrt{a_{n-1}} & \beta_{n-1} \end{pmatrix}.$$

Let  $v_n = (u_1, u_2, \dots, u_n)$  be an eigenvector corresponding to the largest eigenvalue  $x_n$ . By the Hellmann-Feynman Theorem [10],

$$x'_n(q) = \frac{\langle M'_n(q)v_n, v_n \rangle}{\langle v_n, v_n \rangle}.$$

By the Perron-Frobenius Theorem for matrices with non-negative entries [4],  $u_i \geq 0$  for all  $i$ . This implies that  $x'_n(q) \geq 0$ , so  $X(q)$  is a non-decreasing function.

Now, we will show that  $X(q)$  is a continuous function for  $q \in (0, 1)$ . Fix a  $q_0 \in (0, 1)$ . By assumption, there exists a  $C = C_q$  such that  $\frac{d}{dq}\sqrt{a_n(q_0)} < C$  and  $\frac{d}{dq}\beta_n(q_0) = -\frac{d}{dq}b_n(q_0) < C$  for all  $n$ . Hence,

$$\begin{aligned} x'_n(q_0) &= \frac{\langle M'_n(q_0)v_n, v_n \rangle}{\langle v_n, v_n \rangle} \\ &\leq \frac{C(u_1^2 + 2u_1u_2 + u_2^2 + 2u_2u_3 + \dots + 2u_{n-1}u_n + u_n^2)}{u_1^2 + u_2^2 + \dots + u_n^2} \\ &\leq 3C. \end{aligned}$$

Using that  $x'_n(q_0) \leq 3C$ , it is not difficult to show that  $X$  is continuous at  $q_0$ .

Next we will prove that  $X(q) \leq k$  if and only if  $\lambda_n > 0$  for all  $n$ . First, if  $X(q) \leq k$ , then  $P_n(k) > 0$  for all  $n$ , so  $\lambda_n > 0$  for all  $n$ . Next, assume that  $X(q) > k$ . If  $P_n(k) = 0$  for some  $n$ , then  $\lambda_n$  does not exist for all  $n$ . Otherwise  $P_n(k) \neq 0$ , so either  $x_1(q) < k$  or  $x_1(q) > k$ . If  $x_1 > k$ , then  $P_1(k) < 0$ , so  $\lambda_0 < 0$ . If  $x_1 < k$ , then there exists an  $n$  such that  $x_n(q) < k$  and  $x_{n+1}(q) > k$ , so  $P_{n+1}(k) < 0$  and therefore  $\lambda_n < 0$ .

In the cases in which  $X(q) < k$  for all  $q \in (0, 1)$  or  $X(q) > k$  for all  $q \in (0, 1)$ , define  $q_1 := 1$  or  $q_1 := 0$ , respectively. Otherwise, define  $q_1$  to be the largest  $q$  such that  $X(q) = k$ . By the previous paragraph and the properties of  $X(q)$ , these  $q_1$  satisfy the theorem.  $\square$

*Proof of Theorem 1.2.* We will induct on  $n$ . By assumption, we know that

$$\lambda_k \geq \frac{1}{\alpha}\sqrt{a_{k+1}}.$$

Therefore

$$\lambda_{k+1} = b_{k+1} - \frac{a_{k+1}}{\lambda_k} \geq b_{k+1} - \alpha\sqrt{a_{k+1}}.$$

This establishes the base case. For the inductive hypothesis, suppose that

$$\lambda_{n-1} \geq b_{n-1} - \sqrt{a_{n-1}}$$

for  $n - 1 > k$ . Then we have

$$\lambda_n = b_n - \frac{a_n}{\lambda_{n-1}} \geq b_n - \frac{a_n}{b_{n-1} - \alpha\sqrt{a_{n-1}}} \geq b_n - \alpha\frac{a_n}{\sqrt{a_n}} = b_n - \alpha\sqrt{a_n}.$$

By assumption,  $b_n - \alpha\sqrt{a_n} \geq \sqrt{a_{n+1}}/\alpha$ , which is greater than zero since  $\alpha, a_n > 0$ .  $\square$

**2.5. Asymptotic behavior of birth-death processes.** We will now examine the limiting behavior of the birth-death processes to determine whether they will be *recurrent* or *transient*. Let us first define these terms and others used in the statements of the theorems which follow.

**Definition 2.5** ([2], p. 155, p.158). *A process is recurrent if*

$$\int_0^\infty p_n(t)dt = \infty.$$

*A transient process is a process that is not recurrent. Moreover, a recurrent process is positive recurrent if for each state  $n$ ,  $\lim_{t \rightarrow \infty} p_n(t) > 0$  and null recurrent if for each state  $n$ ,  $\lim_{t \rightarrow \infty} p_n(t) = 0$ .*

The *potential coefficients* are defined following the notation in [15]:

$$\pi_0 := 1, \tag{2.19}$$

$$\pi_n := \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \frac{\lambda_0^2 \lambda_1^2 \cdots \lambda_{n-1}^2}{a_1 a_2 \cdots a_n} = \frac{P_n^2(k)}{a_1 a_2 \cdots a_n}, \tag{2.20}$$

where  $k = 1$  if the continued fraction is of the second type.

As in [15], we will use Anderson's notation [2] to study the limiting behavior of the birth-death processes. Define the quantities

$$A := \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n}, \quad B := \sum_{n=0}^{\infty} \pi_n.$$

The process corresponding to these potential coefficients is recurrent if  $A = \infty$ , and is positive recurrent if  $A = \infty$  and  $B < \infty$ .

The following theorems will prove that for  $q < q_1$  the birth-death processes are transient, and for  $q = q_1 \in (0, 1)$  the processes are positive recurrent. Notice that this means that  $f_0(0)$  is well-defined because the integral  $f_0(0) = \int_0^\infty p_0(t)dt$  is finite.

**Theorem 2.6.** *Let  $a_n$ ,  $\beta_n$ , and  $k$  be defined as in Theorem 1.1. Suppose for  $q \in (0, q_1)$  that  $0 < a_n \leq 1$ . Moreover, suppose there exists some  $N \in \mathbb{N}$  such that  $\sup\{a_n\}_{n=N}^\infty = c$  where  $0 < c < 1$  and  $c^{\frac{1}{3}} \leq k - X(q)$ . Then the birth-death process is transient. If we also have  $\lim_{n \rightarrow \infty} a_n = 0$  and  $\lim_{n \rightarrow \infty} \beta_n = \beta$ , then*

$$\lim_{n \rightarrow \infty} \lambda_n = k - \beta \text{ and } \lim_{n \rightarrow \infty} \mu_n = 0.$$

*Proof.* Notice  $P_n(k - x_{n,2}) \cdots (k - x_{n,n})$ , where the  $x_{n,j}$  are the zeros of  $P_n(x)$  such that  $x_{n,i} < x_{n,j}$  for all  $i < j$ . Since the polynomials  $P_n(x)$  are orthogonal polynomials, the zeros interlace (see §2.2 (F1)) and  $(k - x_{n,j}) < (k - x_{n+1,j})$ . Thus we have

$$P_{n+1}(k) > (k - x_{n+1,n+1})P_n(k) > (k - X(q))P_n(k) \geq c^{\frac{1}{3}}P_n(k). \tag{2.21}$$

Recall  $X(q) = \lim_{n \rightarrow \infty} x_n(q)$  where  $x_n$  is the largest zero of  $P_n$ . Notice  $X(q)$  is also the largest element in the spectrum of  $\psi$ , the distribution with respect to which the polynomials  $P_n$  are orthogonal (see §2.2 (F2)). Equation (2.21) implies

$$P_n(k) \geq (c^{\frac{1}{3}})^n \tag{2.22}$$

because  $P_0(x) := 1$ .

We will now show that  $B$  diverges. With the above inequality and the fact that  $a_n \leq c$  for  $n \geq N$ , we have

$$B = 1 + \sum_{n=1}^{\infty} \frac{P_n^2(k)}{a_1 a_2 \cdots a_n} \geq 1 + \sum_{n=1}^{\infty} \frac{(c^{\frac{1}{3}})^{2n}}{a_1 a_2 \cdots a_n} \geq 1 + \sum_{n=1}^{N-1} c^{\frac{2n}{3}} + c^{N-1} \sum_{n=N}^{\infty} (c^{-\frac{1}{3}})^n$$

which diverges because  $c^{-\frac{1}{3}} > 1$ . Thus  $B = \infty$  and the process is not positive recurrent.

Next we will show that the process is not null-recurrent because  $A := \sum_{n=0}^{\infty} 1/(\lambda_n \pi_n) < \infty$ . From the same definition of  $\pi_n$  (2.20), and again using the fact that  $a_n < c$  for all  $n \geq N$  and  $P_n(k) \geq (c^{\frac{1}{3}})^n$ , we arrive at the following inequality:

$$A - \frac{1}{k - \beta_0} \leq \sum_{n=1}^{\infty} \frac{a_1 a_2 \cdots a_n}{P_n(k) P_{n+1}(k)} \leq \sum_{n=1}^{N-1} \frac{1}{(c^{\frac{1}{3}})^{2n+1}} + \sum_{n=N}^{\infty} \frac{c^{n-N+1}}{(c^{\frac{1}{3}})^{2n+1}} = \frac{c^{\frac{2}{3}}(1 - c^{\frac{2N}{3}})}{c^{\frac{2N}{3}}(c^{\frac{1}{3}} - c)} + \frac{c^{\frac{2}{3}(1-N)}}{1 - c^{\frac{1}{3}}}.$$

Since this is finite,  $A$  is finite and the process therefore is not null-recurrent. Thus the process is transient.

We will now show that if  $a_n \rightarrow 0$  and  $\beta_n \rightarrow \beta$ , then the  $\mu_n \rightarrow 0$  and  $\lambda_n \rightarrow k - \beta$ .

Because the zeros of  $P_{n-1}(x)$  and  $P_n(x)$  interlace, we find  $P_n(k)/(k - x_{n,n}) \geq P_{n-1}(k)$ . Recall that for  $q < q_1$ , we have  $k > X(q) \geq x_{n,n}$  where  $x_{n,n}$  is the largest zero of the  $n$ th polynomial  $P_n(x)$ , so the denominator  $(k - x_{n,n}) > 0$ . From the interlacing of the zeros of  $P_n(x)$  and  $P_{n+1}(x)$  (see §2.2 (F1)), the largest zeros of the polynomials  $P_n$  are increasing for a fixed  $q$ , the factors  $(k - x_{n,n})$ s are decreasing as  $n$  increases, and bounded below by  $(k - X) > 0$ . From the recursive definition of the polynomials  $P_n$ , and the fact that  $\lambda_n = P_{n+1}(k)/P_n(k)$ , we have

$$\lambda_n = \beta_n - a_n \frac{P_{n-1}(k)}{P_n(k)}.$$

We will now show that  $\lim_{n \rightarrow \infty} \lambda_n$  is bounded above and below by  $k - \beta$ . Combining the facts deduced in the discussion above about  $\lambda_n$ , we have

$$\lim_{n \rightarrow \infty} \lambda_n = \lim_{n \rightarrow \infty} \left( \beta_n - a_n \frac{P_{n-1}(k)}{P_n(k)} \right) = k - \beta - \lim_{n \rightarrow \infty} a_n \frac{P_{n-1}(k)}{P_n(k)}.$$

But  $a_n P_{n-1}(k)/P_n(k) \leq a_n/(k - x_{n,n}) \leq a_n/(k - X)$ , so  $\lim_{n \rightarrow \infty} \lambda_n \geq k - \beta - (\lim_{n \rightarrow \infty} a_n)/(k - X)$ . Since  $\lambda_n \leq \beta_n$  by definition for all  $n \in \mathbb{N}$ , we have  $\lim_{n \rightarrow \infty} \lambda_n = k - \beta$ . This implies  $\lim_{n \rightarrow \infty} \mu_n = 0$  because  $\lambda_n + \mu_n = \beta_n$  for all  $n$ . Thus, when the numerators  $a_n$  converge to 0, the death rates converge to zero and the birth rates converge to  $k - \beta$ .  $\square$

**Theorem 2.7.** *Suppose  $\{a_n\}$  and  $\{b_n\}$  are defined as in Theorem 1.2, and also that  $\lim_{n \rightarrow \infty} a_n = 0$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then*

$$\lim_{n \rightarrow \infty} \lambda_n = b, \quad \lim_{n \rightarrow \infty} \mu_n = 0,$$

*and the process is transient.*

*Proof.* By the definition of the  $\lambda_n$ , we have  $b_n \geq \lambda_n$  for all  $n$ , and from Theorem 1.2 we have  $\lambda_n \geq b_n - \alpha \sqrt{a_n}$  for sufficiently large  $n$  and some constant  $\alpha > 0$  as in Theorem 1.2. Combining these two facts with the assumption that  $a_n$  and  $b_n$  converge, we have

$$b = \limsup_{n \rightarrow \infty} b_n \geq \limsup_{n \rightarrow \infty} \lambda_n \geq \liminf_{n \rightarrow \infty} \lambda_n \geq \liminf_{n \rightarrow \infty} (b_n - \alpha \sqrt{a_n}) = b,$$

and thus  $\lim_{n \rightarrow \infty} \lambda_n = b$ .

Since  $b_n := \lambda_n + \mu_n$ , we have  $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} (b_n - \lambda_n) = 0$ . Thus  $\lim_{n \rightarrow \infty} \mu_n = 0$ .

Considering again the quantity  $A$ , we will now prove  $A < \infty$  so the process described by these  $\mu_n$  and  $\lambda_n$  is transient.

First, fix some  $\epsilon > 0$ . Since  $\lambda_n$  and  $\mu_n$  converge, there exists some  $N \in \mathbb{N}$  such that for  $n \geq N$  we have  $\mu_n < \epsilon$  and  $\lambda_n < b + \epsilon$ . Then

$$A - \sum_{n=0}^{N-1} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_0 \lambda_1 \cdots \lambda_n} = \frac{\mu_1 \cdots \mu_{N-1}}{\lambda_0 \cdots \lambda_{N-1}} \sum_{n=N}^{\infty} \frac{\mu_n \cdots \mu_n}{\lambda_n \cdots \lambda_n} \leq \frac{\mu_1 \cdots \mu_{N-1}}{\lambda_0 \cdots \lambda_{N-1}} \sum_{n=N}^{\infty} \frac{\epsilon^{n-N+1}}{(b+\epsilon)^{n-N+1}} = \frac{\epsilon}{b}.$$

Thus, since  $A$  is finite, the process described by these  $\mu_n$  and  $\lambda_n$  is transient.  $\square$

**Theorem 2.8.** *Let  $a_n, b_n, \beta_n$ , and  $k$  be defined as in Theorem 1.1. For  $q = q_1 \in (0, 1)$ , if  $\lim_{n \rightarrow \infty} a_n = 0$  and  $\lim_{n \rightarrow \infty} \beta_n = \beta$  for  $a_n$  and  $b_n = k - \beta_n$  defined as in Theorem 1.1, the birth-death process is positive recurrent, and, if  $\lim_{n \rightarrow \infty} \lambda_n$  exists,*

$$\lim_{n \rightarrow \infty} \lambda_n = 0 \text{ and } \lim_{n \rightarrow \infty} \mu_n = k - \beta.$$

*Proof.* Consider the corresponding orthonormal polynomials  $p_n(x) := P_n(x)/\sqrt{a_1 a_2 \cdots a_n}$  mentioned in §2.2 (F3). These polynomials satisfy the recurrence relation

$$\sqrt{a_{n+1}} p_{n+1}(x) = (x - \beta_n) p_n(x) - \sqrt{a_n} p_{n-1}(x)$$

with initial conditions  $p_{-1} = 0$  and  $p_0 = 1$ . The  $p_n$  are orthonormal with respect to the distribution  $\psi$ , and  $X(q_1) = k$  is in the spectrum of  $\psi$ , so by §2.2 (F4) we have

$$B = \sum_{n=0}^{\infty} \pi_n = 1 + \sum_{n=1}^{\infty} \frac{P_n^2(k)}{a_1 a_2 \cdots a_n} = \sum_{n=0}^{\infty} p_n^2(k) \leq \frac{1}{\psi(k+) - \psi(k-)} < \infty.$$

Since  $\sum_{n=0}^{\infty} \pi_n$  converges,  $\lim_{n \rightarrow \infty} \pi_n = 0$ , so  $1/\pi_n$  diverges, and thus  $\sum_{n=0}^{\infty} 1/\pi_n = \infty$ . We know  $\lambda_n \leq b_n \leq k$ , so

$$A = \sum_{n=0}^{\infty} \frac{1}{\lambda_n \pi_n} \geq \frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{\pi_n} = \infty.$$

Thus  $A = \infty$  and  $B < \infty$ , so the process is positive recurrent.

We now use the fact that  $B < \infty$  to determine the asymptotic behavior of the birth and death rates when  $q = q_1$ . We know  $B = 1 + \sum_{n=1}^{\infty} \lambda_0^2 \lambda_1^2 \cdots \lambda_{n-1}^2 / (a_1 a_2 \cdots a_n)$  converges, so, by the ratio test for series convergence, for all  $N \in \mathbb{N}$ , there exists  $n_N \geq N$  such that  $\lambda_{n_N}^2 / a_{n_N+1} \leq 1$ . Thus,  $0 < \lambda_{n_N}^2 \leq a_{n_N+1}$ . By our assumption that  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $\lim_{N \rightarrow \infty} \lambda_{n_N} = 0$ . An infinite subsequence of  $\{\lambda_n\}$  converges to 0, so if  $\{\lambda_n\}$  converges, then  $\lim_{n \rightarrow \infty} \lambda_n = 0$ . Since  $\lambda_n + \mu_n = b_n$ , this implies  $\lim_{n \rightarrow \infty} \mu_n = k - \beta$ .  $\square$

### 3. MODULAR FORMS AND BIRTH-DEATH PROCESSES

In this section we classify modular forms (see [14] for more on modular forms) satisfying relations which will turn out to have very interesting consequences for the transition probabilities of birth-death processes.

**Definition 3.1.** *We say that a modular form  $f_1(\tau)$  is **modularly symmetric** to another modular form  $f_2(\tau)$  if there exist  $\gamma \in GL_2(\mathbb{R})$ , a constant  $\rho \in \mathbb{R}$ , and integers  $\alpha, l > 0$  such that*

$$f_1^l \left( \frac{-1}{\alpha\tau} \right) = \rho \chi(\gamma) \tau^k \gamma (f_2^l(\tau)), \quad (3.1)$$

where  $|\chi(\gamma)| = 1$ ,  $k$  is the weight of the modular form, and  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  acts on  $f$  by linear fractional transformation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f = \frac{af + b}{cf + d}.$$

If  $f_1(\tau) = f_2(\tau)$ , then we call  $f_1(\tau)$  **modularly reflexive**.

*Examples.*

(1) If  $f(\tau)$  is a modular form on  $SL_2(\mathbb{Z})$ , then  $f(\tau)$  is modularly reflexive with  $\alpha = 1$ ,  $l = 1$ ,  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\rho = 1$ , and  $\chi(\gamma) = 1$ .

(2) Slightly abusing notation, let  $r(\tau) = q^{1/5}r(q)$ , where  $q = e^{2\pi i\tau}$ . As we show in §4.2,

$$r\left(\frac{-1}{\tau}\right) = \frac{-\phi r(\tau) + 1}{r(\tau) + \phi}$$

where  $\phi = (1 + \sqrt{5})/2$  is the “golden ratio.” Therefore,  $r(q)$  is a weight 0 modularly reflexive form with  $\alpha = 1$ ,  $l = 1$ ,  $\gamma = \begin{pmatrix} -\phi & 0 \\ 0 & \phi \end{pmatrix}$ ,  $\rho = 1$ , and  $\chi(\gamma) = 1$ .

As we shall see in §4, birth-death processes corresponding to forms that are modularly symmetric satisfy very unexpected symmetry relations themselves. The transformation  $\tau \rightarrow -1/\tau$  on the positive imaginary axis maps  $q = e^{2\pi i\tau}$  from the interval  $(0, 1)$  onto itself, and this is precisely the interval on which our birth-death models are good. In §4.2, we exhibit identities between the Laplace transforms of transition probabilities of different birth-death processes. Fundamentally these identities exist because the modular forms associated to the birth-death processes are modularly symmetric.

## 4. SPECIAL BIRTH-DEATH PROCESSES

### 4.1. Interesting $q$ -continued fractions corresponding to birth-death processes.

Here we examine examples of modular continued fractions due to Ramanujan, Eisenstein, and Jacobi which correspond to birth-death probability models with positivity of the birth rates guaranteed by Theorems 1.1 and 1.2. Throughout, let  $\lambda_n$  and  $\mu_n$ . In all of the following examples, define  $\lambda_0 := 1$  and  $\mu_0 := 0$ . It is straightforward to check that the following continued fractions correspond to good birth-death processes by Theorem 1.2.

1. Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_n = \lambda_n + \mu_n = 1 + q^n, & n \geq 0, \\ a_n = \lambda_{n-1}\mu_n = q^{2n-1}, & n \geq 1. \end{cases}$$

Call the corresponding birth-death model  $\mathcal{B}_1$ . Then

$$f_{\mathcal{B}_1}(q) = \frac{1}{1-} \frac{q}{1+q-} \frac{q^3}{1+q^2-} \frac{q^5}{1+q^3-\dots} = \frac{(q^2; q^3)_\infty}{(q; q^3)_\infty}. \quad (4.1)$$

This continued fraction is Entry 10 in Ramanujan’s second notebook [3, p. 20].

2. Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_n = \lambda_n + \mu_n = 1 + q^{2n}, & n \geq 0, \\ a_n = \lambda_{n-1}\mu_n = q^{2n-1}, & n \geq 1. \end{cases}$$

Call this birth-death model  $\mathcal{B}_2$ . Then

$$f_{\mathcal{B}_2}(q) = \frac{1}{1-} \frac{q}{1+q^2-} \frac{q^3}{1+q^4-} \frac{q^5}{1+q^6-} \dots = \frac{(q^3; q^4)_\infty}{(q; q^4)_\infty}. \quad (4.2)$$

This continued fraction is Entry 11 in Ramanujan's second notebook [3, p. 22]. It corresponds to the case  $q = q^2$ ,  $a = 1/q$ ,  $b = 1$  of Corollary 2.4.

**3.** Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_n = \lambda_n + \mu_n = 1 + q^{2n+1}, & n \geq 0, \\ a_n = \lambda_{n-1}\mu_n = q^{2n+1}, & n \geq 1. \end{cases}$$

Call the corresponding birth-death model  $\mathcal{B}_3$ . Then

$$f_{\mathcal{B}_3}(q) = \frac{1}{1-} \frac{q^3}{1+q^3-} \frac{q^5}{1+q^5-} \dots$$

Note the relation between  $f_{\mathcal{B}_3}(q)$  and the classical Jacobi theta function, a modular form (see [8, p. 290]):

$$\vartheta(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + \frac{2q}{1-} \frac{q^3}{1+q^3-} \frac{q^5}{1+q^5-} \dots = 1 + 2qf_{\mathcal{B}_3}(q).$$

This corresponds to the case  $q = q^2$ ,  $a = -q$ ,  $b = q$  of Corollary 2.4.

**4.** Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_n = \lambda_n + \mu_n = 1 + q^{2n}, & n \geq 0, \\ a_n = \lambda_{n-1}\mu_n = q^{2n}, & n \geq 1. \end{cases}$$

Call the corresponding birth-death model  $\mathcal{B}_4$ . Then

$$f_{\mathcal{B}_4}(q) = \frac{1}{1-} \frac{q^2}{1+q^2-} \frac{q^4}{1+q^4-} \dots$$

Note the relationship to the second Jacobi theta function, a modular form [8, p. 290]:

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} = \frac{2q^{1/4}}{1-} \frac{q^2}{1+q^2-} \frac{q^4}{1+q^4-} \dots = 2q^{1/4}f_{\mathcal{B}_4}(q). \quad (4.3)$$

This corresponds to the case  $q = q^2$ ,  $a = -1$ ,  $b = 1$  of Corollary 2.4.

Before continuing with the next 5 examples, we introduce the notion of an ‘‘even part’’ of a continued fraction (see [11, p. 41]). Let

$$K(a_n/b_n) = \frac{a_0}{b_0+} \frac{a_1}{b_1+} \frac{a_2}{b_2+} \dots \quad (4.4)$$

be a continued fraction. Denote its  $n$ th convergent by  $C_n$ . Then the *even part* of  $K(a_n/b_n)$  is the continued fraction

$$K(a_n^*/b_n^*) = \frac{a_0^*}{b_0^*+} \frac{a_1^*}{b_1^*+} \frac{a_2^*}{b_2^*+} \dots \quad (4.5)$$

if  $C_n^* = C_{2n}$  for  $n \geq 0$ . Clearly if  $K(a_n/b_n)$  converges then  $K(a_n^*/b_n^*)$  converges to the same value. The following examples all converge for  $q < 1$ . Furthermore, we have the equations

$$\begin{cases} a_0^* = a_0 b_1, & b_0^* = a_1 + b_0 b_1 \\ a_k^* = -a_{2k} a_{2k-1} b_{2k+1} / b_{2k}, & k \geq 1 \\ b_k^* = a_{2k} b_{2k+1} / b_{2k-1} + a_{2k+1} + b_{2k} b_{2k+1}, & k \geq 1. \end{cases} \quad (4.6)$$

5. Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_0 = \lambda_0 = 1 + q, \\ b_n = \lambda_n + \mu_n = 1 + q^{2n} + q^{2n+1}, & n \geq 1, \\ a_n = \lambda_{n-1} \mu_n = q^{4n-1}, & n \geq 1. \end{cases}$$

Call the corresponding birth-death model  $\mathcal{B}_5$ . Then

$$f_{\mathcal{B}_5}(q) = \frac{1}{1+q-} \frac{q^3}{1+q^2+q^3-} \frac{q^7}{1+q^4+q^5-} \cdots$$

By (4.6), the even part of the Rogers-Ramanujan continued fraction, a modular form [7], is

$$\frac{q^{1/5}}{1+q-} \frac{q^3}{1+q^2+q^3-} \cdots = q^{1/5} f_{\mathcal{B}_5}(q).$$

Therefore

$$f_{\mathcal{B}_5}(q) = q^{-1/5} \frac{q^{1/5}}{1+} \frac{q}{1+} \frac{q^2}{1+} \cdots = \prod_{n=0}^{\infty} \frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})}. \quad (4.7)$$

The Rogers-Ramanujan continued fraction corresponds to the case  $a = 1, b = 0$  of Corollary 2.4.

6. Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_0 = \lambda_0 = 1 + q + q^2, \\ b_n = \lambda_n + \mu_n = 1 + (q^{2n} + q^{4n}) + (q^{2n+1} + q^{4n+1}), & n \geq 1, \\ a_n = \lambda_{n-1} \mu_n = (q^{2n} + q^{4n})(q^{2n-1} + q^{4n-2}), & n \geq 1. \end{cases}$$

Call the corresponding birth-death process  $\mathcal{B}_6$ . Then

$$f_{\mathcal{B}_6}(q) = \frac{1}{1+q+q^2-} \frac{(q+q^2)(q^2+q^4)}{1+q^2+q^4+q^3+q^6-} \frac{(q^2+q^4)(q^3+q^6)}{1+q^3+q^6+q^4+q^8-} \cdots$$

which by (4.6) is the even part of the continued fraction

$$\frac{1}{1+} \frac{q+q^2}{1+} \frac{q^2+q^4}{1+} \cdots = \prod_{n=0}^{\infty} \frac{(1-q^{2n+1})}{(1-q^{6n+3})^3}.$$

This is Entry 54 in Ramanujan's second notebook [3, p. 59]. Therefore

$$f_{\mathcal{B}_6}(q) = \prod_{n=0}^{\infty} \frac{(1-q^{2n+1})}{(1-q^{6n+3})^3}.$$

7. Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_0 = \lambda_0 = 1 + q, \\ b_n = \lambda_n + \mu_n = 1 + (q^k + q^{2k}) + q^{2k+1}, & n \geq 1, \\ a_n = \lambda_{n-1} \mu_n = q^{2n-1}(q^n + q^{2n}), & n \geq 1. \end{cases}$$

Call the corresponding birth-death process  $\mathcal{B}_7$ . Then

$$f_{\mathcal{B}_7}(q) = \frac{1}{1+q-} \frac{q^2+q^3}{1+q+q^2+q^3-} \frac{q^5+q^7}{1+q^2+q^4+q^5-} \cdots$$

which by (4.6) is the even part of the continued fraction

$$\frac{1}{1+} \frac{q}{1+} \frac{q^2+q}{1+} \frac{q^3}{1+} \frac{q^4+q^2}{1+} \cdots = \prod_{n=0}^{\infty} \frac{(1+q^{2n})}{(1+q^{2n+1})}.$$

This is Entry 12 in Ramanujan's second notebook [3, p. 23]. Therefore

$$f_{\mathcal{B}_7}(q) = \prod_{n=0}^{\infty} \frac{(1+q^{2n})}{(1+q^{2n+1})}. \quad (4.8)$$

By Theorem 7 in [16], the continued fraction (4.1) is equal to

$$\frac{1}{1+} \frac{q}{1+q+} \frac{q^2}{1+q^2+} \cdots$$

This corresponds to the case  $a = 1$ ,  $b = 1$  of Corollary 2.4.

**8.** Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_0 = \lambda_0 = 1 + q + q^2, \\ b_n = \lambda_n + \mu_n = 1 + q^{2k+1} + q^{4k} + q^{4k+2}, \quad n \geq 1, \\ a_n = \lambda_{n-1}\mu_n = q^{4n}(q^{2n-1} + q^{4n-2}), \quad n \geq 1. \end{cases}$$

Call the corresponding birth-death process  $\mathcal{B}_8$ . Then

$$f_{\mathcal{B}_8}(q) = \frac{1}{1+q+q^2-} \frac{q^4(q+q^2)}{1+q^3+q^4+q^6-} \cdots$$

which is the even part of the continued fraction

$$\frac{1}{1+} \frac{q+q^2}{1+} \frac{q^4}{1+} \frac{q^3+q^6}{1+} \frac{q^8}{1+} \cdots = \prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+7})}{(1-q^{8n+3})(1-q^{8n+5})}.$$

This is Entry 13 in Ramanujan's second notebook [3, p. 23]. Therefore

$$f_{\mathcal{B}_8}(q) = \prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+6})}{(1-q^{8n+3})(1-q^{8n+5})}. \quad (4.9)$$

**9.** Let  $\{\lambda_n\}, \{\mu_n\}$  be sequences of real numbers satisfying

$$\begin{cases} b_0 = \lambda_0 = 1, \\ b_n = \lambda_n + \mu_n = 1 + \frac{q^{4n-1}(1-q^{4n+1})}{(1-q^{4n-3})} + q^{8n} - q^{4n-1}, \quad n \geq 1, \\ a_n = \lambda_{n-1}\mu_n = \frac{q^{8n-4}(1-q^{4n+1})}{(1-q^{4n-1})}, \quad n \geq 1. \end{cases}$$

Call the corresponding birth-death process  $\mathcal{B}_9$ . Then

$$f_{\mathcal{B}_9}(q) = \frac{1}{1-} \frac{q^4(1-q^5)/(1-q^3)}{1+q^3(1-q^5)/(1-q)+q^8-q^3-} \cdots$$

Note that  $(1 - q)f_{\mathcal{B}_9}(0)$  is the even part of the continued fraction

$$\vartheta_3(q) = \sum_{n=0}^{\infty} (-1)^n q^{n^2} = \frac{1}{1+} \frac{q}{1-q+} \frac{q^3}{1-q^3+} \frac{q^5}{1-q^5+} \cdots$$

This is another classical Jacobi theta function [8, p. 282]. It corresponds to the case  $q = q^2$ ,  $a = 1/q$ ,  $b = -1/q$  of Corollary 2.4.

$$1 - 2qf_{\mathcal{B}_9}(q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2}. \quad (4.10)$$

**4.2. Symmetry between birth-death processes.** Here we give examples of modularly symmetric forms (see §3) corresponding to birth-death processes exhibited in §4.1.

**Theorem 4.1.** *Let  $q = e^{2\pi i\tau}$  where  $\Im(\tau) > 0$ . The following are true:*

(1)  $\vartheta(q)$  is a weight  $1/2$  modularly reflexive form with  $\alpha = 4$ ,  $l = 1$ ,  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\rho = \frac{1}{\sqrt{2}}$ , and  $\chi(\gamma) = \sqrt{-i}$ .

(2)  $\vartheta_2(q)$  is a weight  $1/2$  modularly symmetric form to  $\vartheta_3(q)$  with  $\alpha = 4$ ,  $l = 1$ ,  $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\rho = \sqrt{2}$ , and  $\chi(\gamma) = \sqrt{i}$ .

(3)  $r(q)$  is a weight  $0$  modularly reflexive form with  $\alpha = 1$ ,  $l = 1$ ,  $\gamma = \begin{pmatrix} -\phi & 0 \\ 0 & \phi \end{pmatrix}$  where  $\phi = \frac{1+\sqrt{5}}{2}$  is the ‘‘golden ratio,’’  $\rho = 1$ , and  $\chi(\gamma) = 1$ .

(4) Let  $u(\tau) = \sqrt{2} q^{1/8} \prod_{n=1}^{\infty} \frac{(1+q^{2n})}{(1+q^{2n+1})}$ . Then  $u(\tau)$  is a weight  $0$  modularly reflexive form with  $\alpha = 1/4$ ,  $l = 2$ ,  $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ ,  $\rho = 1$ , and  $\chi(\gamma) = 1$ .

(5) Let  $v(\tau) = q^{1/2} \prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+6})}{(1-q^{8n+3})(1-q^{8n+5})}$ . Then  $v(\tau)$  is a weight  $0$  modularly reflexive form with  $\alpha = 8$ ,  $l = 2$ ,  $\gamma = \begin{pmatrix} 1 & -\sigma^2 \\ \sigma^2 & -1 \end{pmatrix}$  where  $\sigma = -1 + \sqrt{2}$ ,  $\rho = 1$ , and  $\chi(\gamma) = 1$ .

*Proof.*

(1)

$$\vartheta_3(q) = \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2}. \quad (4.11)$$

We substitute  $\tau = -1/\tau$  in (4.11), and using the transformation law for the Dedekind  $\eta$ -function [14, p. 17] we obtain

$$\frac{\eta(-1/(2\tau))^5}{\sqrt{-2i\tau} \eta(-1/\tau)^2 \eta(-1/(4\tau))^2} = \frac{\eta(2\tau)^5}{\eta(\tau)^2 \eta(4\tau)^2}. \quad (4.12)$$

(2) We have the following  $\eta$ -quotient representations [14, p. 17]:

$$\frac{\eta(\tau)^2}{\eta(2\tau)} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \quad \text{and} \quad \frac{\eta(4\tau)^2}{\eta(2\tau)} = \sum_{n=0}^{\infty} q^{(n+1/2)^2}. \quad (4.13)$$

Now make the transformation  $\tau \rightarrow -1/\tau$  in  $\eta(\tau)^2/\eta(2\tau)$ .

(3) This follows from Proposition 2 in [7]:

$$r\left(\frac{-1}{\tau}\right) = \frac{-\phi r(\tau) + 1}{r(\tau) + \phi} \quad (4.14)$$

where  $\phi = (1 + \sqrt{5})/2$  is the “golden ratio.”

(4) This follows from the identity [7, p. 21]

$$u^2(-4/\tau) = \frac{1 - u^2(\tau)}{1 + u^2(\tau)}. \quad (4.15)$$

(5) This follows from the identity [7, p. 22]

$$v^2(-1/(8\tau)) = \frac{v^2(\tau) - \sigma^2}{\sigma^2 v^2(\tau) - 1} \quad (4.16)$$

where  $\sigma = -1 + \sqrt{2}$ . □

**Corollary 4.2.** *For  $y > 0$ , the following identities are true:*

(1)

$$1 + 2e^{-\pi(2y)} f_{\mathcal{B}_3}(e^{-\pi(2y)}) = \sqrt{2/y} (1 + e^{-\pi/(2y)} f_{\mathcal{B}_3}(e^{-\pi/(2y)})).$$

(2)

$$-2e^{-\pi y/2} f_{\mathcal{B}_4}(e^{-\pi(2y)}) = \frac{1}{\sqrt{2y}} e^{\pi/(2y)} f_{\mathcal{B}_9}(e^{-\pi/(2y)}) - \frac{1}{2\sqrt{2y}}.$$

(3)

$$e^{2\pi/(5y)} f_{\mathcal{B}_5}(e^{-2\pi/y}) = \frac{-\phi e^{2\pi y} f_{\mathcal{B}_5}(e^{-2\pi y}) + 1}{e^{2\pi y} f_{\mathcal{B}_5}(e^{-2\pi y}) + \phi}.$$

(4)

$$\frac{e^{\pi y/2}}{2} f_{\mathcal{B}_7}^2(e^{-8\pi/y}) = \frac{1 - \frac{e^{\pi y/2}}{2} f_{\mathcal{B}_7}^2(e^{-2\pi y})}{1 + \frac{e^{\pi y/2}}{2} f_{\mathcal{B}_5}^2(e^{-2\pi y})}.$$

(5)

$$e^{\pi/(4y)} f_{\mathcal{B}_8}^2(e^{-\pi/(4y)}) = \frac{e^{2\pi y} f_{\mathcal{B}_8}^2(e^{-2\pi y}) - \sigma^2}{\sigma^2 e^{2\pi y} f_{\mathcal{B}_8}^2(e^{-2\pi y}) - 1}.$$

*Proof.*

(1) Recall that for the birth-death process  $\mathcal{B}_1$  we have the equation

$$\vartheta_3(q) = 1 + 2q f_{\mathcal{B}_1}(q). \quad (4.17)$$

Set  $\tau = iy$  in (4.12), and substitute the identity (4.17) in the resultant equation.

(2) Make the transformation  $\tau \rightarrow -1/\tau$  in (4.10), substitute identities (4.3) and (4.10) into (4.13), then set  $\tau = iy$ .

(3) Substitute (4.7) into (4.14) and then set  $\tau = iy$ . As before, we obtain (3) in Corollary 4.2 after some straightforward manipulations.

(4) Substitute (4.8) into (4.15) and set  $y = i\tau$ . As before, we obtain (4) in Corollary 4.2 after some straightforward manipulations.

(5) Substitute (4.9) into (4.16) and set  $\tau = iy$ . As before, we obtain (5) Corollary 4.2 after some straightforward manipulations. □

*Remark.* The identities in 4.2 stemmed from the distinctive behavior of modular forms under the transformation  $\tau \rightarrow -1/\tau$ . There are further results which depend on algebraic equations satisfied by modular forms. For example, let  $f(q) = f_{\mathcal{B}_5}(q)$ . For  $q = e^{2\pi i\tau}$  with  $\Im(\tau) > 0$ , it is easy to show the following (see Theorems 7.5.1, 7.5.3, and 7.5.4 in [5] and Entry 24 and Entry 25 in [3]):

(1)

$$\frac{1}{q}f(q)f(q^2)^2 = \frac{f(q^2) - f(q)^2}{f(q^2) + f(q)^2}$$

(2)

$$\frac{3}{q}f(q)^2f(q^3)^2 = (f(q^3) - f(q)^3) \left(1 + \frac{f(q)f(q^3)^2}{q^2}\right)$$

(3)

$$\begin{aligned} & \frac{5}{q^2}f(q)^2f(q^4)^2 \left(\frac{f(q)f(q^4)}{q} - 1\right)^2 - \frac{f(q)f(q^4)}{q} \\ &= \left(\frac{f(q)^5}{q} + \frac{f(q^4)^5}{q^4}\right) \left(\frac{f(q)f(q^4)}{q} - 1\right) + \frac{f(q)^5f(q^4)^5}{q^5} \end{aligned}$$

(4)

$$\frac{f(q)^5}{f(q^5)} = \frac{1 - 2\frac{f(q^5)}{q} + 4\frac{f(q^5)^2}{q^2} - 3\frac{f(q^5)^3}{q^3} + \frac{f(q^5)^4}{q^4}}{1 + 3\frac{f(q^5)}{q} + 4\frac{f(q^5)^2}{q^2} + 2\frac{f(q^5)^3}{q^3} + \frac{f(q^5)^4}{q^4}}$$

(5)

$$\begin{aligned} & \frac{1}{q^5}f(q)^5f(q^2)^5 (f(q)^{10}f(q^2)^5 + f(q^2)^{10}) + f(q)^{10} - f(q^2)^5 \\ &+ \frac{10}{q}f(q)^5f(q^2)^5 \left(\frac{f(q)f(q^2)^5}{q^3} - \frac{f(q)^5}{q} + \frac{f(q^2)^5}{q^2} + 1\right) = 0. \end{aligned}$$

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38 WYNGATE DRIVE, AVON, CT 06001  
*E-mail address:* [tfeng@college.harvard.edu](mailto:tfeng@college.harvard.edu)

7212 LONGWOOD DRIVE, BETHESDA, MD 20817  
*E-mail address:* [rkirsch@umd.edu](mailto:rkirsch@umd.edu)

146 HARRISON DRIVE, EDINBORO, PA 16412  
*E-mail address:* [elise@mit.edu](mailto:elise@mit.edu)

1411 N BRIARCLIFF DRIVE, APPLETON, WI 54915  
*E-mail address:* [mwage@princeton.edu](mailto:mwage@princeton.edu)