Integration by Substitution

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Abstract

This handout contains material on a very important integration method called integration by substitution. Substitution is to integrals what the chain rule is to derivatives.

1 Integration by Substitution (5.5)

The Fundamental Theorem of Calculus tells us that in order to evaluate an integral, we need to find an antiderivative of the function we are integrating (the integrand). However, the list of antiderivatives we have is rather short, and does not cover all the possible functions we will have to integrate. For example, $\int xe^{x^2}dx$ is not in our list. Neither is $\int 2x\sqrt{1+x^2}dx$. What do we do then? One method, the one we will study in this handout, involves changing the integral so that it looks like one we can do, by doing a change of variable, also called a substitution. Substitution for integrals corresponds to the chain rule for derivatives.We look at some simple examples to illustrate this.

Before we see how to do this, we need to review another concept, the differential.

1.1 The Differential

You will recall from differential calculus that the notation dx meant a small change in the variable x. It has a name, it is called the differential (of the variable x). Now, if $y = f(x)$ and f is a differential function, we may also be interested in finding the differential of y, denoted dy.

Definition 1 *The differential dy is defined by*

$$dy = f'(x)\,dx$$

Example 2 *Find dy if $y = x^2$*
By definition

$$\begin{aligned} dy &= \left(x^2\right)' dx \\ &= 2xdx \end{aligned}$$

Example 3 *Find dy if $y = \sin x$*
By definition

$$\begin{aligned} dy &= (\sin x)' \, dx \\ &= \cos x dx \end{aligned}$$

1.2 The Substitution Rule for Indefinite Integrals

We are now ready to learn how to integrate by substitution. Substitution applies to integrals of the form $\int f(g(x)) g'(x) \, dx$. If we let $u = g(x)$, then $du = g'(x) \, dx$. Therefore, we have

$$\int f(g(x)) g'(x) \, dx = \int f(u) \, du \tag{1}$$

This is the substitution rule formula. Note that the integral on the left is expressed in terms of the variable x. The integral on the right is in terms of u. The key when doing substitution is, of course, to know which substitution to apply. At the beginning, it is hard. With practice, it becomes easier. Also, looking at equation 1 and trying to understand the pattern will make things easier. In that formula, it is assumed that we can integrate the function f. Looking at the integral on the left, one sees the function f. But the integral also has extra expressions. Inside of f, there is an expression in term of x. Outside of f, is the derivative of this expression. When this is the case, the expression will be the substitution. For example, given $\int 2x \sin\left(x^2 dx\right)$, one would use $u = x^2$ as the substitution. Given $\int \cos x \sqrt{\sin x} dx$, one would use $u = \sin x$ as the substitution. Let us look at some examples.

Example 4 *Find $\int 2x \sin\left(x^2\right) dx$*
If $u = x^2$, then $du = 2x dx$, therefore

$$\begin{aligned} \int 2x \sin\left(x^2\right) dx &= \int \sin\left(x^2\right) 2x dx \\ &= \int \sin u du \end{aligned}$$

The last integral is a known formula

$$\int \sin u du = -\cos u + C$$

The original problem was given in terms of the variable x, you must give your answer in terms of x. Therefore,

$$\int 2x \sin\left(x^2\right) dx = -\cos\left(x^2\right) + C$$

Example 5 *Find* $\int xe^{x^2}dx$
If $u = x^2$*, then* $du = 2xdx$*, therefore*

$$\begin{aligned}\int xe^{x^2}dx &= \int e^{x^2}xdx \\ &= \int e^u \frac{du}{2} \\ &= \frac{1}{2}\int e^u du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C\end{aligned}$$

Example 6 *Find* $\int x^3\cos\left(x^4+1\right)dx$
If $u = x^4 + 1$*, then* $du = 4x^3dx$*, therefore*

$$\begin{aligned}\int x^3\cos\left(x^4+1\right)dx &= \int \cos\left(x^4+1\right)x^3dx \\ &= \int \cos u \frac{du}{4} \\ &= \frac{1}{4}\int \cos u du \\ &= \frac{1}{4}\sin u + C \\ &= \frac{1}{4}\sin\left(x^4+1\right) + C\end{aligned}$$

Example 7 *Find* $\int \tan xdx$
If we think of $\tan x$ *as* $\dfrac{\sin x}{\cos x}$ *and let* $u = \cos x$*, then* $du = -\sin xdx$*, therefore*

$$\begin{aligned}\int \tan xdx &= \int \frac{\sin x}{\cos x}dx \\ &= \int \frac{1}{u}(-1)\,du \\ &= -\int \frac{1}{u}du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C\end{aligned}$$

This is not the formula usually remembered. Since $\cos x = \dfrac{1}{\sec x}$*, and, one of* *the properties of logarithmic functions says that* $\ln \frac{a}{b} = \ln a - \ln b$*, we have*

$$\begin{aligned}\int \tan x dx &= -\ln|\cos x| + C \\ &= -\ln\frac{1}{|\sec x|} + C \\ &= -\ln 1 - (-\ln|\sec x|) + C \\ &= \ln|\sec x| + C\end{aligned}$$

Example 8 *Find* $\int 2x\sqrt{x^2+1}dx$
If $u = x^2 + 1$*, then* $du = 2xdx$*, therefore*

$$\begin{aligned}\int 2x\sqrt{x^2+1}dx &= \int \sqrt{u}du \\ &= \int u^{\frac{1}{2}}du \\ &= \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} + C\end{aligned}$$

1.3 The Substitution Rule for Definite Integrals

With definite integrals, we have to find an antiderivative, then plug in the limits of integration. We can do this one of two ways:

1. Use substitution to find an antiderivative, express the answer in terms of the original variable then use the given limits of integration.

2. Change the limits of integration when doing the substitution. This way, you won't have to express the antiderivative in terms of the original variable. More precisely,

$$\int_a^b f\left(g\left(x\right)\right)g'\left(x\right)dx = \int_{g(a)}^{g(b)} f\left(u\right)du$$

We illustrate these two methods with examples.

Example 9 *Find* $\int_1^e \frac{\ln x}{x}dx$ *using the first method.*
First, we find an antiderivative of the integrand, and express it in term of x*. If*

$u = \ln x$, *then* $du = \frac{1}{x}dx$. *Therefore*

$$\begin{aligned} \int \frac{\ln x}{x} dx &= \int u du \\ &= \frac{u^2}{2} \\ &= \frac{(\ln x)^2}{2} \end{aligned}$$

It follows that

$$\begin{aligned} \int_1^e \frac{\ln x}{x} dx &= \left. \frac{(\ln x)^2}{2} \right|_1^e \\ &= \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} \\ &= \frac{1}{2} \end{aligned}$$

Example 10 *Same problem using the second method.*
The substitution will be the same, but we won't have to express the antiderivative in terms of x. *Instead, we will find what the limits of integration are in terms of* u. *Since* $u = \ln x$, *when* $x = 1$, $u = \ln 1 = 0$. *When* $x = e$, $u = \ln e = 1$. *Therefore,*

$$\begin{aligned} \int_1^e \frac{\ln x}{x} dx &= \int_0^1 u du \\ &= \left. \frac{u^2}{2} \right|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

Remark 11 *A special case of substitution is renaming a variable in an integral. You will recall that* $\int_a^b f(x)\, dx = \int_a^b f(u)\, du$. *In this case, we just performed the trivial substitution* $u = x$, *in other words, we simply renamed the variable. This can always be done, however, it does not accomplish anything. Sometimes we do it for display purposes, as we will see in the next theorem.*

1.4 Integrating Even and Odd Functions

Definition 12 *A function* f *is even if* $f(-x) = f(x)$. *It is odd if* $f(-x) = -f(x)$

Example 13 $f(x) = x^2$ *is even. In fact* $f(x) = x^n$ *is even if* n *is even.*

Example 14 $f(x) = x^3$ *is odd. In fact* $f(x) = x^n$ *is odd if* n *is odd.*

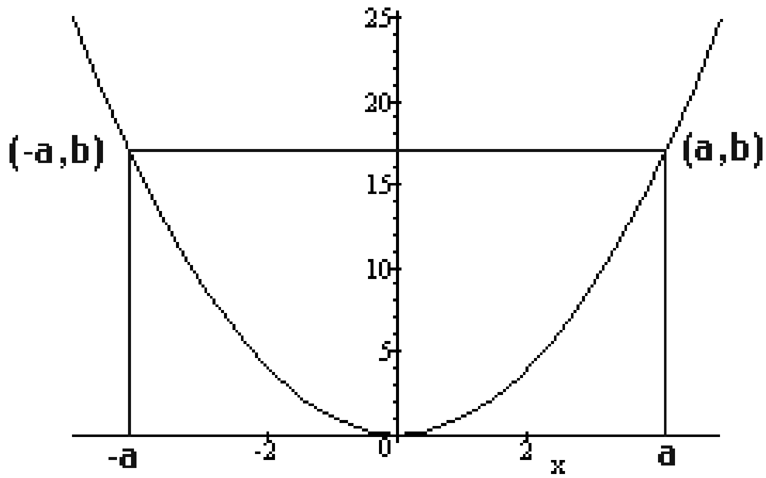

Figure 1: Even Function

Example 15 $\sin(-x) = -\sin x$, *therefore* $\sin x$ *is odd.*

Example 16 $\cos(-x) = \cos x$, *therefore* $\cos x$ *is even.*

Example 17 *From the previous two examples, it follows that* $\tan x$ *and* $\cot x$ *are odd*

The graph of an even function is symmetric with respect to the y-axis. The graph of an odd function is symmetric with respect to the origin. Another way of thinking about it is the following. If f is even and (a, b) is on the graph of f, then $(-a, b)$ is also on the graph of f. If f is odd and (a, b) is on the graph of f, then $(-a, -b)$ is also on the graph of f. This is illustrated on figure 1 for even functions, and on figure 2 for odd functions.

Knowing if a function is even or odd can make integrating it easier.

Theorem 18 *Suppose that* f *is continuous on* $[-a, a]$ *then:*

1. *If* f *is even, then* $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. *If* f *is odd, then* $\int_{-a}^{a} f(x)\,dx = 0$

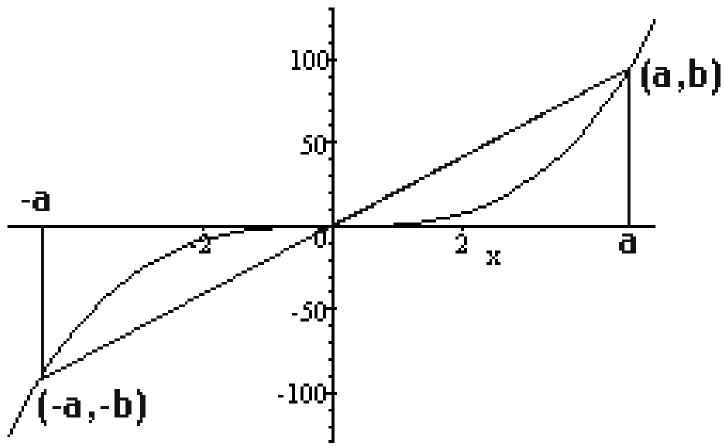

Figure 2: Odd Function

Proof. *Using the properties of integrals, we have:*

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx\end{aligned}$$

In the first integral, we use the substitution $u = -x$ so that $du = -dx$, we obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx$$

For clarity, use the substitution $u = x$ to obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx \tag{2}$$

We now consider the cases f is even and odd separately.

- *case 1: f is even. In this case, $f(-x) = f(x)$. Therefore, equation 2 becomes*

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 2\int_{0}^{a} f(x)\,dx\end{aligned}$$

- *case 2: f is odd. In this case, $f(-x) = -f(x)$. Therefore, equation 2 becomes*

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx \\ &= 0\end{aligned}$$

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Remark 19 *The first part of the theorem does not save us a lot of work. We still have to be able to find an antiderivative in order to evaluate the integral. However, in the second part, we only need to know the function is odd. If it is, then the integral will be 0, there is no need to be able to find an antiderivative of the integrand.*

Example 20 *Find* $\int_{-1}^{1} \frac{\tan x}{x^4 + x^2 + 1} dx$

Let $f(x) = \frac{\tan x}{x^4 + x^2 + 1}$. *The reader can verify that f is an odd function, therefore*

$$\int_{-1}^{1} \frac{\tan x}{x^4 + x^2 + 1} dx = 0$$

1.5 Things to Know

- Be able to integrate using the substitution method.
- Know what odd and even functions are, be able to recognize them and know how to integrate them on an interval of the form $[-a, a]$.
- Related problems assigned: # 1, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 29, 31, 37, 41, 49, 61,63, 64, 65 on pages 392, 393
- Try this more challenging problem: $\int \frac{xe^{2x}}{(2x+1)^2} dx$