1. Let $R$ be a UFD.
   a. If $I$ is a nonzero prime ideal of $R$, show that $I$ is generated by prime elements.
   b. If $\pi \in R$ is prime, show that $I = (\pi)$ is a prime ideal.
   c. Prove that the polynomial ring $R[x_1, x_2, \ldots]$, in infinitely many indeterminants, is a UFD. You can quote lemmas from class.

2. Let $R$ be a PID and let $I$ be a nonzero ideal of $R$.
   a. Show that $I$ is prime if and only if $I = (\pi)$ for some prime element $\pi$. Conclude that the nonzero prime ideals of $R$ are maximal.
   b. Show that $I$ is primary if and only if it is a prime power.

3. Let $R$ be a commutative Noetherian ring.
   a. Prove that $r \in R$ is a zero divisor (that is, $rs = 0$ for some $0 \neq s \in R$) if and only if $r$ is contained in one of the prime ideals that occur in a normal primary decomposition of the zero ideal.
   b. If all the prime ideals of $R$ are maximal, prove that $R$ is Artinian.

4. Let $F$ be a field, set $R = F[x, y]$ and $M = (x, y)$.
   a. Show that the $F$-vector space $V_n = M^{n-1}/M^n$ has dimension $n$.
   b. If $r_1, r_2, ..., r_k \in R$ generate the ideal $M^{n-1}$, use part (a) to prove that $k \geq n$.

5. Let $R$ be a domain and let $M$ be a multiplicatively closed subset. If $I$ is an ideal of $R$ define $I' = I : M = \{r \in R \mid rm \in I \text{ for some } m \in M\}$.
   a. Show that $I'$ is an ideal of $R$, $I' \supseteq I$ and $I'' = I'$. Thus $'$ is a closure operator.
   b. If $I$ is primary, show that either $I' = I$ or $I' = R$.
   c. Show that there is a one-to-one correspondence between the closed ideals of $R$ and the ideals of the localization $R_M$.

6. Let $R$ be a commutative ring.
   a. Suppose $A$ and $B$ are ideals of $R$ with $A$ finitely generated and with $R/A$ and $R/B$ both Noetherian rings. Prove that $A/AB$ is a Noetherian $R$-module and conclude that $R/AB$ is a Noetherian ring.
   b. If $R$ is not Noetherian, show that there exists an ideal $P$ of $R$ maximal with the property of being not finitely generated. Use (a) to deduce that $P$ is a prime ideal.
   c. Let $V$ be a nonzero right module for the polynomial ring $R[t]$. If $V$ has $n < \infty$ generators as an $R$-module, use the determinental trick to prove that there exists a monic polynomial in $R[t]$ of degree $n$ that annihilates $V$. This is the Cayley-Hamilton theorem from linear algebra.