Math 521 - Midterm Exam

RULES. Exams are due in class Wednesday, March 8.
Do not leave exams in your professor’s mailbox.
You may not discuss this exam with anyone but your professor.
You can use your text and class notes, but no other reference material.

1. a) Prove that every real number is a limit of a sequence of rational numbers.
   b) Use part (a) to show that every real number is a limit of a sequence of irrational numbers.

2. Let $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ be countably many nonempty subsets of $\mathbb{R}^n$, each containing the next.
   a) Show, by example, that it all $C_i$ are closed, then the intersection $\bigcap_i C_i$ can be empty.
   b) If $C_1$ is bounded and if all $C_i$ are closed, prove that $\bigcap_i C_i$ is nonempty. (Hint. You can use the Bolzano-Weierstrass theorem.)
   c) Show, by example, that if all $C_i$ are open and if $C_1$ is bounded, then the intersection $\bigcap_i C_i$ can be empty.

3. Let $B$ be the subset of $\mathbb{R}^n$ consisting of all points $p \in \mathbb{R}^n$ whose distance from the origin is at most 1. In other words, $|p| \leq 1$ or $d(p, 0) \leq 1$.
   a) Prove that $B$ is a closed subset of $\mathbb{R}^n$.
   b) Show that the point $q = (1, 0, \ldots, 0) \in \mathbb{R}^n$ belongs to the boundary $\partial B$ of $B$.

4. Given a point $p$ in $\mathbb{R}^n$, let $f: \mathbb{R}^n \to \mathbb{R}$ be the function defined by $f(x) = |x - p| = d(x, p)$.
   a) Prove that $f: \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous.
   b) Suppose $S$ is a compact subset of $\mathbb{R}^n$ with $p \notin S$. Prove that there is a real number $r > 0$ such that $|x - p| \geq r$ for all $x \in S$.
   c) Let $T$ be a connected subset of $\mathbb{R}^n$. Suppose there exist points $a, b \in T$ with $|a - p| = 1$ and $|b - p| = 3$. Show that there exists a point $c \in T$ with $|c - p| = 2$. 