Let $G$ be a finite, nonabelian simple group. Suppose $H$ is a proper subgroup of $G$ with index $|G : H| = n > 1$.

1. Show that $G$ embeds isomorphically in $\text{Alt}_n$, and that $n \geq 5$.

2. If $n \leq 8$, prove that $G$ has no elements of order 10.

3. Viewing $G$ as a subgroup of $\text{Alt}_n$, show that either $G = \text{Alt}_n$ or $|\text{Alt}_n : G| \geq n$. Hence either $|G| = n!/2$ or $|G| \leq (n - 1)!/2$.

4. Show that there is no nonabelian simple group of order 120. Indeed, if $G$ is such a group, deduce from the above that $n \geq 8$, so $|H| \leq 15$. Now determine the number of Sylow 5-subgroups of $G$.

5. Show that the only nonabelian simple group of order 60 is $\text{Alt}_5$. To this end, suppose by way of contradiction that $G$ is such a group and that $G$ is not isomorphic to $\text{Alt}_5$. First, deduce from the above that $n \geq 6$, so $|H| \leq 10$. Next, show that $G$ contains 24 elements of order 5. Finally, prove that a Sylow 2-subgroup $P$ of $G$ is a self-normalizing T.I. set, so $G$ has 45 elements of order 2 or 4. (Recall that $P$ is a T.I. set if distinct conjugates of $P$ are disjoint. To prove that $P$ is a T.I. set, you might consider the centralizer of the intersection of two such conjugates.)