INTEGRALITY PROPERTIES OF QUOTIENTS OF WRONSKIANS OF THE ANDREWS-GORDON SERIES

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ABSTRACT. For positive integers $k \geq 2$, we consider the integrality of quotients of Wronskians involving certain normalizations of the Andrews-Gordon q-series

$$\prod_{1 \leq n \not\equiv 0, \pm i (\mathrm{mod}\, 2k+1)} \frac{1}{1-q^n}.$$

This study is motivated by the appearance of these series in conformal field theory as irreducible characters of Virasoro minimal modules. We establish an upper bound on primes p for which such quotients of Wronskians are non-p-integral. Also, for primes p below this bound, we give a criteria for determining the p-integrality of these quotients which depends only on their first few coefficients.

1. Introduction and Statement of Results

Modular functions sometimes appear as characters of infinite-dimensional Lie algebras. They play this role specifically in two-dimensional conformal field theory and vertex operator algebra theory. A particularly well-known example of this is the character equation of the Moonshine Module, which is the classical modular function

$$j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots ,$$

where $q := e^{2\pi i z}$ throughout. Although these characters are not always modular in this way, it can instead be the case that the vector spaces spanned by these irreducible characters are invariant under actions of the modular group. For example, modular forms arise from Wronskians of bases for $SL_2(\mathbb{Z})$-modules (for a more in-depth discussion of these topics, see [2]). Here, we explore Wronskians from Virasoro vertex algebras.

We begin by defining some notation. Throughout, let $k \geq 2$ be an integer. Define c_k and $h_{i,k}$ by

$$c_k := 1 - \frac{3(2k-1)^2}{(2k+1)},$$

$$h_{i,k} := \frac{(2(k-i)+1)^2 - (2k-1)^2}{8(2k+1)},$$

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where i is an integer such that $1 \leq i \leq k$. Then let $a_{i,k}$ be defined by

$$a_{i,k} := h_{i,k} - \frac{c_k}{24} = \frac{(2(k-i)+1)^2}{8(2k+1)} - \frac{1}{24}. \tag{1.1}$$

Now, the character equation $\mathrm{ch}_{i,k}(q)$ for an irreducible lowest-weight module for the Virasoro algebra of central charge c_k and weight $h_{i,k}$ is given by (see [2])

$$\mathrm{ch}_{i,k}(q) = q^{a_{i,k}} \cdot \left(\prod_{1 \leq n \not\equiv 0, \pm i (\mathrm{mod} 2k+1)} \frac{1}{1-q^n} \right). \tag{1.2}$$

Notice that if $p_b(a,n)$ is the number of partitions of n into parts not congruent to 0 or $\pm a$ mod b, then $\mathrm{ch}_{i,k}(q)/q^{a_{i,k}}$ is the generating function for $p_{2k+1}(i,n)$. Such series have been studied extensively by Andrews and Gordon.

We then define the Wronskian and its derivative on the complete set of characters as follows. Let the $k \times k$ matrices $W_k(q)$ and $W_k'(q)$ be defined by

$$W_k(q) := \begin{pmatrix} \mathrm{ch}_{1,k}(q) & \mathrm{ch}_{2,k}(q) & \cdots & \mathrm{ch}_{k,k}(q) \\ \mathrm{ch}'_{1,k}(q) & \mathrm{ch}'_{2,k}(q) & \cdots & \mathrm{ch}'_{1,k}(q) \\ \vdots & \vdots & & \vdots \\ \mathrm{ch}^{(k-1)}_{1,k}(q) & \mathrm{ch}^{(k-1)}_{2,k}(q) & \cdots & \mathrm{ch}^{(k-1)}_{1,k}(q) \end{pmatrix}, \tag{1.3}$$

and

$$W_k'(q) := \begin{pmatrix} \mathrm{ch}'_{1,k}(q) & \mathrm{ch}'_{2,k}(q) & \cdots & \mathrm{ch}'_{k,k}(q) \\ \mathrm{ch}^{(2)}_{1,k}(q) & \mathrm{ch}^{(2)}_{2,k}(q) & \cdots & \mathrm{ch}^{(2)}_{1,k}(q) \\ \vdots & \vdots & & \vdots \\ \mathrm{ch}^{(k)}_{1,k}(q) & \mathrm{ch}^{(k)}_{2,k}(q) & \cdots & \mathrm{ch}^{(k)}_{1,k}(q) \end{pmatrix}, \tag{1.4}$$

where differentiation is given by

$$\left(\sum a(n) q^n \right)' := \sum n a(n) q^n.$$

Notice that this is equivalent to evaluating $\frac{1}{2\pi i} \cdot \frac{d}{dz}$ when $q := e^{2\pi i z}$. Given these matrices, we define the Wronskians $\mathcal{W}_k(q)$ and $\mathcal{W}_k'(q)$ by

$$\mathcal{W}_k(q) := \alpha(k) \cdot \det W_k(q),$$
$$(1.5) \qquad \mathcal{W}'_k(q) := \beta(k) \cdot \det W'_k(q),$$

respectively, where $\alpha(k)$ and $\beta(k)$ are rational numbers defined such that the resulting corresponding power series has leading coefficient of 1 whenever it is non-vanishing.

As referenced in [2], $\mathcal{W}_k(q)$ can be easily expressed in terms of Dedekind's eta-function, which is defined for z in the upper half-plane of $\mathbb{C}$ by

$$\eta(z) := q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n).$$

Namely, if $k \geq 2$, then $\mathcal{W}_k(q) = \eta(z)^{2k(k-1)}$. In contrast, there is no known closed formula for $\mathcal{W}'_k(q)$. So instead, we investigate the quotient

$$(1.6) \qquad \mathcal{F}_k(q) := \frac{\mathcal{W}'_k(q)}{\mathcal{W}_k(q)}.$$

This function is itself a power series with rational coefficients. As $\mathcal{W}_k(q)$ (resp. $\mathcal{W}'_k(q)$) is defined so to have leading coefficient of 1 (resp. 1 when it is non-vanishing), $\mathcal{F}_k(q)$ also has leading coefficient 1 when it is non-vanishing. Natural questions pertaining to this power series are to determine when $\mathcal{F}_k(q)$ vanishes, and what form its coefficients take when they are non-zero. Our first task is to characterize primes p for which $\mathcal{F}_k(q)$ is p-integral (meaning that p does not divide the denominators of its coefficients).

Theorem 1.1. *If $k \geq 2$ and*

$$B(k) = 6k^2 - 31k + 37,$$

then $\mathcal{F}_k(q)$ is p-integral for every prime $p > B(k)$.

This bound is achieved when $k = 6$, where the largest prime appearing in the denominators is $B(6) = 67$. This is not to say, however, that all primes up to this bound can be found in the denominators of the coefficients of $\mathcal{F}_k(q)$. For example, for small k, $\mathcal{F}_k(q)$ has integer coefficients. In particular, for $2 \leq k \leq 5$, we have

$$\mathcal{F}_2(z) = E_4(z), \quad \mathcal{F}_3(z) = E_6(z),$$
$$\mathcal{F}_4(z) = E_4(z)^2, \quad \mathcal{F}_5(z) = E_4(z)E_6(z),$$

where E_{2k} is the standard Eisenstein series (see p. 12 of [1]). Specifically, $E_4(z)$ and $E_6(z)$ are given by

$$(1.7) \qquad E_4(z) = 1 + 240\sum_{n\geq 1}\sum_{d|n} d^3q^n \quad \text{and} \quad E_6(z) = 1 - 504\sum_{n\geq 1}\sum_{d|n} d^5q^n.$$

Therefore, $\mathcal{F}_k(q)$ has integer coefficients for $2 \leq k \leq 5$.

In general, we can construct $\mathcal{F}_k(q)$ from these two Eisenstein series. From [2] we know that $\mathcal{F}_k(q)$ is a weight $2k$ holomorphic modular form on $SL_2(\mathbb{Z})$. These modular forms make a vector space whose dimension is given by (see p. 119 in [1]):

$$\dim(M_{2k}) = \begin{cases} \left\lfloor \frac{2k}{12} \right\rfloor & 2k \equiv 2 (\text{mod } 12), \\ \left\lfloor \frac{2k}{12} \right\rfloor + 1 & 2k \not\equiv 2 (\text{mod } 12). \end{cases} \tag{1.8}$$

In Section 3, we make use of a diagonalized basis with integer coefficients for these forms. This allows us to easily describe the coefficients of $\mathcal{F}_k(q)$ in terms of the first few of its coefficients. In particular, we have the following theorem.

Theorem 1.2. *If the first* $\dim(M_{2k})$ *coefficients of* $\mathcal{F}_k(q)$ *are p-integral, then* $\mathcal{F}_k(q)$ *is p-integral.*

Remark. By explicitly expanding the quotient using techniques similar to those outlined in [2] (see the proof of Lemma 5.1), one can show that when $k \not\equiv 1 \mod 4$ (resp. when $k \not\equiv 1 \mod 3$), $\mathcal{F}_k(q)$ is 2-integral (resp. 3-integral) and $\mathcal{F}_k(q) \equiv 1 \mod 2$ (resp. $\mathcal{F}_k(q) \equiv 1 \mod 3$). In view of Theorem 1.2, such congruences should not come as a surprise, as the Eisenstein series are well-behaved modulo. In particular, is easy to see from (1.7) that both $E_4(z)$ and $E_6(z)$ are congruent to 1 modulo 2 and 3.

The following sections present proofs of the theorems above. In Section 2, we characterize sources of primes for which $\mathcal{F}_k(q)$ is not p-integral and prove Theorem 1.1. In Section 3, we provide a useful basis for M_{2k} and prove Theorem 1.2.

2. Proof of Theorem 1.1

Here we investigate the quotient

$$\mathcal{F}_k(q) := \frac{\mathcal{W}_k'(q)}{\mathcal{W}_k(q)},$$

characterize the source of primes in the denominators of its coefficients, and prove Theorem 1.1. We begin with a calculation of $\alpha(k)$ and $\beta(k)$ as given in (1.5).

Lemma 2.1. $\mathcal{W}_k(q)$ is non-vanishing for all values of k, and the scaling factor $\alpha(k)$ is

$$\begin{aligned} \alpha(k) &= \left(\prod_{1 \le i < j \le k} (a_{j,k} - a_{i,k}) \right)^{-1} \\ &= \frac{(2(2k+1))^{\frac{k(k-1)}{2}}}{\prod_{1 \le i < j \le k} (j-i)(j+i-(2k+1))}. \end{aligned} \tag{2.1}$$

Additionally,

$$\beta(k) = \begin{cases} 0 & \text{if } a_{i,k} = 0 \text{ for some } 1 \le i \le k, \\ \alpha(k)\left(\prod_{1\le i\le k} a_{i,k}\right)^{-1} & \text{otherwise.} \end{cases} \tag{2.2}$$

Remark. Note that when $\beta(k)$ is non-zero, $\mathcal{W}'_k(q)$ (and thus $\mathcal{F}_k(q)$) is non-zero. Alternatively, $\mathcal{W}'_k(q)$ vanishes precisely when $\beta(k)$ vanishes. Thus $\mathcal{F}_k(q)$ vanishes precisely when $a_{i,k} = 0$ for some $1 \le i \le k$.

Proof. By applying the differentiation operator and expanding the terms of $W_k(q)$ we have

$$\mathcal{W}_k(q) = \alpha(k)\cdot\det\begin{vmatrix} q^{a_{1,k}} + \cdots & \cdots & q^{a_{k,k}} + \cdots \\ a_{1,k}q^{a_{1,k}} + \cdots & \cdots & a_{k,k}q^{a_{k,k}} + \cdots \\ \vdots & & \vdots \\ a_{1,k}^{k-1}q^{a_{1,k}} + \cdots & \cdots & a_{k,k}^{k-1}q^{a_{k,k}} + \cdots \end{vmatrix}.$$

Notice that the smallest power of q in the resulting power series can be calculated exclusively in terms of these leading values. Therefore the leading term of $\mathcal{W}_k(q)$ can be calculated exclusively in terms of the leading coefficients of each of these entries. In other words, the leading coefficient c of $\mathcal{W}_k(q)$ (if $\mathcal{W}_k(q) \neq 0$, then $c = 1/\alpha(k)$ by definition) can be evaluated by

$$c = \det\begin{vmatrix} 1 & \cdots & 1 \\ a_{1,k} & \cdots & a_{k,k} \\ \vdots & & \vdots \\ a_{1,k}^{k-1} & \cdots & a_{k,k}^{k-1} \end{vmatrix}.$$

This is a Vandermonde determinant, whose evaluation is well known. Precisely, we have

$$c = \prod_{1\le i<j\le k}(a_{j,k} - a_{i,k}).$$

In particular, we can clearly see from this relationship that since each $a_{i,k}$ is distinct as i varies, c never evaluates to zero. Thus, $\mathcal{W}_k(q)$ itself never evaluates to zero, and

$$\begin{aligned} \alpha(k) &= \left(\prod_{1\le i<j\le k}(a_{j,k} - a_{i,k})\right)^{-1} \\ &= \frac{(2(2k+1))^{\frac{k(k-1)}{2}}}{\prod_{1\le i<j\le k}(j-i)(j+i-(2k+1))}. \end{aligned}$$

Observe that the leading term of each entry of $W_k'(q)$ has the coefficient of the corresponding entry in $W_k(q)$, but scaled by a factor of $a_{i,k}$ in column i. Thus, the leading term c' of $\mathcal{W}_k'(q)$ is

$$c' = \frac{1}{\alpha(k)} \prod_{1\le i\le k} a_{i,k}.$$

Since $\alpha(k)$ is finite, c' clearly evaluates to zero precisely when $a_{i,k} = 0$ for some $1 \le i \le k$. Thus,

$$\beta(k) = \begin{cases} 0 & \text{if } a_{i,k} = 0 \text{ for some } 1 \le i \le k, \\ \alpha(k)\left(\prod_{1\le i\le k} a_{i,k}\right)^{-1} & \text{otherwise.} \end{cases}$$

Thus concludes our proof. □

Lemma 2.2. The primes p exceeding $2k+1$ for which the coefficients of $\mathcal{F}_k(q)$ are non-p-integral are bounded by the prime factors of the numerators of $\{a_{i,k}\}_{1\le i\le k}$.

Proof. First note that the coefficients of the character functions $\mathrm{ch}_{i,k}(q)$, as given in (1.2), are integral. Thus, the only source of non-integral coefficients in the entries of $W_k(q)$ and $W_k'(q)$ are powers of $a_{i,k}$ introduced by differentiation.

Let R_k be the power series ring $P_k[[q]]$, where $P_k := \mathbb{Z}[\{\frac{1}{p_i} \mid p_i \le (2k+1), p \text{ prime }\}]$. In other words, R_k is the ring of power series whose coefficients are rational and have denominators whose prime factors are bounded above by $2k+1$. The prime factors of the denominators of each $a_{i,k}$ are bounded above by $2k+1$, so $\mathrm{ch}_{i,k}^{(j)}(q) \in R_k$ for all i and j. The determinants of $W_k(q)$ and $W_k'(q)$ are both linear combinations (with integral coefficients) of these character equations, so $\det(W_k(q)), \det(W_k'(q)) \in R_k$. By Lemma 2.1, $\alpha(k)^{-1}$ is an element of P_k, so $\mathcal{W}_k(q) \in R_k$. $\mathcal{W}_k(q)$ has leading coefficient 1 by definition, so its inverse is also in R_k. So if $F_k(q)$ is defined by

$$F_k(q) := \det(W_k'(q)) \cdot (\alpha(k) \cdot \mathcal{W}_k(q))^{-1},$$

(so $\mathcal{F}_k(q) = \left(\prod_{1\le i\le k} a_{i,k}\right)^{-1} \cdot F_k(q)$ by Lemma 2.1), then $F_k(q) \in R_k$.

However, we are not guaranteed that $\left(\prod_{1\le i\le k} a_{i,k}\right)^{-1}$ is an element of P_k. Indeed, for even $k = 2$, $\left(\prod_{i\le 2} a_{i,2}\right)^{-1} = -3600/11 \notin P_2$. Therefore, there may be primes p dividing the denominators of the coefficients of $\mathcal{F}_k(q)$ which exceed $2k+1$. If there are, it follows then that p must be a prime divisor of the numerator of some $a_{i,k}$. □

Proof of Theorem 1.1. Following from Lemmas 2.1 and 2.2, we need only find bounds for primes resulting from numerators of $a_{i,k}$. From (1.1) we can see that $a_{i,k}$ is strictly decreasing over $1 \le i \le k$, so is maximal at $i = 1$. Simplifying $a_{i,k}$ we have

$$a_{i,k} = \frac{6i^2 - 6(2k+1)i + (2k+1)(3k+1)}{12(2k+1)}.$$

Thus, the largest prime appearing in the denominators of $\mathcal{F}_k(q)$ are bounded above by the largest value of

$$b_k(i) := 6i^2 - 6(2k+1)i + (2k+1)(3k+1).$$

However

$$b_k(1) = (6k-1)(k-1)$$

and

$$b_k(2) = (6k-13)(k-1),$$

which do not have large prime factors when evaluated at k since the polynomials themselves factor. The largest $b_k(i)$ which does not factor as a polynomial in k is

$$b_k(3) = 6k^2 - 31k + 37.$$

Note that for $k > 5$, $b_k(3) > (6k-1)$, and as demonstrated in the example following theorem 1.1, $\mathcal{F}_k(q)$ is p-integral for all primes when $k \leq 5$. Thus we have that all primes p for which $\mathcal{F}_k(q)$ is *not* p-integral are bounded above by $B(k) := b_k(3) = 6k^2 - 31k + 37$ for all $k \geq 2$. □

3. Proof of Theorem 1.2

Here we use facts about modular forms over $SL_2(\mathbb{Z})$ to understand the coefficients of $\mathcal{F}_K(q)$ (see [1] for background). As mentioned in Section 1, we can describe M_{2k} (the set of holomorphic modular forms of weight $2k$ over $SL_2(\mathbb{Z})$) as a vector space over $\mathbb{C}$. We know that the dimension of M_{2k} is given by (1.8). Recall that E_4 and E_6, the standard Eisenstein series given by (1.7), are elements of M_4 and M_6 respectively, and both have integer coefficients. Note also that these each have leading term 1. We will also make use of the standard delta function, given by

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + 252q^3 \dots . \tag{3.1}$$

This function is also modular, and is an element of M_{12} with integer coefficients. The integrality of its coefficients follow from the formulas for $E_4(z)$ and $E_6(z)$, or from its infinite product expansion, given by

$$\Delta(z) = q \cdot \prod_{n=1}^{\infty} (1-q^n)^{24}.$$

From these forms, we will define a basis $\mathcal{B}_k$ for M_{2k} as follows.

For $2 \leq k \leq 7$, let $\mathcal{B}_k$ be defined by

$$\begin{array}{lll} \mathcal{B}_2 := \{E_4(z)\}, & \mathcal{B}_3 := \{E_6(z)\}, & \mathcal{B}_4 := \{E_4(z)^2\}, \\ \mathcal{B}_5 := \{E_4(z)E_6(z)\}, & \mathcal{B}_6 := \{E_4(z)^3, \Delta(z)\}, & \mathcal{B}_7 := \{E_4(z)^2E_6(z)\}. \end{array}$$

Notice that each of these bases are diagonalized. That is, for each $0 \leq i \leq \dim(M_{2k})-1$, there is one basis element which has minimal degree i. From these, we define $\mathcal{B}_k$ recursively as follows. If $\mathcal{B}_k$ is given by

$$\mathcal{B}_k = \{b_0(z),\ \ldots,\ b_{\dim(M_{2k})-1}(z)\},$$

where the leading degree of $b_i(z)$ is i, then let $\mathcal{B}_{k+6}$ be defined by

$$\mathcal{B}_{k+6} := \{E_4(z)^3 b_0(z),\ \Delta(z)b_0(z),\ \Delta(z)b_1(z),\ \ldots,\ \Delta(z)b_{\dim(M_{2k})-1}(z)\}. \tag{3.2}$$

As a quick check, we can see first that by (1.8) that $\mathcal{B}_{k+6}$ should have exactly one more element than $\mathcal{B}_k$. Also, since both $E_4(z)^3$ and $\Delta(z)$ have weight 12, multiplying by these will increase the weights of basis elements by the desired amount. Finally, if $\mathcal{B}_k$ is diagonalized, then $\mathcal{B}_{k+6}$ is diagonalized, and thus contains linearly independent elements. Using this basis, we can easily proceed into a proof of Theorem 1.2.

Proof of Theorem 1.2. We know that $\mathcal{F}_k(q)$ is an element of M_{2k}, and thus can be expressed as a linear combination of elements of $\mathcal{B}_k$. Let

$$\mathcal{F}_k(q) = f_0 b_0(z) + \cdots + f_{\dim(M_{2k})-1} b_{\dim(M_{2k})-1}(z),$$

where $f_i \in \mathbb{Q}$. Since every element of $\mathcal{B}_k$ has integer coefficients, if $\mathcal{F}_k(q)$ is not p-integral for some prime p, then there must be some f_i which is not p-integral. Since $\mathcal{B}_k$ is diagonalized, then it follows that the i^{th} coefficient of $\mathcal{F}_k(q)$ must not be p-integral. Therefore, if $\mathcal{F}_k(q)$ is p-integral for the first $\dim(M_{2k})$ coefficients, then it must be p-integral overall. □

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