

Ring Theory Problems

1. Given the commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\sigma} & B \\
 \alpha \downarrow & & \downarrow \beta \\
 A' & \xrightarrow{\sigma'} & B'
 \end{array}$$

show that $\alpha: \ker \sigma \rightarrow \ker \sigma'$ and that $\beta: \operatorname{coker} \sigma \rightarrow \operatorname{coker} \sigma'$. Here $\operatorname{coker} \sigma = B/\sigma(A)$.

2. Let K be a field, let V be an infinite dimensional K -vector space with basis $\{v_0, v_1, v_2, \dots\}$, and set $R = \operatorname{End}_K(V)$. Prove that $R_R \cong R_R \oplus R_R$ by constructing a 2×1 invertible R -matrix using suitable shift maps.

3. Given the commutative diagram

$$\begin{array}{ccccccccc}
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
 \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \epsilon \downarrow \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
 \end{array}$$

with exact rows and with α, β, δ and ϵ isomorphisms. Prove that γ is an isomorphism. This is the five lemma.

4. Let $R = \mathbb{Z}$. If F_R is free, show that F contains no infinitely divisible element. Conclude that F does not contain \mathbb{Q}_R and hence that \mathbb{Q}_R is not projective.

5. Again let $R = \mathbb{Z}$, set $V = \prod \mathbb{Z}$, a countable direct product of copies of \mathbb{Z} , and let $W = \sum \mathbb{Z}$ be the direct sum submodule. Suppose, by way of contradiction, that V is contained in a free \mathbb{Z} -module F with basis \mathcal{B} . (a) Show that there is a countable subset \mathcal{B}' of \mathcal{B} so that $W \subseteq F' \subseteq F$, where F' has free basis \mathcal{B}' . (b) Notice that F/F' is a free \mathbb{Z} -module containing $V/(V \cap F')$. (c) For each sequence $\epsilon = (\epsilon_1, \epsilon_2, \dots)$ of \pm signs, define $v_\epsilon \in V$ by $v_\epsilon = \prod_{i=1}^{\infty} \epsilon_i i!$ and show that $v_\epsilon + (V \cap F')$ is infinitely divisible in F/F' . (d) Conclude that V cannot be contained in a free \mathbb{Z} -module.

6. Show that all right R -modules are free if and only if R is a division ring.

7. Given V_R , show that there is a one-to-one correspondence between internal direct sums $V = X \oplus Y$ and idempotents e in $\text{End}(V_R)$.

8. A ring R is semihereditary if all finitely generated right ideals are projective R -modules. Show that, for such a ring R , any finitely generated submodule of a free module is projective.

9. Define

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Q} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{pmatrix}.$$

(a) Show that V and N are minimal right ideals of R , with V_R projective and $N_R \cong V_R$. (b) Show that any right ideal I of R strictly larger than $N + V$ is of the form zR for some nonzero integer z and hence $I_R \cong R_R$ is projective. (c) If I is a right ideal of R and I does not contain N , observe that $I + N = I \oplus N$. In particular if $I + N$ is projective, then so is I . (d) Conclude that R is right hereditary. (e) Let $A = N + V$, so that A is a 2-sided ideal of R . Prove that $\bigcap_{n=1}^{\infty} nR = A = AR$ and deduce that if F is a free left R -module, then $\bigcap_{n=1}^{\infty} nF = AF$. (f) If ${}_R N$ is projective, then there is a free left R -module F with $F = N \oplus W$. Now note that

$$N = \bigcap_{n=1}^{\infty} nN \subseteq \bigcap_{n=1}^{\infty} nF = AF = AN \oplus AW = AW$$

and this is a contradiction. Conclude that R is not left hereditary.

10. Let R be the ring of upper triangular $n \times n$ matrices over a field K . (a) Show that each row of R is a projective right R -module. (b) Show that each row has a unique chain of R -submodules and that each submodule is projective. (c) Deduce that R is right hereditary.

11. Let a and b be positive integers that are not relatively prime and set $R = \mathbb{Z}/(ab)$. Show that the R -modules aR and bR both have infinite projective dimension.

12. Let R be a ring that has no commutative homomorphic image and let A_R and B_R be right R -modules. Let $\phi: A \times B \rightarrow U$ be a map to the abelian group U that is bilinear and satisfies $\phi(ar, b) = \phi(a, br)$ for all $a \in A$, $b \in B$ and $r \in R$. Prove that ϕ is the 0 map.

13. Let I be an ideal of R and let A_R and ${}_R B$ be R -modules with $AI = 0 = IB$. Show that $A \otimes_R B$ is naturally isomorphic to $A \otimes_{R/I} B$.

14. Let I be a left ideal of R and let J be a right ideal. (a) If ${}_R I$ is flat, show that $J \otimes I \cong JI$. (b) If R/I is flat, prove that $J \cap I = JI$.

15. Let $R = K[x, y]$ be a polynomial ring in two variables over the field K and set $I = Rx + Ry$. Suppose by way of contradiction that ${}_R I$ is flat. (a) Deduce that $I \otimes I \cong I^2$ as an (R, R) -bimodule. (b) Note that I^2/I^3 is the largest homomorphic image of I^2 that is annihilated on both sides by I and that $\dim_K I^2/I^3 = 3$. (c) On the other hand, observe that $I \otimes I$ maps onto $(I/I^2) \otimes_R (I/I^2) \cong (I/I^2) \otimes_{R/I} (I/I^2)$ and this image has K -dimension 4.

16. Let R be a right Noetherian ring. (a) If the following is an exact sequence of finitely generated right R -modules

$$0 \rightarrow A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow 0$$

prove that $\sum_{i=0}^n (-1)^i [A_i] = 0$ in $G_0(R)$. (b) If

$$0 = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A$$

is a finite sequence of right submodules of the finitely generated module A , show that $[A] = \sum_{i=1}^n [A_i/A_{i-1}]$ in $G_0(R)$.

17. The graded ring $S = \bigoplus_{i=0}^{\infty} S_i$ is said to be graded right Noetherian if the set of graded right ideals satisfies the ascending chain condition. Prove that S is right Noetherian if and only if it is graded right Noetherian. Note that if I is any right ideal of S and if I_n denotes the set of degree n components of elements of $I \cap (S_0 + S_1 + \cdots + S_n)$, then $\sum_{i=0}^{\infty} I_i$ is a graded ideal of S . One can now proceed as in the proof of the Hilbert basis theorem.

18. Show that S_0 Noetherian does not imply that S is Noetherian.

19. Find all finitely generated graded $K[x]$ -modules up to abstract isomorphism. Here K is a field and recall that $K[x]$ is a PID.

20. In the following four problems we let J denote the Jacobson radical of the ring R . Prove Nakayama's lemma: Let V be a finitely generated R -module and let W be a submodule. (a) If $W < V$, show that there exists a

maximal submodule M of V with $W \subseteq M < V$. (b) If $V = W + VJ$, prove that $V = W$. In particular, if $V = VJ$, conclude that $V = 0$.

21. Let P be a finitely generated projective R -module and let $\mu: V \rightarrow P$ be an R -module homomorphism. Let $\bar{\mu}: V/VJ \rightarrow P/PJ$ be the natural map. (a) If $\bar{\mu}$ is onto, conclude that $P = \mu(V) + PJ$ and then that μ is onto. (b) If $\bar{\mu}$ is an isomorphism, use the splitting of $\mu: V \rightarrow P$ to deduce that μ is an isomorphism.

22. Suppose P is a finitely generated projective R -module, with P/PJ a free R/J -module. (a) Show that there exists a finitely generated free R -module F with $F/FJ \cong P/PJ$. (b) Use the fact that $P \rightarrow P/PJ$ is an epimorphism and that F is projective, to construct a map $\mu: F \rightarrow P$ with $\bar{\mu}: F/FJ \rightarrow P/PJ$ an isomorphism. (c) Conclude that P is free.

23. If all finitely generated projective R/J -modules are free, show that all finitely generated projective R -modules are free.

24. Let K be a field and set $R = K[x \mid x^m = 0]$. Show that the Cartan map c is not surjective.

25. Write $A(a, b)$ and $B(a, b)$ for the 4×4 triangular matrices

$$A(a, b) = \begin{pmatrix} a & * & 0 & 0 \\ & b & 0 & 0 \\ & & b & * \\ & & & a \end{pmatrix} \quad B(a, b) = \begin{pmatrix} a & * & * & * \\ & b & * & * \\ & & b & * \\ & & & a \end{pmatrix}$$

with the $*$ entries arbitrary. Now let R and S be the four and eight dimensional subalgebras of $M_4(K)$ given by

$$R = \{A(a, b) \mid a, b \in K\} \quad S = \{B(a, b) \mid a, b \in K\}$$

Observe that $R/\text{Rad}(R) \cong K \oplus K \cong S/\text{Rad}(S)$ and describe the Cartan matrices of R and S . Conclude that c need not be injective in general.

26. Show that $[X] = [Y]$ in $G_0(R)$ if and only if there exist exact sequences $0 \rightarrow A \rightarrow B \oplus X \rightarrow C \rightarrow 0$ and $0 \rightarrow A \rightarrow B \oplus Y \rightarrow C \rightarrow 0$ with A , B , and C finitely generated R -modules.

27. If R is a right Noetherian ring, prove that the same is true of the power series ring $R[[x]]$.

28. Let R be a right Noetherian ring. Use Baer's Criterion to prove that a direct sum of injective R -modules is injective.

29. Let K be a field and suppose R is a K -algebra. Then any R -module V is a K -vector space on either side and thus V_R is a (K, R) -bimodule. Show that the proof of Baer's Theorem carries over in this context using $\text{Hom}_K(-, K)$ rather than $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. Conclude that if $\dim_K R < \infty$ and $\dim_K V < \infty$, then V embeds in an injective R -module that is also finite dimensional over K . In particular, the injective hull of V is finite dimensional over K .

30. Let $R = \begin{pmatrix} K & K \\ & K \end{pmatrix}$. Use the above to find $\mathbf{E}(e_{11}R)$, $\mathbf{E}(e_{22}R)$ and $\mathbf{E}(R)$.

31. Let $W_i \text{ ess } V_i$ for $i = 1, 2, \dots$. Prove that $\bigoplus \sum_i W_i$ is essential in $\bigoplus \sum_i V_i$. On the other hand, show by example that $\prod_i W_i$ need not be essential in $\prod_i V_i$.

32. A ring R is self-injective if R_R is an injective R -module. Use Baer's criterion to show that a Wedderburn ring is self-injective. Also show that $\mathbb{Z}/n\mathbb{Z}$ is self-injective for any integer $n \geq 2$.

33. If R is a ring with no zero divisors, prove that R is self-injective if and only if it is a division ring.

34. Let K be a field and set

$$R = \begin{pmatrix} K & K[x] \\ 0 & K[x] \end{pmatrix}.$$

Show that R has a classical right ring of quotients, but not a classical left ring of quotients.