Integration Involving Trigonometric Functions and Trigonometric Substitution

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Abstract

This handout describes techniques of integration involving various combinations of trigonometric functions. It also describes a technique known as trigonometric substitution. Students may want to review some basic trigonometric identities before reading further. The following trigonometric identities will be used:

• \( \sin^2 x + \cos^2 x = 1 \)
• \( 1 + \tan^2 x = \sec^2 x \)
• \( \sin^2 x = \frac{1 - \cos 2x}{2} \)
• \( \cos^2 x = \frac{1 + \cos 2x}{2} \)
• \( \sin 2x = 2 \sin x \cos x \)

In addition, students need to remember the following:

• \( \sin x \) is invertible when \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \). Its inverse is denoted \( \sin^{-1} x \).
• \( \tan x \) is invertible when \( -\frac{\pi}{2} < x < \frac{\pi}{2} \). Its inverse is denoted \( \tan^{-1} x \).
• \( \sec x \) is invertible when \( 0 \leq x < \frac{\pi}{2} \) or \( \pi \leq x < \frac{3\pi}{2} \). Its inverse is denoted \( \sec^{-1} x \).

1 Powers of Sine and Cosine

Before we explain the technique, let us recall that we can integrate integrals of the form \( \int \sin^n x \cos x \, dx \) or \( \int \cos^n x \sin x \, dx \) where \( n \) is a positive integer, by using substitution. For example, to integrate \( \int \sin^n x \cos x \, dx \), we let \( u = \sin x \).
then \( du = \cos x \, dx \). Therefore

\[
\int \sin^n x \cos x \, dx = \int u^n \, du = \frac{u^{n+1}}{n+1} = \frac{\sin^{n+1} x}{n+1}
\]

The other integral is done similarly.

The technique used here depends on whether one of the powers is odd or both are even. We summarize the techniques, then do some examples.

**Proposition 1** Suppose we have an integral of the form \( \int \sin^m x \cos^n x \, dx \).

1. If \( n \) is odd, that is \( n = 2k + 1 \), then save one cosine factor, and use the identity \( \sin^2 x + \cos^2 x = 1 \) to express the remaining factors in terms of sine. Then, use the substitution \( u = \sin x \). In other words

\[
\int \sin^m x \cos^n x \, dx = \int \sin^m x \cos^{2k+1} x \, dx
\]

\[
= \int \sin^m x \cos^{2k} x \cos x \, dx
\]

\[
= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx
\]

2. If \( m \) is odd, that is \( m = 2k + 1 \), then save one sine factor, and use the identity \( \sin^2 x + \cos^2 x = 1 \) to express the remaining factors in terms of cosine. Then, use the substitution \( u = \cos x \). In other words

\[
\int \sin^m x \cos^n x \, dx = \int \sin^{2k+1} x \cos^n x \, dx
\]

\[
= \int \sin^{2k} x \cos^n x \sin x \, dx
\]

\[
= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx
\]

3. If both \( m \) and \( n \) are even, we use the half-angle identities

\[
\sin^2 x = \frac{1 - \cos 2x}{2}
\]

\[
\cos^2 x = \frac{1 + \cos 2x}{2}
\]

as well as the identity

\[
\sin x \cos x = \frac{\sin 2x}{2}
\]
Example 2 Find $\int \cos^3 x dx$

This is the case where the power of cosine is odd. We save one cosine factor and write

$$\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x$$

Therefore,

$$\int \cos^3 x dx = \int (1 - \sin^2 x) \cos x dx$$

$$= \int \cos x dx - \int \sin^2 x \cos x dx$$

(1)

The first integral is known

$$\int \cos x dx = \sin x$$

(2)

The second integral can be evaluated using the substitution $u = \sin x \implies du = \cos x dx$ and therefore

$$\int \sin^2 x \cos x dx = \int u^2 du$$

(3)

$$= \frac{u^3}{3}$$

$$= \frac{\sin^3 x}{3}$$

Using equation 2 and equation 3 in equation 1 gives us

$$\int \cos^3 x dx = \sin x - \frac{\sin^3 x}{3} + C$$

Example 3 Find $\int \sin^5 x \cos^2 x dx$

This is the case where the power of sine is odd. We save one sine factor and write

$$\sin^5 x \cos^2 x = \sin^4 x \cos^2 x \sin x$$

$$= (\sin^2 x)^2 \cos^2 x \sin x$$

$$= (1 - \cos^2 x)^2 \cos^2 x \sin x$$

$$= (1 - 2 \cos^2 x + \cos^4 x) \cos^2 x \sin x$$

$$= \cos^2 x \sin x - 2 \cos^4 x \sin x + \cos^6 x \sin x$$
Therefore,

\[ \int \sin^5 x \cos^2 x \, dx = \int (\cos^2 x - 2 \cos^4 x + \cos^6 x) \sin x \, dx \]

We then use the substitution \( u = \cos x \implies du = -\sin x \, dx \) to get

\[ \int \sin^5 x \cos^2 x \, dx = -\int (u^2 - 2u^4 + u^6) \, du \]

\[ = -\left( \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C \]

\[ = -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C \]

**Example 4** Find \( \int \sin^2 x \, dx \)

This is the case when the powers of sine and cosine are even (the power of cosine being 0). We use the half angle identity \( \sin^2 x = \frac{1 - \cos 2x}{2} \) to obtain

\[ \int \sin^2 x \, dx = \frac{1}{2} \int (1 - \cos 2x) \, dx \]

We use the substitution \( u = 2x \implies du = 2 \, dx \) to get

\[ \int \sin^2 x \, dx = \frac{1}{4} \int (1 - \cos u) \, du \]

\[ = \frac{1}{4} (u - \sin u) + C \]

\[ = \frac{1}{4} (2x - \sin 2x) + C \]

\[ = \frac{x}{2} - \frac{\sin 2x}{4} + C \]

Similar techniques can be applied to powers of tangent and secant. We will not cover them here. They can be found in most Calculus books.

### 2 Trigonometric Substitution

The techniques we are about to describe apply to integrals containing expressions of the form

\[ \sqrt{a^2 - x^2} \]
\[ \sqrt{a^2 + x^2} \]
\[ \sqrt{x^2 - a^2} \]

for which the other techniques have failed. For example, if we were given \( \int x \sqrt{1 - x^2} \, dx \), the substitution \( u = 1 - x^2 \) would work. However, if we were
given \( \int \sqrt{1 - x^2} \, dx \), it would be much more difficult to do. We will look at each case separately. Before we do this, it is important to keep in mind an important difference between the substitution technique learned before and the one we are about to explain. In the traditional substitution, we define the new variable in terms of the old. For example, \( u = 1 - x^2 \). In trigonometric substitution, we redefine the given variable.

**Remark 5** In order to be able to do this substitution successfully, you must be able to find all the trigonometric functions, knowing one of them. This can be done either by using trigonometric identities or a triangle. This technique can be found in any book dealing with trigonometric functions. It can also be found on the handout linked to on the web site for the class.

### 2.1 Integral Containing \( \sqrt{a^2 - x^2} \)

We use the substitution \( x = a \sin \theta \), with \(- \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\) and \( a > 0 \). We impose this restriction on \( \theta \) so that \( \sin \theta \) will have an inverse. This substitution is based on the identity \( 1 - \sin^2 \theta = \cos^2 \theta \) and works as follows:

\[
x = a \sin \theta \implies x^2 = a^2 \sin^2 \theta \\
\implies a^2 - x^2 = a^2 - a^2 \sin^2 \theta \\
= a^2 (1 - \sin^2 \theta) \\
= a^2 \cos^2 \theta
\]

Therefore

\[
\sqrt{a^2 - x^2} = \sqrt{a^2 \cos^2 \theta} \\
= |a \cos \theta| \\
= |a| |\cos \theta| \\
= a \cos \theta
\]

We were able to remove the absolute value because \( a > 0 \) and \( \cos \theta \geq 0 \) when \(- \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\). We illustrate this with examples.

**Example 6** Find \( \int \frac{\sqrt{9 - x^2}}{x^2} \, dx \)

We let \( x = 3 \sin \theta \), with \(- \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\). Then \( dx = 3 \cos \theta \). Also, as noted above,
\[
\sqrt{9-x^2} = 3 \cos \theta. \text{ Therefore, }
\]
\[
\int \frac{\sqrt{9-x^2}}{x^2} \, dx = \int \frac{3 \cos \theta}{9 \sin^2 \theta} \, 3 \cos \theta \, d\theta
\]
\[
= \int \frac{\cos^2 \theta}{\sin^2 \theta} \, d\theta
\]
\[
= \int \cot^2 \theta \, d\theta
\]
\[
= \int (\csc^2 \theta - 1) \, d\theta
\]
\[
= - \cot \theta - \theta + C
\]

We need to express our answer in terms of \(x\). Since \(x = 3 \sin \theta\), it follows that \(\theta = \sin^{-1} \frac{x}{3}\). Also, either using trigonometric identities, or a triangle, we find that \(\cot \theta = \frac{\sqrt{9-x^2}}{x}\). Therefore,
\[
\int \frac{\sqrt{9-x^2}}{x^2} \, dx = - \frac{\sqrt{9-x^2}}{x} - \sin^{-1} \frac{x}{3} + C
\]

Example 7 Find \(\int_0^2 \sqrt{4-x^2} \, dx\)

**Method 1** We recognize that \(\sqrt{4-x^2}\) is the upper half circle of radius 2 centered at the origin. The integral of it between 0 and 2 corresponds to the area of the first quadrant of this circle. Therefore
\[
\int_0^2 \sqrt{4-x^2} \, dx = \frac{1}{4} \text{ (area of a circle of radius 2)}
\]
\[
= \frac{1}{4} (2 \pi)
\]
\[
= \pi
\]

This method is very quick and easy. However, it would not work if the problem had been to find an antiderivative. We show another technique, using trigonometric substitution.

**Method 2** According to what was explained above, we let \(x = 2 \sin \theta\). Then, \(\sqrt{4-x^2} = 2 \cos \theta\). Also, \(dx = 2 \cos \theta \, d\theta\). To find the value of this integral, we will first find an antiderivative, then use the given limits of integration. Therefore,
\[
\int \sqrt{4-x^2} \, dx = 4 \int \cos \theta \cos \theta \, d\theta
\]
\[
= 4 \int \cos^2 \theta \, d\theta
\]
Remembering the techniques of the previous section, we use $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$. Therefore,

$$\int \sqrt{4 - x^2} \, dx = 4 \int \frac{1 + \cos 2\theta}{2} \, d\theta$$

$$= 2 \int (1 + \cos 2\theta) \, d\theta$$

If we let $u = 2\theta$, then $du = 2 \, d\theta$ and we have

$$\int \sqrt{4 - x^2} \, dx = \int (1 + \cos u) \, du$$

$$= u + \sin u$$

$$= 2\theta + \sin 2\theta$$

$$= 2\theta + 2 \sin \theta \cos \theta$$

We obtained the last equality using the identity $\sin 2\theta = 2 \sin \theta \cos \theta$. Now, we write everything back in terms of $x$. First, since $x = 2 \sin \theta$, we see that

$$\sin \theta = \frac{x}{2}$$

and

$$\theta = \sin^{-1} \frac{x}{2}$$

To express $\cos \theta$ in terms of $x$, we use $\cos^2 \theta = 1 - \sin^2 \theta$ and since $\cos \theta \geq 0$, we have

$$\cos \theta = \sqrt{1 - \sin^2 \theta}$$

$$= \sqrt{1 - \left(\frac{x}{2}\right)^2}$$

$$= \sqrt{\frac{4 - x^2}{4}}$$

$$= \sqrt{\frac{1}{4} (4 - x^2)}$$

$$= \frac{\sqrt{4 - x^2}}{2}$$

Therefore

$$\int \sqrt{4 - x^2} \, dx = 2 \sin^{-1} \frac{x}{2} + 2 \frac{x \sqrt{4 - x^2}}{2}$$

$$= 2 \sin^{-1} \frac{x}{2} + \frac{x \sqrt{4 - x^2}}{2}$$
We can now find the definite integral
\[
\int_0^2 \sqrt{4-x^2} \, dx = 2 \sin^{-1} \frac{x}{2} + \frac{x \sqrt{4-x^2}}{2} \bigg|_0^2
\]
\[= \left( 2 \sin^{-1} 1 + \frac{2\sqrt{0}}{2} \right) - \left( 2 \sin^{-1} 0 + \frac{0 \sqrt{4}}{2} \right)
\]
\[= \left( \frac{\pi}{2} + 0 \right) - (0 + 0)
\]
\[= \pi
\]

2.2 Integral Containing \( \sqrt{a^2 + x^2} \)

We use the substitution \( x = a \tan \theta \), with \( a > 0 \) and \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\). We impose this restriction on \( \theta \) so that \( \tan \theta \) will have an inverse. The substitution is based on the identity \( 1 + \tan^2 \theta = \sec^2 \theta \) and works as follows:

\[
a^2 + x^2 = a^2 + a^2 \tan^2 \theta
\]
\[= a^2 (1 + \tan^2 \theta)
\]
\[= a^2 \sec^2 \theta
\]

Therefore
\[
\sqrt{a^2 + x^2} = \sqrt{a^2 \sec^2 \theta}
\]
\[= a \sqrt{\sec^2 \theta}
\]
\[= a |\sec \theta|
\]
\[= a \sec \theta
\]

Because \( a > 0 \) and \( \sec \theta \geq 0 \) if \(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\).

Example 8 Find \( \int \frac{1}{x^2 \sqrt{x^2 + 4}} \, dx \)

We let \( x = 2 \tan \theta \), \( dx = 2 \sec^2 \theta \, d\theta \). Also, \( \sqrt{x^2 + 4} = 2 \sec \theta \). Therefore:

\[
\int \frac{1}{x^2 \sqrt{x^2 + 4}} \, dx = \int \frac{2 \sec^2 \theta \, d\theta}{(4 \tan^2 \theta) (2 \sec \theta)}
\]
\[= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} \, d\theta
\]

Now,

\[
\frac{\sec \theta}{\tan^2 \theta} = \frac{1}{\cos \theta \sin^2 \theta}
\]
\[= \frac{\cos \theta}{\cos^2 \theta}
\]
\[= \frac{\cos \theta}{\sin^2 \theta}
\]
If we make the substitution \( u = \sin \theta \), then \( du = \cos \theta d\theta \) and we get:

\[
\int \frac{1}{x^2\sqrt{x^2 + 4}} \, dx = \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} \, d\theta \\
= \frac{1}{4} \int \frac{du}{u^2} \\
= \frac{1}{4} \int u^{-2} \, du \\
= -\frac{1}{4u} + C \\
= -\frac{1}{4\sin \theta} + C
\]

We express \( \sin \theta \) in terms of \( x \) and obtain

\[
\sin \theta = \frac{x}{\sqrt{4 + x^2}}
\]

Therefore

\[
\int \frac{1}{x^2\sqrt{x^2 + 4}} \, dx = \frac{-\sqrt{4 + x^2}}{4x} + C
\]

2.3 Integral Containing \( \sqrt{x^2 - a^2} \)

We use the substitution \( x = a \sec \theta \), with \( a > 0 \) and \( 0 \leq \theta < \frac{\pi}{2} \) or \( \pi \leq \theta < \frac{3\pi}{2} \).

We impose this restriction on \( \theta \) so that \( \sec \theta \) will be invertible. This substitution is based on the identity \( \sec^2 \theta - 1 = \tan^2 \theta \) and works as follows:

\[
x^2 - a^2 = a^2 \sec^2 \theta - a^2 \\
= a^2 (\sec^2 \theta - 1) \\
= a^2 \tan^2 \theta
\]

Therefore

\[
\sqrt{x^2 - a^2} = \sqrt{a^2 \tan^2 \theta} \\
= |a| |\tan \theta| \\
= a \tan \theta
\]

because \( a > 0 \) and \( \tan \theta \geq 0 \) when \( 0 \leq \theta < \frac{\pi}{2} \) or \( \pi \leq \theta < \frac{3\pi}{2} \).

Example 9 Find \( \int \frac{dx}{\sqrt{x^2 - a^2}} \), where \( a > 0 \).

According to the explanation above, we let \( x = a \sec \theta \). Then, \( dx = a \sec \theta \tan \theta \, d\theta \).
Also, $\sqrt{x^2 - a^2} = a \tan \theta$. Therefore

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec \theta \tan \theta d\theta}{a \tan \theta}$$

$$= \int \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| + C$$

(see homework 1). Now, we need to write everything in terms of $x$. $\sec \theta = \frac{x}{a}$ and $\tan \theta = \frac{\sqrt{x^2 - a^2}}{a}$. Therefore,

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C$$

$$= \ln |x + \sqrt{x^2 - a^2}| - \ln a + C$$

$$= \ln |x + \sqrt{x^2 - a^2}| + C$$

3 Problems

1. Find $\int \sec \theta d\theta$. (hint: multiply both numerator and denominator by $\sec \theta + \tan \theta$)

2. Find $\int \cos^5 x \sin^4 x dx$

3. Find $\int \cos^4 x dx$

4. Using the technique of example 7, find $\int \sqrt{a^2 - x^2} dx$

5. Find $\int \frac{x}{\sqrt{1 - x^2}} dx$

6. Find $\int \sqrt{1 - 4x^2} dx$

7. Find $\int \frac{x}{\sqrt{x^2 + 3}} dx$

8. Find $\int e^{\sqrt{9 - e^{2t}}} dt$

4 Answers

1. $\int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C$

2. $\int \cos^5 x \sin^5 x dx = \frac{1}{6} \sin^6 x - \frac{1}{4} \sin^8 x + \frac{1}{10} \sin^{10} x + C$

3. $\int \cos^4 x dx = \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x$
4. $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$

5. $\int \frac{x}{\sqrt{1-x^2}} \, dx = -\sqrt{1-x^2} + C$

6. $\int \sqrt{1-4x^2} \, dx = \frac{1}{4} \sin^{-1} 2x + \frac{1}{2} x \sqrt{1-4x^2} + C$

7. $\int \frac{x}{\sqrt{x^2+3}} \, dx = \sqrt{x^2+3} + C$

8. $\int e^{\sqrt{9-e^{2t}}} \, dt = \frac{1}{2} \left[ e^{\sqrt{9-e^{2t}}} + 9 \sin^{-1} \frac{e^{t}}{3} \right] + C$