Abstract

This handout is a summary of the most commonly used tests which are used to determine if a series converges.

1 Testing a Series for Convergence (8.3)

1.1 Review of what is known up to now

- In theory, testing a series \( \sum_{i=1}^{\infty} a_i \) for convergence amounts to finding the sequence of partial sums \( S_n = \sum_{i=1}^{n} a_i \) and testing it for convergence. We know that \( \lim_{n \to \infty} S_n = \sum_{i=1}^{\infty} a_i \). So, if the sequence of partial sums converges, so will the series and they converge to the same number. If the sequence of partial sums diverges, so will the series. In theory, this sounds like the problem is solved, since we know how to study the convergence of sequences. In practice though it is not as easy. In general, we will not have a nice formula for \( S_n \), therefore it may be difficult to determine if it converges. This technique does work in some examples.

- When the sequence of partial sums cannot be used, there are several alternatives which we list here:
  - If the sequence is a known series, use your knowledge of that series. This is why, as you learn new series, it is important to remember them. So far, we know the following series:
    1. Harmonic series: \( \sum_{i=1}^{\infty} \frac{1}{i} \). It diverges.
2. Geometric series: $\sum_{i=1}^{\infty} ar^{i-1}$. It converges to $\frac{a}{1-r}$ if $|r| < 1$ and diverges otherwise.

- If the sequence is not known, the first test that should be applied is the test for divergence. It says that if $\lim_{n \to \infty} a_n \neq 0$, then the series will diverge. If this limit is zero, the test provides no conclusion. Other tests must be used.

**Remark 1** It should be obvious to the reader that convergence of a series is not affected by the first finitely many terms of the series. In other words, if $\sum_{i=1}^{\infty} a_i$ converges, so will $\sum_{i=2}^{\infty} a_i$, $\sum_{i=3}^{\infty} a_i$, ..., $\sum_{i=p}^{\infty} a_i$ where $p$ is any positive integer. The same is true for divergence. What will change however, is the sum. Obviously, if $\sum_{i=1}^{\infty} a_i$ converges say to $L$, that is if $\sum_{i=1}^{\infty} a_i = L$, then we also know that $\sum_{i=2}^{\infty} a_i \neq L$. It is easy to find what it is though. $\sum_{i=2}^{\infty} a_i = L - a_1$. (why?)

The rest of this document presents some of the tests which can be used to test series for convergence. We first begin with tests which only apply to series having positive terms. We then look at tests for series having mixed terms.

### 1.2 Testing Series With Positive Terms

#### 1.2.1 The Integral Test

This test compares a series to an integral (improper). It establishes that under certain conditions, converges of a series can be established by studying the convergence of an improper integral. It even goes further. The value of the integral can also be used to approximate the infinite series. However, unlike for geometric series, this test will not give us an exact value for convergent infinite series.

**Theorem 2 (Integral test)** If $f$ is a continuous, positive and decreasing function on $[1, \infty)$ and $a_n = f(n)$ then $\sum_{i=1}^{\infty} a_i$ converges if and only if $\int_{1}^{\infty} f(x) \, dx$ converges.

**Remark 3** The conditions in the integral test do not have to hold starting at $n = 1$, as long as they hold from some point on.

**Remark 4** Before applying this test, make sure that the series satisfies all the conditions of the theorem.
Remark 5 The theorem does not indicate what the series converges to. You should not assume that the value of the integral and the series are the same.

Example 6 Test $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence.

The function defining the general term of this series is $f(x) = \frac{\ln x}{x}$. It is continuous and positive on $[1, \infty)$. To determine if it is decreasing, we study the sign of its derivative. $f'(x) = \frac{1 - \ln x}{x^2}$. Since the denominator is a square, it is always positive. So the sign of the derivative is uniquely determined by its numerator.

$$f'(x) < 0$$
$$1 - \ln x < 0$$
$$1 < \ln x$$
$$\ln x > 1$$
$$x > e$$

Therefore, $f$ is not decreasing on $[1, \infty)$ but it is on $[3, \infty)$, which is good enough. Remember, this property only needs to hold from some point on, which it does here. Having verified that all the hypotheses of the theorem were satisfied, we can now use the integral test. We study the convergence of $\int_1^{\infty} \frac{\ln x}{x} \, dx$. This is an improper integral.

$$\int_1^{\infty} \frac{\ln x}{x} \, dx = \lim_{t \to \infty} \int_1^{t} \frac{\ln x}{x} \, dx$$
$$= \lim_{t \to \infty} \left[ \frac{(\ln t)^2}{2} - \frac{(\ln 1)^2}{2} \right]$$
$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2}$$
$$= \infty$$

The integral diverges, therefore the series diverges.

Remark 7 The above integral was evaluated using the substitution $u = \ln x$ therefore $du = \frac{dx}{x}$.

Example 8 Test $\sum_{i=1}^{\infty} \frac{1}{i^p}$ for convergence.

If $p < 0$, then $\lim_{i \to \infty} \frac{1}{i^p} = \infty$, so the series diverges. If $p = 0$, then $\frac{1}{i^p} = \frac{1}{i^0} = 1$. 

3
hence, \( \lim_{i \to \infty} \frac{1}{i^p} = 1 \). The series also diverges. If \( p > 0 \), then the function \( f(x) = \frac{1}{x^p} \) is continuous, decreasing (since its denominator is increasing) and positive on \([1, \infty)\). So, instead of studying the convergence of the series, we study the convergence of the integral \( \int_1^\infty \frac{dx}{x^p} \). From a theorem about improper integrals, we know that this integral converges if \( p > 1 \) and diverges otherwise. Combining all the cases, we see that the series converges when \( p > 1 \) and diverges otherwise.

The above example is very important. Its result should be remembered. Series which appear in the above example have a special name.

**Definition 9** A **p-series** is a series of the form \( \sum_{i=1}^{\infty} \frac{1}{i^p} \)

**Example 10** Here are some examples of p-series.

1. \( \sum_{i=1}^{\infty} \frac{1}{i^2} \) (\( p = 2 \))
2. \( \sum_{i=1}^{\infty} \frac{1}{i} \) (\( p = \frac{1}{2} \))
3. \( \sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}} \) (\( p = \frac{3}{2} \))

In example 8, we prove the following:

**Theorem 11** A p-series converges if \( p > 1 \). It diverges otherwise.

**Example 12** Test \( \sum_{i=1}^{\infty} \frac{1}{i^2} \) for convergence.

*This is a p-series with \( p = 2 \), therefore it converges.*

**Example 13** Test \( \sum_{i=1}^{\infty} \frac{1}{i^{1/2}} \) for convergence.

*This is a p-series with \( p = \frac{1}{2} \), therefore it diverges.*

### 1.2.2 Comparison Tests

The idea here is to compare an unknown series to a known series. Series such as the harmonic series, geometric series and p-series are used a lot. The question is, given an unknown series, how do we know what to compare it to. Series which look like a p-series or a geometric series should be compared with such series.

For example, \( \sum_{i=1}^{\infty} \frac{1}{n^2 + 1} \) should be compared with \( \sum_{i=1}^{\infty} \frac{1}{n^2} \) because for large \( n \),
$n^2 + 1 \approx n^2$, thus $\frac{1}{n^2 + 1} \approx \frac{1}{n^2}$. Similarly, $\sum_{i=1}^{\infty} \frac{1}{2^n - 1}$ should be compared to $\sum_{i=1}^{\infty} \frac{1}{2^n}$.

In addition, the following facts are often used and should be remembered.

- If $a, b, c$ are positive numbers then $b > c$ implies that $\frac{a}{b} < \frac{a}{c}$.
- $\ln x < x < e^x$.

One of two comparison tests can be used:

**Theorem 14 (Standard comparison test)** Suppose that $\sum a_n$ and $\sum b_n$ are two series with positive terms such that $a_n \leq b_n$ for every $n$ from some point on. Then,

- If $\sum b_n$ converges, so does $\sum a_n$ (i.e. if the series with larger terms converges, so will the series with smaller terms).
- If $\sum a_n$ diverges, so does $\sum b_n$ (i.e. if the series with smaller terms diverges, so will the series with larger terms).

Intuitively this theorem should make sense. Since we have series of positive terms, either the series is finite, in which case it converges or it is infinite, in which case it diverges. So, if $0 \leq a_n \leq b_n$ then

$$0 \leq \sum a_n \leq \sum b_n$$

So, we see that if the series of larger terms, $\sum b_n$ converges that is is finite, then $\sum a_n$ must also be finite hence converge. Equivalently, if $\sum a_n$ diverges, that is is infinite, then $\sum b_n$ must also be infinite that is diverge. These are the only two situations under which we can conclude.

**Remark 15** It is easier to remember the theorem in this form: Given two series of positive terms. If the series of larger terms converges, so does the series of smaller terms. Equivalently, if the series of smaller terms diverges, so does the series of larger terms.

The idea here is that if we suspect that the given series converges, we need to find a series with larger terms which we know converges. Conversely, if we suspect the given series diverges, we need to find a series with smaller terms which we know diverges.

It is also important to note that the inequality does not have to hold for every $n$. It just has to hold from some point on.

**Example 16** Test $\sum \frac{1}{n^2 + 5}$ for convergence.

For large $n$, $n^2 + 5$ behaves like $n^2$. So, we compare $\sum \frac{1}{n^2 + 5}$ to $\sum \frac{1}{n^2}$ which
is a $p$-series with $p = 2$, thus it converges. At this point, we cannot conclude yet that the given series converges. For this, we need to prove that the terms of $\sum \frac{1}{n^2 + 5}$ are smaller than the terms of $\sum \frac{1}{n^2}$. This is easy to see because $n^2 + 5 \geq n^2$ and therefore $\frac{1}{n^2 + 5} \leq \frac{1}{n^2}$. By the standard comparison test, it follows that $\sum \frac{1}{n^2 + 5}$ converges.

**Example 17** Test $\sum \frac{\ln n}{n}$ for convergence.

Since $\ln n \geq 1$ for $n \geq 3$, it follows that $\frac{\ln n}{n} \geq \frac{1}{n}$. We know that $\sum \frac{1}{n}$ diverges. By the standard comparison test, $\sum \frac{\ln n}{n}$ diverges.

**Example 18** Test $\sum \frac{1}{2^n + 1}$ for convergence.

For large $n$, $2^n + 1$ behaves like $2^n$. We also know that $\sum \frac{1}{2^n}$ converges (geometric series with $r = \frac{1}{2}$). Thus we suspect our series converges. Since $\frac{1}{2^n + 1} \leq \frac{1}{2^n}$, we can use the standard comparison test to conclude that it does.

**Example 19** Test $\sum \frac{1}{n^2 - 5}$ for convergence.

Since $n^2 - 5$ behaves like $n^2$, we suspect our series converges since $\sum \frac{1}{n^2}$ converges. However, $\frac{1}{n^2 - 5} \geq \frac{1}{n^2}$. So, we cannot use the standard comparison test. The next comparison test addresses this issue.

**Theorem 20 (Limit comparison test)** Suppose that $\sum a_n$ and $\sum b_n$ are two series with positive terms such that $L = \lim_{n \to \infty} \frac{a_n}{b_n}$ exists or is $\infty$ then:

1. If $0 < L < \infty$, the two series behave alike.

2. If $L = 0$ then convergence of $\sum b_n$ implies convergence of $\sum a_n$. Divergence of $\sum a_n$ implies divergence of $\sum b_n$.

3. If $L = \infty$ then convergence of $\sum a_n$ implies convergence of $\sum b_n$. Divergence of $\sum b_n$ implies divergence of $\sum a_n$.

Intuitively, the theorem should make sense. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ then this indicates that $b_n$ is much larger than $a_n$. Thus, $\sum b_n$ is the series with larger terms. We can use remark 15 to help us remember what the theorem allows us to conclude. Similarly, if $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$, then $a_n$ is much larger than $b_n$. So, this time $\sum a_n$ is the series with larger terms.
Example 21 Test \( \sum \frac{1}{n^2 - 5} \) for convergence.

This is the series we could not handle with the standard comparison test. Again, we compare it to \( \sum \frac{1}{n^2} \). This time, we look at

\[
\lim_{n \to \infty} \frac{1}{n^2 - 5} = \lim_{n \to \infty} \frac{n^2}{n^2 - 5}
\]

\[= 1 \text{ by l'Hôpital's rule.}
\]

Since \( \sum \frac{1}{n^2} \) we conclude by the limit comparison test that \( \sum \frac{1}{n^2 - 5} \) also converges.

1.2.3 Using the Integral Test to Estimate Sums

As pointed out above, the integral test can be used to study the convergence of certain series. But, the integral and the series do not have the same value. However, the integral can be used to find the error when approximating the sum of a series by its sequence of partial sums.

Let us introduce some notation. Let us assume that we have a series \( \sum_{i=1}^{\infty} a_i \), which satisfies the conditions of the integral test. Assume we have proven that this series converges. We would like to know what it converges to. Call \( S = \sum_{i=1}^{\infty} a_i \). Obviously, we cannot add up all the terms; there is an infinite amount. But we know from the test for divergence that because the series converges, \( \lim_{n \to \infty} a_n = 0 \). Therefore, the terms we are adding become smaller and smaller. Eventually, they will be so small that they will almost not count. This suggests that we could approximate this infinite sum by adding a finite number of terms and ignore the remaining ones. In other words, we could approximate \( S = \sum_{i=1}^{\infty} a_i \) by \( S_n = \sum_{i=1}^{n} a_i \). Of course, the two will not be exactly equal. When we approximate something with something else, it is important to know the error we are making. Let us denote the error by \( R_n \). In other words, \( R_n = S - S_n \).

It turns out that not only we can find the error, we can also find how many terms we need to add if we need the error to be less than a certain number. This is the next theorem, which we will state without proof. The proof can be found in any calculus book.

\textbf{Theorem 22} If \( \sum_{i=1}^{\infty} a_i \) converges and satisfies the conditions of the integral test
and if we set $R_n = S - S_n$, then
\[
\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx
\]
(1)
and therefore
\[
S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S \leq S_n + \int_{n}^{\infty} f(x) \, dx
\]
(2)

We illustrate with a few examples how this theorem can be used to find the error when we approximate $S$ with $S_n$ as well as to find out how many terms must be added so that the error is less than a certain amount. The second idea is extremely important. We know that when we build something, whether it is a bridge, or tiny parts for a computer, we can never build these parts to the exact measurements. Hence, there is the idea of "tolerance", that is the error we are allowed for the part to still function. Obviously, the tolerance for a bridge is much larger than for computer parts. This idea of tolerance exists in pretty much everything we do. So, it may happen that a real problem has an infinite series as its solution. Because we may not be able to find exactly what the infinite series is, we may have to approximate it. The tolerance of the problem we are solving will tell us the maximum error our approximation can have.

If you think about it, this is pretty amazing. We do not know what the exact sum of the series is, yet we can tell how far from the exact value our approximation is.

**Example 23** Consider the series $\sum_{i=1}^{\infty} \frac{1}{i^3}$. Suppose we approximate this series with $S_{10}$ (the sum of the first 10 terms). What will be the error? Let $S$ denote the exact sum. We are approximating $S$ with $S_{10}$. Using the notation above, the error is $R_{10}$. From the theorem, we see that
\[
\int_{11}^{\infty} \frac{1}{x^3} \, dx \leq R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3} \, dx
\]
We begin by computing the integrals.
\[
\int_{n}^{\infty} \frac{1}{x^3} \, dx = \lim_{t \to \infty} \int_{n}^{t} \frac{1}{x^3} \, dx
\]
\[
= \lim_{t \to \infty} \left[ \frac{-1}{2x^2} \right]_{n}^{t}
\]
\[
= \lim_{t \to \infty} \left[ \frac{-1}{2t^2} + \frac{1}{2n^2} \right]
\]
\[
= \frac{1}{2n^2}
\]
Therefore,
\[
\int_{11}^{\infty} \frac{1}{x^3} \, dx = \frac{1}{2 (11)^2} = \frac{1}{242} = .0041322314
\]

And
\[
\int_{10}^{\infty} \frac{1}{x^3} \, dx = \frac{1}{2 (10)^2} = \frac{1}{200} = .005
\]

It follows that if we approximate \( S \) by \( S_{10} \), the error \( R_{10} \) will satisfy

\[ .0041322314 \leq R_{10} \leq .005 \]

In other words, the error is at least .0041322314 but no greater than .005.

In fact, we can even do better, using the second inequality of the theorem, and the fact that

\[ S_{10} = \sum_{i=1}^{10} \frac{1}{i^3} = \frac{19164113947}{16003008000} = 1.197531986, \]

we see that

\[
S_{10} + \int_{11}^{\infty} \frac{1}{x^3} \, dx \leq S \leq S_{10} + \int_{10}^{\infty} \frac{1}{x^3} \, dx
\]

\[ 1.197531986 + .0041322314 \leq S \leq 1.197531986 + .005 \]

\[ 1.201664217 \leq S \leq 1.202531986 \]

Thus, if we approximate \( S \) by the midpoint of these two values, we have

\[
S \approx \frac{1.201664217 + 1.202531986}{2} = 1.202098102
\]

And the error is less than half the difference between these two values that is

\[
R_{10} \leq \frac{1.202531986 - 1.201664217}{2} \leq 0.0004338845
\]

**Remark 24** What the above example suggests is that we can approximate \( S \) by using the midpoint of the interval \( \left( S_n + \int_{n+1}^{\infty} f(x) \, dx, S_n + \int_{n}^{\infty} f(x) \, dx \right) \).
doing so, the error is less than half the width of the same interval, that is

\[ R_n \leq \frac{n}{2} \sum_{n+1}^{\infty} f(x) dx - \int_{n}^{\infty} f(x) dx \]

(3)

This provides a better approximation as the example below will confirm.

**Example 25** Approximate the series \( \sum_{i=1}^{\infty} \frac{1}{i^2} \) within .001.

In other words, we want to find \( n \) (how many term we should add) so that \( R_n < .001 \).

1. Using the first approach. We want to approximate \( \sum_{i=1}^{\infty} \frac{1}{i^2} \) with \( S_n \), so that

\[ R_n \leq .001 \]  

From the theorem and equation 1, we know that \( R_n \leq \int_{n}^{\infty} \frac{1}{x^2} dx \).

Thus, if we solve \( \int_{n}^{\infty} \frac{1}{x^2} dx \leq .001 \), what we want will follow. We begin by computing the integral.

\[
\int_{n}^{\infty} \frac{1}{x^2} dx = \lim_{t \to \infty} \int_{n}^{t} \frac{1}{x^2} dx
\]

\[
= \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_{n}^{t}
\]

\[
= \lim_{t \to \infty} \left[ -\frac{1}{t} + \frac{1}{n} \right]
\]

\[
= \frac{1}{n}
\]

To have \( \frac{1}{n} \leq .001 \), we must have \( n \geq \frac{1}{.001} = 1000 \). To approximate \( \sum_{i=1}^{\infty} \frac{1}{i^2} \) within .001 we use \( \sum_{i=1}^{1000} \frac{1}{i^2} \). Indeed, \( \sum_{i=1}^{1000} \frac{1}{i^2} = \frac{1}{6} \pi^2 = 1.6449 \) and \( \sum_{i=1}^{1000} \frac{1}{i^2} = 1.6439 \). With this approach, we had to add the first 1000 terms of the series.

2. Using equation 3, we want to find \( n \) so that

\[ \frac{n}{2} \sum_{n+1}^{\infty} f(x) dx - \int_{n}^{\infty} f(x) dx \leq .001 \]
This happens when

\[
\frac{1}{n} - \frac{1}{n+1} \leq .001 \\
\frac{1}{2n(n+1)} \leq .001
\]

Using your calculator, you can try values of \( n \) until the inequality is satisfied. When \( n = 21 \),

\[
\frac{1}{2n(n+1)} = \frac{1}{42(22)} = 1.0823 \times 10^{-3} > .001.
\]

But when \( n = 22 \),

\[
\frac{1}{2n(n+1)} = \frac{1}{44(23)} = 9.8814 \times 10^{-4} < .001.
\]

This means that if we approximate \( S \) by the midpoint of \( \left( S_{22} + \int_{23}^{\infty} f(x) \, dx, S_{22} + \int_{22}^{\infty} f(x) \, dx \right) \), the error will be less than .001. You may think this is more work, but it is not the case. Most of the work is in adding the terms of the series. Above, we added the first 1000 terms. Here, we only add the first 22 terms.

\[
S_{22} = \sum_{i=1}^{22} \frac{1}{i^2} = 1.6005
\]

Furthermore,

\[
\int_{23}^{\infty} f(x) \, dx = \frac{1}{23} = 4.3478 \times 10^{-2}
\]

and

\[
\int_{22}^{\infty} f(x) \, dx = \frac{1}{22} = 4.5455 \times 10^{-2}
\]

Therefore, the midpoint of the interval \( \left( S_{22} + \int_{23}^{\infty} f(x) \, dx, S_{22} + \int_{22}^{\infty} f(x) \, dx \right) \) is

\[
\frac{S_{22} + \int_{23}^{\infty} f(x) \, dx + S_{22} + \int_{22}^{\infty} f(x) \, dx}{2} = \frac{1.6005 + 4.3478 \times 10^{-2} + 1.6005 + 4.5455 \times 10^{-2}}{2}
\]

This is within the desired accuracy.
1.3 Things to Know

Students should know and be able to use the following tests

- Integral test
- p-series, what they are, when they converge
- Standard and limit comparison tests
- Students should also be able to approximate the sum of a series, and determine the error of the approximation.

Related problems assigned:

- #3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 35 on pages 585, 586.