ULTRAPRODUCTS AND THE GROUP RING SEMIPRIMITIVITY PROBLEM

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Abstract. This expository paper is a slightly expanded version of the final talk I gave at the group rings conference celebrating the 25th anniversary of the Institute of Mathematics and Statistics at the University of São Paulo. The talk concerned an application of ultraproducts to the solution of the semiprimitivity problem for group algebras $K[G]$ of locally finite groups G. It explained why the special cases of G locally solvable and of G infinite simple turn out to be the critical factors which must be studied.

§1. Ultrafilters

Let $\mathcal{N}$ be a set and let $\mathcal{F}$ be a nonempty collection of subsets of $\mathcal{N}$. We say that $\mathcal{F}$ is a *filter* on $\mathcal{N}$ if:

i. $F_1, F_2 \in \mathcal{F}$ implies that $F_1 \cap F_2 \in \mathcal{F}$.
ii. $F \in \mathcal{F}$ and $\mathcal{N} \supseteq G \supseteq F$ imply that $G \in \mathcal{F}$.
iii. $\emptyset \notin \mathcal{F}$.

These conditions can be paraphrased by saying that: (i) $\mathcal{F}$ is closed under finite intersections, (ii) $\mathcal{F}$ is closed under supersets, and (iii) $\mathcal{F}$ is nontrivial. Notice that if $\emptyset \in \mathcal{F}$, then (ii) implies that $\mathcal{F} = \mathcal{P}(\mathcal{N})$ is the full power set of $\mathcal{N}$.

If $\mathcal{N}$ is an infinite set and if $\mathcal{F}$ is the collection of all cofinite subsets of $\mathcal{N}$, then $\mathcal{F}$ is easily seen to be a filter on $\mathcal{N}$.

We remark that filters which are maximal under inclusion are of key importance and are called *ultrafilters*. For example, if $a \in \mathcal{N}$, then $\{\, F \subseteq \mathcal{N} \mid a \in F \,\}$ is easily seen to be an ultrafilter. These are, of course, rather trivial examples. The existence of nontrivial examples requires a standard Zorn's lemma argument.

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Lemma 1.1. *Any filter $\mathcal{F}'$ on $\mathcal{N}$ can be extended to an ultrafilter $\mathcal{F}$ on $\mathcal{N}$.*

Note that if $\mathcal{F}$ is a filter and if A is a subset of $\mathcal{N}$, then we cannot have both A and its complement A^c in $\mathcal{F}$. Otherwise, $\emptyset = A \cap A^c \in \mathcal{F}$ and this contradicts nontriviality. The surprising property of ultrafilters is

Lemma 1.2. *Let $\mathcal{F}$ be an ultrafilter on $\mathcal{N}$ and let $A \subseteq \mathcal{N}$. Then either A or A^c is contained in $\mathcal{F}$.*

Proof. Say $A \notin \mathcal{F}$ and define $\mathcal{G}$ to be the set of all subsets G of $\mathcal{N}$ which contain $F \cap A$ for some $F \in \mathcal{F}$. It is easy to see that $\mathcal{G}$ is closed under finite intersections and supersets. Furthermore, $\mathcal{G} \supseteq \mathcal{F}$ and $A \in \mathcal{G}$. Thus, the maximality of $\mathcal{F}$ implies that $\mathcal{G}$ cannot be nontrivial. In particular, $A \cap F = \emptyset$ for some $F \in \mathcal{F}$, so $F \subseteq A^c$ and $A^c \in \mathcal{F}$. $\square$

If $\mathcal{F}$ is an ultrafilter, it is convenient to think of the elements of $\mathcal{F}$ as having measure 1 and their complements as having measure 0. Thus every set has measure 0 or 1, and a finite union of sets of measure 0 has measure 0.

Finally, suppose $\mathcal{N}$ is a partially ordered set with *upper bounds*. By this we mean that if $x, y \in \mathcal{N}$, then there exists $z \in \mathcal{N}$ with $z \geq x$ and $z \geq y$. In this case, we define the *cone* of x to be $C(x) = \{\, t \in \mathcal{N} \mid t \geq x \,\}$. Note that, if $z \geq x, y$ as above, then $C(x) \cap C(y) \supseteq C(z)$ and therefore

$$\mathcal{F}' = \{\, F \subseteq \mathcal{N} \mid F \supseteq C(x) \text{ for some } x \in \mathcal{N} \,\}$$

is a filter on $\mathcal{N}$. In particular, Lemma 1.1 yields

Lemma 1.3. *Let $\mathcal{N}$ be a partially ordered set with upper bounds. Then there exists an ultrafilter $\mathcal{F}$ on $\mathcal{N}$ which contains all cones.*

§2. Ultraproducts

Suppose we are given a collection of sets $\{\, \Omega_i \mid i \in \mathcal{N} \,\}$ indexed by $\mathcal{N}$. Then we let $\bar{\Omega} = \prod_{i \in \mathcal{N}} \Omega_i$ be their *complete direct product*. Thus, $\bar{\Omega}$ consists of all elements $\otimes_i w_i$ with $w_i \in \Omega_i$. Now, if $\mathcal{F}$ is an ultrafilter on $\mathcal{N}$, then we define $\otimes_i w_i \equiv_{\mathcal{F}} \otimes_i w'_i$ if and only if $\{\, i \mid w_i = w'_i \,\}$ is contained in $\mathcal{F}$. In other words, the two elements are congruent if and only if their components agree almost everywhere, that is, except on a set of measure 0. It is easy to see that $\equiv_{\mathcal{F}}$ is an equivalence relation and we write $\Omega = \prod_{\mathcal{F}} \Omega_i$ for the set of equivalence classes. Ω is called the *ultraproduct* of the Ω_i with respect to $\mathcal{F}$.

Note that, if the Ω_i have some algebraic structure, then this structure is frequently inherited by their ultraproduct $\prod_{\mathcal{F}} \Omega_i$. For example, if $\{\, G_i \mid i \in \mathcal{N} \,\}$ is a family of groups, then $\prod_{i \in \mathcal{N}} G_i$ is a group with multiplication $(\otimes_i g_i)(\otimes_i h_i) = \otimes_i g_i h_i$. Furthermore, it is easy to see that the equivalence relation $\equiv_{\mathcal{F}}$ respects this operation and thus $\prod_{\mathcal{F}} G_i$ is also a group. Similarly, if $\{\, R_i \mid i \in \mathcal{N} \,\}$ is a family of rings, then $\prod_{i\in\mathcal{N}} R_i$ is a ring with arithmetic given by $(\otimes_i r_i)+(\otimes_i s_i) = \otimes_i(r_i+s_i)$ and $(\otimes_i r_i)(\otimes_i s_i) = \otimes_i r_i s_i$. Again, $\equiv_{\mathcal{F}}$ resects these operations and therefore the ultraproduct $\prod_{\mathcal{F}} R_i$ inherits a natural ring structure. Some basic observations are as follows.

Lemma 2.1. *Let $\mathcal{F}$ be an ultrafilter on $\mathcal{N}$. Suppose $\{ F_i \mid i \in \mathcal{N} \}$ is a family of fields and set $F = \prod_{\mathcal{F}} F_i$. Furthermore, let $\{ G_i \mid i \in \mathcal{N} \}$ be a family of groups and set $G = \prod_{\mathcal{F}} G_i$.*

i. *$F = \prod_{\mathcal{F}} F_i$ is a field.*
ii. *$\prod_{\mathcal{F}} \mathrm{M}_n(F_i) \cong \mathrm{M}_n(F)$, the ring of $n \times n$ matrices over F.*
iii. *If $G_i \subseteq GL_n(F_i)$, then $G \subseteq GL_n(F)$.*
iv. *If each G_i is an ordered group, then so is G.*

Proof. (i) Certainly, F is a commutative ring with 1. Now let $\otimes_i f_i$ be a representative of the element $f \in F$ and write $A = \{ i \mid f_i = 0 \}$ and $B = \{ i \mid f_i \neq 0 \}$. Then $A \cup B = \mathcal{N}$, so precisely one of these sets has measure 1. If $A \in \mathcal{F}$, then clearly $f = 0$. On the other hand, if $B \in \mathcal{F}$, then f is also represented by an element $\otimes_i h_i$ with all $h_i \neq 0$, so f is invertible and F is a field.

(iv) Define $\otimes_i g_i \prec \otimes_i h_i$ to mean that $g_i < h_i$ almost everywhere. Then $\prec$ is certainly a partial order on G. Furthermore, if $g = \otimes_i g_i$ and $h = \otimes_i h_i$ are arbitrary elements of G, write $A = \{ i \mid g_i < h_i \}$, $B = \{ i \mid g_i > h_i \}$ and $C = \{ i \mid g_i = h_i \}$. Thus $A \cup B \cup C = \mathcal{N}$ is a disjoint union, and precisely one of these sets has measure 1. If $A \in \mathcal{F}$, then $g \prec h$, if $B \in \mathcal{F}$ then $g \succ h$, and if $C \in \mathcal{F}$ then $g = h$. Thus $\prec$ is indeed a linear ordering on G. □

§3. Malcev's Theorem

The following result is Malcev's theorem for groups. It has obvious analogs for most algebraic systems.

Theorem 3.1. *If G is a group, then G embeds in an ultraproduct of its finitely generated subgroups. In particular, if each finitely generated subgroup is a linear group of fixed degree d or an ordered group, then the same is true of G.*

Proof. Let $\mathcal{N}$ be the collection of all nonempty finite subsets of G. If $n \in \mathcal{N}$, then $\langle n \rangle$ is a finitely generated subgroup of G which we denote by G_n. Note that $\mathcal{N}$ is partially ordered by inclusion and has upper bounds. Thus, by Lemma 1.3, there exists an ultrafilter $\mathcal{F}$ on $\mathcal{N}$ which contains all cones. For each $n \in N$ we define a map $\theta_n : G \to G_n$ by $\theta_n(x) = x$ if $x \in G_n$ and $\theta_n(x) = 1$ otherwise. Using these, we let $\theta : G \to \prod_{\mathcal{F}} G_n$ be given by $\theta(g) = \otimes_n \theta_n(g)$. We claim that θ is a group homomorphism and an embedding.

To this end, let $x, y \in G$. Then the cone $C = C(\{ x, y \})$ is contained in $\mathcal{F}$. Furthermore, if $n \in C$, then $x, y \in n$ so $x, y, xy \in G_n$. In particular, $\theta_n(x)\theta_n(y) = xy = \theta_n(xy)$ and hence $\theta(x)\theta(y) = \otimes_n \theta_n(x)\theta_n(y)$ and $\theta(xy) = \otimes_n \theta_n(xy)$ agree on C, a set of measure 1. Consequently $\theta(x)\theta(y) = \theta(xy)$ and θ is indeed a group homomorphism. Finally, if $x \neq 1$, then $\theta_n(x) = x \neq 1$ for all $n \in C$ and therefore $\theta(x) \neq 1$. Thus θ is one-to-one and the remaining observations follow from Lemma 2.1. $\square$

§4. Locally Subnormal Subgroups

Suppose now that G is a locally finite group and that K is a field of characteristic $p > 0$. If A is a finite subgroup of G, we say that A is *locally subnormal* in G and write $A \operatorname{lsn} G$ if and only if $A \lhd\lhd B$ for all finite subgroups B of G which contain A. The following elementary observation relates the Jacobson radical $JK[A]$ of $K[A]$ to that of $K[G]$.

Lemma 4.1. *If $A \operatorname{lsn} G$, then $JK[A] \subseteq JK[G]$. In particular, if p divides $|A|$, then $JK[G] \neq 0$.*

Proof. Suppose that $A \subseteq B \subseteq G$ with B finite. If $A \lhd B$, then it is trivial to see that $JK[A] \subseteq JK[B]$. Thus, by induction, the same is true if $A \lhd\lhd B$ and, in particular, $JK[A] \cdot K[B]$ is nilpotent. Thus, since G is locally finite and $A \operatorname{lsn} G$, it follows that $JK[A] \cdot K[G] = \bigcup_{B \supseteq A} JK[A] \cdot K[B]$ is a nil right ideal and hence it is contained in $JK[G]$. The remaining observation follows from the converse of Maschke's theorem. $\square$

Note that a ring R is said to be *semiprimitive* if $JR = 0$. Surprisingly, the above result actually characterizes when $K[G]$ is semiprimitive. Specifically, we have

Theorem 4.2. *Let G be a locally finite group and let K be a field of characteristic $p > 0$. Then the group algebra $K[G]$ is semiprimitive if and only if G has no locally subnormal subgroups of order divisible by p.*

As indicated in the abstract, the goal of this talk is to show how ultraproducts were used in the proof of the preceding result.

Note that, if H is a subgroup of G, then its *normal closure* H^G is the smallest normal subgroup of G containing H. Obviously, H^G is the subgroup of G generated by all conjugates $H^g = g^{-1}Hg$ of H. Of course, $H \lhd G$ if and only if $H = H^G$.

Now suppose $H \subseteq X$ are both finite groups. Since the family of subnormal subgroups of X is closed under intersections, it follows that there exists a unique minimal subnormal subgroup S of X with $H \subseteq S$. As usual, S is called the *subnormal closure* of H in X and we denote this group by $S = H^{[X]}$. Since $H \subseteq H^S \lhd S \lhd\lhd X$, the minimality of S implies that $S = H^S$. Furthermore, if $H \subseteq X \subseteq Y$ are all finite, then $H \subseteq H^{[Y]} \cap X \lhd\lhd X$, so $H^{[X]} \subseteq H^{[Y]} \cap X \subseteq H^{[Y]}$, and this allows us to define the *locally subnormal closure* of H in G. Specifically, if H is a finite subgroup of the locally finite group G, then we let $H^{[G]} = \bigcup_X H^{[X]}$ where X runs through the finite subgroups of G containing H. It is now trivial to see that

Lemma 4.3. *If H is a finite subgroup of G, then $S = H^{[G]}$ is a subgroup of G. Furthermore, $S = H^S$ and $H \operatorname{lsn} G$ if and only if $H = S$.*

We remark that subnormal closures do not exist in general for arbitrary subgroups of infinite groups.

As it turns out, the proof of Theorem 4.2 can be reduced to the study of locally subnormal closures of certain finite subgroups of G. Thus, in view of the above lemma, we need to consider the structure of locally finite groups G with $G = H^G$ for some finite subgroup $H \subseteq G$.

§5. Wielandt's Theorem

If Ω is any set, we let Sym_Ω denote the full group of permutations on Ω. In other words, the elements of Sym_Ω are allowed to move arbitrarily many points. On the other hand, the set of all permutations moving only finitely many points is call the *finitary symmetric group* and is denoted by FSym_Ω. Obviously, $\mathrm{FSym}_\Omega \triangleleft \mathrm{Sym}_\Omega$ and FSym_Ω is a locally finite group. Furthermore, the set of all even permutations in FSym_Ω is FAlt_Ω, the *finitary alternating group.* Of course, if $|\Omega| \geq 5$, then FAlt_Ω is a normal simple subgroup of FSym_Ω of index 2.

Now let G be any group which acts as permutations on Ω, and assume that G is *transitive* so that Ω is a single orbit. In this situation, we say that $\emptyset \neq \Delta \subseteq \Omega$ is a *block* for G if, for every $g \in G$, either $\Delta g = \Delta$ or $\Delta g \cap \Delta = \emptyset$. Note that Ω and singleton subsets of Ω are blocks for G and these are considered to be trivial. If all blocks are trivial, then the action of G is said to be *primitive.* The following is a beautiful and deep result of Wielandt.

Theorem 5.1. *Suppose that Ω is an infinite set and that $G \subseteq \mathrm{FSym}_\Omega$. If G is primitive on Ω, then $G = \mathrm{FSym}_\Omega$ or FAlt_Ω.*

Recall that a group G is said to be *locally normal* if it is a union of finite normal subgroups. Equivalently, G is locally normal if and only if every finite subset of G is contained in a finite normal subgroup. The following is an easy consequence of the preceding result.

Corollary 5.2. *Let $G \subseteq \mathrm{Sym}_\Omega$ and suppose that $G = H^G$ for some finite subgroup H. If $H \subseteq \mathrm{FSym}_\Omega$, then G has a finite subnormal series*

$$\langle 1 \rangle = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

with each factor G_{i+1}/G_i either locally normal or isomorphic to $\mathrm{FAlt}_{\Lambda_i}$ for some infinite set Λ_i.

Proof. Since $H \subseteq \mathrm{FSym}_\Omega \triangleleft \mathrm{Sym}_\Omega$, it follows that $G = H^G \subseteq \mathrm{FSym}_\Omega$. Now suppose that H moves k points of Ω. Then H can act nontrivially on at most k orbits of G and thus $G = H^G$ implies that G has at most k nontrivial orbits.

For simplicity, let us just consider the case where G is transitive on the infinite set Ω, and let Δ be a block for G. If $|\Delta| > k$, then Δ contains a point fixed by H and hence $\Delta = \Delta H$. Furthermore, each conjugate H^g of H also moves k points, so $\Delta = \Delta H^g$. Thus Δ is stabilized by $\langle H^g \mid g \in G\rangle = H^G = G$, so Δ is an orbit of G and hence $\Delta = \Omega$. In other words, all nontrivial blocks have size $\leq k$ and therefore we can choose one, say Δ, of maximal size.

Now if Λ denotes the set $\{\, \Delta g \mid g \in G \,\}$ of distinct translates of Δ, then it follows that $|\Lambda| = \infty$ and that G acts in a primitive manner on Λ. In particular, if N is the kernel of this action, then Theorem 5.1 implies that $G/N \cong \mathrm{FSym}_\Lambda$ or FAlt_Λ. Furthermore, N stabilizes all Λg and acts faithfully on the disjoint union $\Omega = \bigcup \Delta g$ with $\Delta g \in \Lambda$. Thus, since $N \subseteq G \subseteq \mathrm{FSym}_\Omega$, it follows that N embeds in the direct sum of the finite symmetric groups $\mathrm{Sym}_{\Delta g}$ and therefore N is locally normal. $\square$

§6. Lifting Permutations

Let $G_1 \subseteq G_2 \subseteq \cdots$ be finite subgroups of G with $G = \bigcup_{i=1}^\infty G_i$ and let $\mathcal{N} = \{\, 1, 2, \dots \,\}$. By Lemma 1.1, there exists an ultrafilter $\mathcal{F}$ on $\mathcal{N}$ containing the cofinite subsets, and consequently all members of $\mathcal{F}$ are infinite. Now suppose that, for each i, G_i acts as permutations on a set Ω_i with kernel N_i. Then $\prod_\mathcal{F} G_i$ acts on $\Omega = \prod_\mathcal{F} \Omega_i$ via $\otimes_i w_i \cdot \otimes_i g_i = \otimes_i w_i g_i$. Furthermore, as in the proof of Malcev's theorem, we can define a homomorphism $\theta : G \to \prod_\mathcal{F} G_i$ by $\theta(g) = \otimes_i \theta_i(g)$ where $\theta_i(g) = g$ if $g \in G_i$ and $\theta_i(g) = 1$ otherwise. In this way, we obtain a permutation action of G on Ω which satisfies

Lemma 6.1. *Let G and Ω be as above.*

i. *If N is contained in the kernel of the action of G on Ω, then there exists a subsequence $\mathcal{M} \subseteq \mathcal{N}$ such that N is the ascending union of the subgroups $N \cap N_i$ with $i \in \mathcal{M}$.*
ii. *If $g \in G$ and if $\theta_i(g)$ moves at most k points of Ω_i for each i, then g moves at most k points of Ω and hence is finitary on Ω.*

Proof. (i) Suppose $x \in N$ and let $S(x) = \{\, i \mid \theta_i(x)$ acts nontrivially on $\Omega_i \,\}$. For each $i \in S(x)$ choose $w_i \in \Omega_i$ moved by $\theta_i(x)$, and if $i \notin S(x)$ let $w_i \in \Omega_i$ be arbitrary. Then $w = \otimes_i w_i \in \Omega$ and, since $x \in N$, we have

$$\otimes_i w_i \equiv_\mathcal{F} (\otimes_i w_i)x = \otimes_i w_i \theta_i(x).$$

Thus $w_i = w_i \theta_i(x)$ almost everywhere and consequently $S(x)$ must have measure 0. Furthermore, if X is any finite subset of N, then $S(X) = \bigcup_{x \in X} S(x)$ also has measure 0 and therefore the complement of $S(X)$ is contained in $\mathcal{F}$ and is infinite. Thus we can choose $i \in S(X)^c$ sufficiently large so that $X \subseteq G_i$. But then $\theta_i(X) = X$ acts trivially on Ω_i and hence $X \subseteq N_i$. In other words, every finite subset of N is contained in some $N \cap N_i$ and the result follows easily since each such N_i is finite.

(ii) Suppose for example that $k = 3$ so that $\theta_i(g)$ moves at most 3 points of Ω_i. Then, for each i, we can choose $a_i, b_i, c_i \in \Omega_i$, not necessarily distinct, with $\theta_i(g)$ fixing the remaining points. Now let $a = \otimes_i a_i$, $b = \otimes_i b_i$ and $c = \otimes_i c_i$ be the elements of Ω determined by these choices. We claim that these are the only possible points moved by g. To this end, let $w = \otimes_i w_i \in \Omega$ and define $A = \{\, i \mid w_i = a_i \,\}$, $B = \{\, i \mid w_i = b_i \,\}$, $C = \{\, i \mid w_i = c_i \,\}$, and $D = \{\, i \mid w_i \neq a_i, b_i, c_i \,\}$. Then $A \cup B \cup C \cup D = \mathcal{N}$ and hence at least one of these four sets must have measure 1. Now, if $A \in \mathcal{F}$, then $w = \otimes_i w_i \equiv_{\mathcal{F}} \otimes_i a_i = a$ and similarly $B \in \mathcal{F}$ yields $w = b$ and $C \in \mathcal{F}$ yields $w = c$. Finally, if $D \in \mathcal{F}$, then since $\theta_i(g)$ acts trivially on $\Omega_i \setminus \{\, a_i, b_i, c_i \,\}$, we have $wg = \otimes_i w_i\theta_i(g) \equiv_{\mathcal{F}} \otimes_i w_i = w$ and g fixes w. $\square$

§7. Action on the Socle

Now, what might the groups G_i act on? To understand our choice, let us first assume that $G_i = L$ is a finite group with no nonidentity solvable normal subgroup. Let $S = \operatorname{soc} L$ be the *socle* of L, so that S is generated by the minimal normal subgroups of L. Since any two distinct minimal normal subgroups commute, it follows that $\operatorname{soc} L$ is in fact the direct product of certain of these subgroups. Furthermore, any minimal normal subgroup is either an elementary abelian q-group for some prime q, or it is *semisimple*, namely a direct product of nonabelian simple groups. This proves (i) below and, of course, parts (ii) and (iii) are routine consequences.

Lemma 7.1. *Let L be a finite group with no nonidentity solvable normal subgroup and set $S = \operatorname{soc} L$.*

i. *$S = M_1 \times M_2 \times \cdots \times M_k$ is a finite direct product of the nonabelian simple groups M_i. Thus S is semisimple.*
ii. *$\mathbb{C}_L(S) = \langle 1 \rangle$, so L acts faithfully as automorphisms on S.*
iii. *The groups M_i are precisely the minimal normal subgroups of S. Thus L permutes the set $\Omega = \{\, M_1, M_2, \dots, M_k \,\}$ by conjugation.*
iv. *If N is the kernel of the action of L on Ω, then $S = N^{(4)}$ where the latter is the fourth derived subgroup of N.*

Proof. (iv) Note that $N = \bigcap_i \mathbb{N}_L(M_i)$, so $N \supseteq S$ and $N^{(4)} \supseteq S^{(4)} = S$. Furthermore, since $\mathbb{C}_L(S) = \langle 1 \rangle$, it follows that N embeds in $\prod_i \operatorname{Aut}(M_i)$. But under this embedding, S corresponds to $\prod_i \operatorname{Inn}(M_i)$, so N/S embeds in $\prod_i \operatorname{Out}(M_i)$. Finally, the precise version of the Schreier conjecture, using the classification of finite simple groups, implies that each outer automorphism group $\operatorname{Out}(M_i)$ is solvable of derived length ≤ 4, and hence $N^{(4)} \subseteq S$, as required. $\square$

If L is an arbitrary finite group, we let $\operatorname{sol} L$ denote the unique largest normal solvable subgroup of L. Then $\bar{L} = L/\operatorname{sol} L$ has no nonidentity solvable normal subgroup, so the above lemma applies to this group. In particular, if we define $\operatorname{rad} L \supseteq \operatorname{sol} L$ by $\operatorname{rad} L/\operatorname{sol} L = \operatorname{soc} \bar{L}$, then $\operatorname{rad} L$ is *solvable-by-semisimple* and L permutes the set $\Omega(L)$ of simple factors of $\operatorname{rad} L/\operatorname{sol} L$ by conjugation. Indeed, if N

is the kernel of this action, then $N^{(4)} \triangleleft \operatorname{rad} L$, so $N^{(4)}$ is also solvable-by-semisimple. For convenience, we call $|\Omega(L)|$ the *width* of L.

§8. Critical Factors

It is time to put all these ingredients together. Our goal is to understand the structure of locally finite groups G satisfying $G = H^G$ for some finite subgroup H and to apply this to the group ring semiprimitivity problem. Specifically, we want to demonstrate why infinite simple groups and locally solvable groups turn out to be critical factors in this process.

Note that, in terms of the semiprimitivity problem, it suffices to assume that G is countably infinite. In particular, we can write $G = \bigcup_{i=1}^{\infty} G_i$ where the G_i are finite subgroups of G satisfying $H \subseteq G_1 \subseteq G_2 \subseteq \cdots$. Now, as in the preceding section, each G_i acts as permutations on the set Ω_i of simple factors of $\operatorname{rad} G_i/\operatorname{sol} G_i$. Indeed, if N_i is the kernel of this action, then Lemma 7.1(iv) implies that $N_i^{(4)}$ is a normal subgroup of $\operatorname{rad} G_i$ and hence it is solvable-by-semisimple. Furthermore, if we choose the ultrafilter $\mathcal{F}$ as in Section 6, then G acts as permutations on the ultraproduct $\Omega = \prod_{\mathcal{F}} \Omega_i$ with kernel N. We study G by considering N and $\bar{G} = G/N \subseteq \operatorname{Sym}_\Omega$ in turn.

To start with, Lemma 6.1(i) implies that there exists a subsequence $\mathcal{M}$ of the natural numbers $\mathcal{N} = \{1, 2, \dots\}$ such that $L = N^{(4)}$ is the ascending union of its finite subgroups $L \cap N_i^{(4)}$ with $i \in \mathcal{M}$. Furthermore, note that $(L \cap N_i^{(4)}) \triangleleft N_i^{(4)}$ and that $N_i^{(4)}$ is solvable-by-semisimple. Thus $L \cap N_i^{(4)}$ is also solvable-by-semisimple, and $N^{(4)} = L = \bigcup_{i \in \mathcal{M}}^{\infty} (L \cap N_i^{(4)})$ is *locally solvable-by-semisimple*. There are now two cases to consider according to whether the widths which occur here are bounded or not. For the bounded case, we have

Lemma 8.1. *Let L be the ascending union of the finite subgroups $L_1 \subseteq L_2 \subseteq \cdots$ and suppose that each L_i is solvable-by-semisimple. If the widths of the various subgroups L_i are uniformly bounded, then L has a finite subnormal series*

$$\langle 1 \rangle = M_0 \triangleleft M_1 \triangleleft \cdots \triangleleft M_n = L$$

with each factor M_{i+1}/M_i either simple or locally solvable.

This follows easily by induction on the given upper bound for the widths. For example, if all L_i are solvable, which occurs when all widths are equal 0, then L is certainly locally solvable. On the other hand, if each L_i is a simple group, then clearly the same is true of L.

Using this lemma and the known semiprimitivity results for the critical factors, namely the infinite simple groups and the locally solvable groups, we can easily settle the semiprimitivity problem for $N^{(4)} = L$ in the case of bounded widths. The unbounded case builds upon this, but also requires the construction of certain group elements called *p-insulators*.

Finally, consider $\bar{G} = G/N \subseteq \mathrm{Sym}_\Omega$, and notice that $\bar{G} = \bar{H}^{\bar{G}}$. Again, there are two cases to deal with according to the nature of the action of H on the various Ω_i. Suppose first that H moves a bounded number of points in each Ω_i. Then Lemma 6.1(ii) implies that $\bar{H} \subseteq \mathrm{FSym}_\Omega$ and the corollary to Wielandt's theorem applies. In particular, $\bar{G}$ has a finite subnormal series with factors which are either locally normal or isomorphic to FAlt_∞, the countably infinite finitary alternating group. Again, we see that the critical factors here are either infinite simple or locally normal, and latter groups are quite easy to handle.

The last case, where H moves arbitrarily large numbers of points in its various actions, requires an entirely new approach based on the representation theory of finite wreath products. The key ideas here were developed during my previous trip to São Paulo a year ago, and I am pleased to thank my hosts Cesar Polcino Milies and Jairo Gonçalves for their kind hospitality on both occasions.

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