

## Math 742 Midterm - Passman

RULES. You may not discuss this exam with anyone but me.

You can use your class notes, but no other reference material.

Exams are due in class Tuesday, March 25. Do not leave exams in my mailbox.

Each problem is worth 10 points, except for #9 which is worth 20 points.

Good mathematics is written neatly, precisely, and succinctly.

### An Intersection Theorem for Primary Ideals

Let  $R$  be a commutative Noetherian ring that is not necessarily a domain, let  $P$  be a prime ideal of  $R$ , and let  $M = R \setminus P$ . For each integer  $n \geq 1$  define

$$P^{[n]} = \{r \in R \mid ar \in P^n \text{ for some } a \in M\}.$$

Furthermore, let

$$K_P = \{r \in R \mid ar = 0 \text{ for some } a \in M\} \quad \text{and} \quad D_P = \bigcap_n P^{[n]}.$$

1. Verify that  $M$  is multiplicatively closed with  $0 \notin M$ . Prove (briefly) that  $P^{[n]}$  and  $K_P$  are ideals of  $R$ .
2. Show that  $P \supseteq P^{[n]} \supseteq P^n$ ,  $P^{[n]} \supseteq P^{[n+1]}$ , and that  $P^{[n]} \supseteq D_P \supseteq K_P$ .
3. Prove directly that  $P^{[n]}$  is  $P$ -primary.
4. Show that  $P^{[n]}$  is the  $P$ -primary component in the normal primary decomposition of the ideal  $P^n$ .
5. If  $Q$  is a  $P$ -primary ideal and  $Q \supseteq P^n$ , show that  $Q \supseteq P^{[n]}$ .
6. If  $Q$  is a primary ideal and  $Q \supseteq P^n$ , must  $Q \supseteq P^{[n]}$ . Prove or give a counterexample.
7. Let  $D$  be an ideal. By the Artin-Rees Lemma, there exists an integer  $n$  such that  $PD \supseteq D \cap P^n$ . If  $r \in D \cap P^{[n]}$ , show that there exists  $a \in M$  with  $ar \in PD$ .
8. If  $r \in D_P$ , show that there exists  $a \in M$  with  $ar \in PD_P$ .
9. Use a variant of the determinantal trick to prove that  $D_P = K_P$ .