REU 2010 TOPICS

The 2010 REU will concentrate on the following topics (note. Like past years, some students will probably write papers on other topics of their choice):

- Hypergeometric invariants of elliptic curves.
- p-adic interplay between Catalan numbers and elliptic curves.
- Lehmer's Conjecture on Ramanujan's tau-function.

Here we briefly describe these topics.

1. Hypergeometric invariants of elliptic curves

The study of elliptic curves is central in modern number theory. Indeed, the classical "Congruent Number Problem" is often given as motivation for the Birch and Swinnerton-Dyer Conjecture. The problem asks for the complete determination of those positive integers N which are areas of right triangles with rational sidelengths.

Elliptic curves are studied in many settings. They are studied by complex analysis using the theory of lattices and periods. They are studied over finite fields by means of character sums and Galois theory. They are also studied by means of differential equations and special functions. It turns out that some elliptic curves can be studied in all three settings by appropriately defining analogs of suitable hypergeometric functions. This is well known for the Legendre normal form elliptic curves:

$$E_\lambda : \ y^2 = x(x-1)(x-\lambda).$$

One team will investigate and uncover new families of such elliptic curves.

2. p-adic interplay between Catalan numbers and elliptic curves

The nth Catalan number is the combinatorial number

$$C(n) := \frac{1}{n+1}\binom{2n}{n}.$$

These numbers play many important roles in combinatorics. For example, $C(n)$ is the number of permutations of the multiset $\{n \cdot 1, n \cdot (-1)\}$ whose partial sums are all non-negative.

It turns out that these numbers are closely related to elliptic curves and other number theoretic objects with rich structure. Students will investigate how these numbers arise in the study of elliptic curves and related objects over finite fields. These students will make use (and perhaps find new proofs) of the modularity of certain elliptic curves, and they will probably prove conjectures which have been raised recently in the subject.

3. Lehmer's Conjecture on Ramanujan's tau-function

As usual, let

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n := q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 - \cdots \tag{3.1}$$

(note. $q := e^{2\pi i z}$ throughout) be the unique normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Ramanujan investigated its coefficients, the aptly named Ramanujan tau-function, and his work provided tantalizing examples of phenomena which are now woven into fabric of modern number theory. Simply put, his work on $\tau(n)$ provided examples of some of the deepest phenomena in the theory of modular forms. Simply put, $\tau(n)$ has served as an excellent prototype for the theory.

The following seemingly simple conjecture has remained open.

Conjecture. (D. H. Lehmer , 1947)
If n is a positive integer, then $\tau(n)$ is non-zero.

Recent theorems shed light on the vanishing of Fourier coefficients. Here we indicate, in the case of Ramanujan's tau-function, how these results are related to criteria obtained using only elementary considerations.

We consider a sequence of polynomials $A_m(x) \in \mathbb{Q}[x]$ whose behavior at $x = 0$ and 1728 dictates the vanishing of $\tau(m)$. To make this precise, let

$$E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$
$$E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

denote the normalized weight 4 and 6 Eisenstein series, and as usual let

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884q + \cdots . \tag{3.2}$$

We recall a special sequence of polynomials $J_m(x)$ which were first considered by Faber. Define these monic degree m polynomials in $\mathbb{Z}[x]$ by the generating function

$$\sum_{m=0}^{\infty} J_m(x) q^m := \frac{E_4(z)^2 E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - x} = 1 + (x - 744)q + \cdots . \tag{3.3}$$

One easily verifies that

$$J_0(z) = 1,$$
$$J_1(z) = x - 744,$$
$$J_2(z) = x^2 - 1488x + 159768,$$
$$J_3(z) = x^3 - 2232x^2 + 1069956x - 36866976.$$

Using these polynomials, for each positive integer m we define the monic degree m polynomial $A_m(x)$ by

$$A_m(x) := -\frac{65520}{691}\sigma_{11}(m) + J_m(x) - 264\sum_{n=1}^{m}\sigma_9(n)J_{m-n}(x). \tag{3.4}$$

For the first primes p, we have

$$A_2(x) = x^2 - 1752x + \frac{18289152}{691},$$
$$A_3(x) = x^3 - 2496x^2 + 1327356x - \frac{192036096}{691},$$
$$A_5(x) = x^5 - 3984x^4 + 5201172x^3 - 2400355072x^2 + 257661670110x - \frac{3680691840}{691}.$$

The following theorem is true.

Theorem 3.1. *For positive integers m, we have that*

$$A_m(0) + \frac{762048}{691}\cdot\tau(m) = A_m(1728) - \frac{432000}{691}\cdot\tau(m) = 0.$$

In particular, we have that $\tau(m) = 0$ if and only if $A_m(0) = A_m(1728) = 0$.

The numerics for $p \leq 5$ reveals this phenomenon.

Prime p	Zeros of $A_p(x)$
2	$15.2396\ldots,\quad 1736.7603\ldots$
3	$0.2094\ldots,\quad 767.8855\ldots,\quad 1727.9050\ldots$
5	$0.00002\ldots,\quad 151.5551\ldots,\quad 701.0707\ldots,\quad 1403.3741\ldots,\quad 1727.9999\ldots$

It is natural to ask whether the criterion in Theorem 3.1 is a consequence of deeper phenomena which sheds further light on the nonvanishing of $\tau(n)$. Students will study these polynomials and investigate the rich theory of a new class of functions which underlies all such questions of this type.