Math 542 Homework Problems

Let d be a positive integer.

1. Show that $\mathbb{Z}[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Z}\}$ is a subring of the complex numbers and that $\mathbb{Q}[\sqrt{-d}] = \{a + b\sqrt{-d} \mid a, b \in \mathbb{Q}\}$ is a subfield of the complex numbers.

2. Find all units, that is invertible elements, in $\mathbb{Z}[\sqrt{-d}]$. If $d > 2$, find all elements of the ring having norm 2 or 4.

3. Prove that $\mathbb{Z}[\sqrt{-2}]$ is Euclidean.

4. If d is odd, prove that there is a homomorphism $\theta\colon \mathbb{Z}[\sqrt{-d}] \to \mathbb{Z}/2\mathbb{Z}$ given by $a + b\sqrt{-d} \mapsto a + b + 2\mathbb{Z}$. Conclude that $I = \ker\theta$ is a proper ideal of $\mathbb{Z}[\sqrt{-d}]$.

5. If d is odd and $d > 3$, prove that I is not a principal ideal. (Hint. First note that $2 \in I$ and $1 - \sqrt{-d} \in I$. If I is principal, generated by α, then 2 and $1 - \sqrt{-d}$ are multiples of α. Thus $4 = N(2)$ is a multiple of $N(\alpha)$.)

6. If $d = 3$, prove that I is not principal.

Since a Euclidean domain is a PID, we conclude from the above that if d is odd and $d \geq 3$, then $\mathbb{Z}[\sqrt{-d}]$ is not Euclidean. Here is a way to show that $\mathbb{Z}[\sqrt{-d}]$ is not Euclidean without checking to see whether ideals are principal. Let d be a positive integer and, for convenience, assume that $d \geq 10$.

7. Find all divisors of 2 or 3 in $\mathbb{Z}[\sqrt{-d}]$. (Hint. Note that if α divides 2 or 3, then $N(\alpha)$ divides $N(2) = 4$ or $N(3) = 9$.)

We know that the units of $R = \mathbb{Z}[\sqrt{-d}]$ are ± 1. Suppose, by way of contradiction that R is Euclidean with size function $\nu\colon R \setminus \{0\} \to \mathbb{Z}^+$. Set $A = R \setminus \{0, \pm 1\}$ and fix $\alpha \in A$ so that $\nu(\alpha)$ is minimal in $\nu(A)$.

8. If $\beta \in R$, then $\beta = q\alpha + r$ with $r = 0$ or $\nu(r) < \nu(\alpha)$. Conclude that $r = 0$ or $r = \pm 1$ and hence that α divides β or $\beta + 1$ or $\beta - 1$.

9. First take $\beta = 2$ and deduce that $\alpha = \pm 2$ or ± 3. Then take $\beta = \sqrt{-d}$ and derive a contradiction.