REU 2009 TOPICS

The 2009 REU will revolve around three main topics (and of course, most years some students write papers on further topics of their choice):

- Continued fractions of modular forms with applications to probability.
- The Birch and Swinnerton-Dyer Conjecture and ranks of elliptic curves
- Lehmer's Conjecture on the nonvanishing of Ramanujan's tau-function

1. Continued fractions and probability

Toss a fair coin repeatedly until it lands heads up. If one flips n tails before the first head, what is the probability that n is even? Since the probability of flipping n tails before the first head is $\frac{1}{2^{n+1}}$, the solution is

$$\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\cdots=\frac{1}{1-\frac{1}{2}}-\frac{1}{1-\frac{1}{4}}=\frac{2}{3}.$$

Instead of continuing until the first head, consider the situation where a coin is repeatedly flipped: once, then twice, then three times, and so on. What is the probability of the outcome that each nth turn, where n is odd, has at least one head?

More generally, let $0 < p < 1$, and let $\{C_1, C_2, \ldots\}$ be a sequence of independent events where the probability of C_n is given by

$$\mathbb{P}_p(C_n) := 1 - p^n. \tag{1.1}$$

For each pair of integers $0 \le r < t$, we let

$$A(r,t) := \{\text{set of sequences where } C_n \text{ occurs if } n \not\equiv \pm r \pmod t\}. \tag{1.2}$$

In the case where $p = 1/2$, one can think of C_n as the event where at least one of n tosses of a coin is a head. Therefore, if $p = 1/2$, the problem above asks for the probability of the outcome $A(0,2)$.

We give two situations where one plainly sees very strange phenomena. For example, consider the limiting behavior of the quotient $L(p) := \frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))}$, as $p \to 1$. The table below is very suggestive.

Theorem 1.1. *The following limits are true:*

$$\lim_{p\to 1}\frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))}=\frac{-1+\sqrt{5}}{2},$$

$$\lim_{p\to 1}\frac{\mathbb{P}_p(A(3,8))}{\mathbb{P}_p(A(1,8))}=-1+\sqrt{2}.$$

Table 1. Values of $L(p)$

p	$\mathbb{P}_p(A(2,5))$	$\mathbb{P}_p(A(1,5))$	$L(p)$
0.3	$0.692\dots$	$0.883\dots$	$0.61607\dots$
0.4	$0.576\dots$	$0.776\dots$	$0.61778\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.97	$6.43\dots \times 10^{-14}$	$1.03\dots \times 10^{-13}$	$0.61803\dots$
0.98	$5.65\dots \times 10^{-21}$	$9.11\dots \times 10^{-21}$	$0.61803\dots$
0.99	$3.11\dots \times 10^{-42}$	$5.03\dots \times 10^{-42}$	$0.61803\dots$

Remark. Notice that $\frac{1}{2}(-1+\sqrt{5}) = -1+\phi$, where ϕ is the *golden ratio*

$$\phi := \frac{1}{2}(1+\sqrt{5}) = 1.61803\dots.$$

Theorem 1.1 is a consequence of a deep fact which relates these probabilities to modular forms and continued fractions. Using these ideas, one may first address the problem of obtaining *algebraic* formulas for all of these ratios, not just the limiting values. As a function of p, one may compute the ratios of these probabilities in terms of continued fractions. To ease notation, we let

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots \tag{1.3}$$

denote the continued fraction

$$\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots}}}.$$

The following continued fractions are well known:

$$\begin{aligned} \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots &= \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} = \frac{-1+\sqrt{5}}{2}, \\ \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots &= \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}} = -1+\sqrt{2}. \end{aligned} \tag{1.4}$$

Theorem 1.1 is the limit of the following exact formulas.

Theorem 1.2. *If $0<p<1$ and $\tau_p := -\log(p)\cdot\frac{i}{2\pi}$, then we have that*

$$\begin{aligned} \frac{\mathbb{P}_p(A(2,5))}{\mathbb{P}_p(A(1,5))} &= \left(\frac{1}{1+}\frac{q_{\tau_p}}{1}+\frac{q_{\tau_p}^2}{1}+\frac{q_{\tau_p}^3}{1}+\cdots\right), \\ \frac{\mathbb{P}_p(A(3,8))}{\mathbb{P}_p(A(1,8))} &= \left(\frac{1}{1+q_{\tau_p}}+\frac{q_{\tau_p}^2}{1+q_{\tau_p}^3}+\frac{q_{\tau_p}^4}{1+q_{\tau_p}^5}+\frac{q_{\tau_p}^6}{1+q_{\tau_p}^7}+\cdots\right). \end{aligned}$$

Theorem 1.2 can also be used to obtain many further beautiful expressions, not just those pertaining to the limit as $p\to 1$. For example, we obtain the following simple corollary.

Corollary 1.3. *For $p_1 = \frac{1}{e^{2\pi}}$ and $p_2 = \frac{1}{e^{\pi}}$, we have that*

$$\frac{\mathbb{P}_{p_1}(A(2,5))}{\mathbb{P}_{p_1}(A(1,5))} = e^{-\frac{2\pi}{5}} \left(-\phi + \sqrt{\frac{1}{2}(5+\sqrt{5})} \right),$$

$$\frac{\mathbb{P}_{p_2}(A(3,8))}{\mathbb{P}_{p_2}(A(1,8))} = e^{-\frac{\pi}{2}} \left(\sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}} \right).$$

Project 1.4. *Some students will investigate generalizations of these results using:*

- *Combinatorics*
- *Modular form theory*
- *Theory of singular moduli and complex multiplication.*

2. The Birch and Swinnerton-Dyer Conjecture and ranks of elliptic curves

We begin by recalling a famous open problem. A positive integer N is called a *congruent number* if it is the area of a right triangle with rational sidelengths. Obviously, $N = 6$ is congruent thanks to the triangle with sidelengths $(3, 4, 5)$. However, $N = 7$ is not congruent, and $N = 157$ is congruent. Try to prove these last two claims!

The classical problem of classifying all the congruent numbers remains open to this day, and it is often presented as motivation for the celebrated Birch and Swinnerton-Dyer Conjecture. Indeed, we have the following well-known criterion.

Theorem 2.1. *A positive integer N is congruent if and only if the elliptic curve*

$$E_N : y^2 = x^3 - N^2 x$$

has infinitely many points over $\mathbb{Q}$.

Here we are employing a classical theorem of Poincaré which asserts that the rational points on an elliptic curve form a finitely generated abelian group. Using standard facts about these curves, one has the following criterion.

Corollary 2.2. *A positive integer N is congruent provided either of the following equivalent conditions are satisfied:*

1. *The rank of E over $\mathbb{Q}$ is positive.*
2. *There is a rational point $(x, y) \in \mathbb{Q}^2$ with $xy \neq 0$ on E.*

To every elliptic curve we may attach an L-function $L(E, s)$. These L-functions have analytic continuations to $\mathbb{C}$, and their behavior at $s = 1$ is the content of the BSD Conjecture.

Conjecture (Birch and Swinnerton-Dyer)**.** *If $E/\mathbb{Q}$ is an elliptic curve, then*

$$\mathrm{ord}_{s=1}(L(E,s)) \ = \ \text{"rank of } E''$$

The congruent number elliptic curves are examples of very special curves: *elliptic curves with complex multiplication.* It is not difficult to explain some consequences of complex multiplication. If p is an odd prime, then let $a(p) := p + 1 - N(p)$, where $N(p)$

denotes the number of points on E over the finite field with p elements (including the point at infinity). It turns out that the L-function for E, denoted $L(E,s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ is related to the formal power series

$$A(q) = \sum_{n=1}^{\infty} a(n)q^n := q\prod_{n=1}^{\infty}(1-q^{4n})^2(1-q^{8n})^2.$$

Thanks to the following classical identities of Jacobi, it is known that

$$J_1(q) := q\prod_{n=1}^{\infty}(1-q^{8n})^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2},$$

$$J_2(q) := q\prod_{n=1}^{\infty}\frac{(1-q^{16n})^2}{(1-q^{8n})} = \sum_{n=0}^{\infty}q^{(2n+1)^2}.$$

Therefore, we find that

$$A(q) = J_1(q^{1/2})J_2(q^{1/2}).$$

There are other natural families of elliptic curves with complex multiplication which arise naturally.

Project 2.3. *Some students will consider BSD and the nonvanishing of L-functions for an important class of elliptic $\mathbb{Q}$-curves with complex multiplication which were originally defined by Benedict Gross in his Ph.D. thesis. This project requires a strong background in analysis.*

3. Lehmer's Conjecture on Ramanujan's tau-function

As usual, let

$$\Delta(z) = \sum_{n=1}^{\infty}\tau(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - \cdots \tag{3.1}$$

(note. $q := e^{2\pi iz}$ throughout) be the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Ramanujan investigated its coefficients, the aptly named Ramanujan tau-function, and his work provided tantalizing examples of phenomena which are now woven into fabric of modern number theory. Simply put, his work on $\tau(n)$ provided examples of some of the deepest phenomena in the theory of modular forms. He conjectured the relations

$$\tau(mn) = \tau(m)\tau(n) \qquad \text{if } \gcd(m,n) = 1,$$

$$\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1}) \qquad \text{if } p \text{ is prime and } n \geq 1,$$

which served as a prototype for the theory of Hecke operators. His congruences provided a testing ground for Serre's original formulation of the theory of modular ℓ-adic Galois representations, which has famously been employed in the proof of Fermat's Last Theorem. Lastly, his speculation that

$$|\tau(p)| \leq 2p^{\frac{11}{2}},$$

for primes p, inspired the Ramanujan-Petersson Conjecture, which was later proved by Deligne in his Fields Medal award winning work on the Weil Conjectures.

Despite these incredible advances, the following seemingly simple conjecture has remained open.

Conjecture. (D. H. Lehmer , 1947)
If n is a positive integer, then $\tau(n)$ is non-zero.

Recent theorems shed light on the vanishing of Fourier coefficients. Here we indicate, in the case of Ramanujan's tau-function, how these results are related to criteria obtained using only elementary considerations.

We consider a sequence of polynomials $A_m(x) \in \mathbb{Q}[x]$ whose behavior at $x = 0$ and 1728 dictates the vanishing of $\tau(m)$. To make this precise, let

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$
$$E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

denote the normalized weight 4 and 6 Eisenstein series, and as usual let

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots . \tag{3.2}$$

We recall a special sequence of polynomials $J_m(x)$ which were first considered by Faber. Define these monic degree m polynomials in $\mathbb{Z}[x]$ by the generating function

$$\sum_{m=0}^{\infty} J_m(x) q^m := \frac{E_4(z)^2 E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - x} = 1 + (x - 744)q + \cdots . \tag{3.3}$$

One easily verifies that

$$\begin{aligned} J_0(z) &= 1, \\ J_1(z) &= x - 744, \\ J_2(z) &= x^2 - 1488x + 159768, \\ J_3(z) &= x^3 - 2232x^2 + 1069956x - 36866976. \end{aligned}$$

Using these polynomials, for each positive integer m we define the monic degree m polynomial $A_m(x)$ by

$$A_m(x) := -\frac{65520}{691} \sigma_{11}(m) + J_m(x) - 264 \sum_{n=1}^{m} \sigma_9(n) J_{m-n}(x). \tag{3.4}$$

For the first primes p, we have

$$\begin{aligned} A_2(x) &= x^2 - 1752x + \frac{18289152}{691}, \\ A_3(x) &= x^3 - 2496x^2 + 1327356x - \frac{192036096}{691}, \\ A_5(x) &= x^5 - 3984x^4 + 5201172x^3 - 2400355072x^2 + 257661670110x - \frac{3680691840}{691}. \end{aligned}$$

The following theorem is true.

Theorem 3.1. *For positive integers m, we have that*

$$A_m(0) + \frac{762048}{691} \cdot \tau(m) = A_m(1728) - \frac{432000}{691} \cdot \tau(m) = 0.$$

In particular, we have that $\tau(m) = 0$ if and only if $A_m(0) = A_m(1728) = 0$.

The zeros of the $A_m(x)$ tend to a nice distribution in the interval $[-\epsilon, 1728 + \epsilon]$ as $m \to +\infty$. In view of Lehmer's Conjecture, we restrict our attention to those $m = p$ which are prime. The numerics for $p \leq 5$ reveals this phenomenon.

Prime p	Zeros of $A_p(x)$				
2		15.2396 . . . ,	1736.7603 . . .		
3		0.2094 . . . ,	767.8855 . . . ,	1727.9050 . . .	
5	0.00002 . . . ,	151.5551 . . . ,	701.0707 . . . ,	1403.3741 . . . ,	1727.9999 . . .

It is natural to ask whether the criterion in Theorem 3.1 is a consequence of deeper phenomena which sheds further light on the nonvanishing of $\tau(n)$. This is indeed the case, and the answer lies in the theory "harmonic Maass forms". Instead of the polynomials $A_p(x)$, it turns out that the heart of the matter are the modular forms

$$\widehat{A}_p(z) := A_p(j(z)). \tag{3.5}$$

The vanishing of $\tau(p)$ is equivalent to an identity involving $\widehat{A}_p(z)$ and a weight -10 harmonic Maass form.

Throughout, let $z = x + iy \in \mathbb{H}$, the upper-half of the complex plane, with $x, y \in \mathbb{R}$. Also, throughout suppose that $k \in \mathbb{N}$. We define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \tag{3.6}$$

Suppose that χ is a Dirichlet character modulo N. Then a *harmonic Maass form of weight k on* $\Gamma_0(N)$ with Nebentypus χ is any smooth function on $\mathbb{H}$ satisfying:

(i) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;
(ii) $\Delta_k f = 0$;
(iii) There is a polynomial $P_f = \sum_{n \leq 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

The polynomial $P_f \in \mathbb{C}[q^{-1}]$ is called the *principal part* of f at the corresponding cusp. We denote the vector space of these harmonic Maass forms by $H_k(\Gamma_0(N), \chi)$

The coefficients of harmonic Maass forms can be used to detect vanishing coefficients of newforms such as $\Delta(z)$. The differential operator

$$\xi_w := 2iy^w \cdot \overline{\frac{\partial}{\partial \overline{z}}},$$

plays a central role, and the key fact is that

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N), \chi) \longrightarrow S_k(\Gamma_0(N), \overline{\chi}), \tag{3.7}$$

where $S_w(\Gamma_0(N), \chi)$ denotes the subspace of cusp forms. It is not difficult to make this more precise using Fourier expansions. Every weight $2-k$ harmonic Maass form $f(z)$ has a Fourier expansion of the form

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n, \tag{3.8}$$

where $\Gamma(a, x)$ is the incomplete Gamma-function. A straightforward calculation shows that $\xi_{2-k}(f)$ has the Fourier expansion

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n. \tag{3.9}$$

As (3.8) reveals, $f(z)$ naturally decomposes into two summands

$$f^+(z) := \textstyle\sum_{n \gg -\infty} c_f^+(n) q^n, \tag{3.10}$$

$$f^-(z) := \textstyle\sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n. \tag{3.11}$$

Therefore, $\xi_{2-k}(f)$ is given simply in terms of $f^-(z)$, the *non-holomorphic part* of f. It turns out that the coefficients of the complementary *holomorphic part* $f^+(z)$ is the key player in our results on the vanishing of coefficients.

Suppose that $g \in S_k(\Gamma_0(N), \overline{\chi})$ is a normalized integer weight newform, and let F_g be the number field obtained by adjoining the coefficients of g to $\mathbb{Q}$. We say that a harmonic Maass form $f \in H_{2-k}(\Gamma_0(N), \chi)$ is *good for g* if it satisfies the following properties:

(i) The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$.
(ii) The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant.
(iii) We have that $\xi_{2-k}(f) = \|g\|^{-2} g$, where $\|g\|$ denotes the usual Petersson norm.

Remark. For every newform g, there is a harmonic Maass form f which is good for g.

Theorem 3.2. *Suppose that $g = \sum_{n=1}^{\infty} c_g(n) q^n \in S_k(\Gamma_0(N), \overline{\chi})$ is a normalized integer weight newform, and suppose that $f \in H_{2-k}(\Gamma_0(N), \chi)$ is good for g. If any of the coefficients of f^+ are transcendental, then there are no primes $p \nmid N$ for which $c_g(p) = 0$.*

One can produce a harmonic Maass form $f_\Delta(z) \in H_{-10}(\Gamma_0(1))$ which is good for $\Delta(z)$. The following theorem gives a closed formula for the coefficients of its holomorphic part f_Δ^+ in terms of the classical Kloosterman sums

$$K(m, n, c) := \sum_{v(c)^\times} e\left(\frac{m\overline{v} + nv}{c}\right), \tag{3.12}$$

where v runs through the primitive residue classes modulo c, and $\overline{v}$ denotes the multiplicative inverse of v modulo c.

Theorem 3.3. *There is a weight -10 harmonic Maass form $f_\Delta(z)$ which is good for $\Delta(z)$. For convenience, denote its holomorphic part by*

$$f_\Delta^+(z) = \sum_{n=-1}^{\infty} a_\Delta(n)q^n = \Gamma(12)q^{-1} - \frac{2^{12}\pi^{12}}{\zeta(12)} + \cdots,$$

where for positive integers n we have

$$a_\Delta(n) = -2\pi\Gamma(12)n^{-\frac{11}{2}} \cdot \sum_{c=1}^{\infty} \frac{K(-1,n,c)}{c} \cdot I_{11}\left(\frac{4\pi\sqrt{n}}{c}\right).$$

Here $I_{11}(x)$ is the usual I_{11}-Bessel function.

It turns out that

$$\frac{1}{11!} \cdot f_\Delta^+(z) \sim q^{-1} - \frac{65520}{691} - 1842.89472q - 23274.07545q^2 - 225028.75877q^3 - \cdots.$$

We note that the first two coefficients are exact, while the others are numerical approximations. The coefficients $a_\Delta(n)$ for positive n do not appear to be simple rational numbers. In this case, the explicit form of Theorem 3.2 is:

Theorem 3.4. *If there are any positive integers n for which $a_\Delta(n)$ is irrational, then Lehmer's Conjecture is true. Moreover, if Lehmer's Conjecture is false, then there is a positive integer K such that $Kn^{11}a_\Delta(n)$ is an integer for every positive integer n.*

Project 3.5. *Some students will investigate the following two problems:*

Problem 1. *Show that there are no primes p for which $A_p(0) \cdot A_p(1728) = 0$.*

Problem 2. *Prove that*

$$\pi n^{-\frac{11}{2}} \cdot \sum_{c=1}^{\infty} \frac{K(-1,n,c)}{c} \cdot I_{11}\left(\frac{4\pi\sqrt{n}}{c}\right)$$

is irrational **for at least** *one positive integer n.*