BURNSIDE'S THEOREM FOR HOPF ALGEBRAS

D. S. PASSMAN AND DECLAN QUINN

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ABSTRACT. A classical theorem of Burnside asserts that if \( \chi \) is a faithful complex character for the finite group \( G \), then every irreducible character of \( G \) is a constituent of some power \( \chi^n \) of \( \chi \). Fifty years after this appeared, Steinberg generalized it to a result on semigroup algebras \( K[G] \) with \( K \) an arbitrary field and with \( G \) a semigroup, finite or infinite. Five years later, Rieffel showed that the theorem really concerns bialgebras and Hopf algebras. In this note, we simplify and amplify the latter work.

Let \( K \) be a field and let \( A \) be a \( K \)-algebra. A map \( \Delta: A \to A \otimes A \) is said to be a comultiplication on \( A \) if \( \Delta \) is a coassociative \( K \)-algebra homomorphism. For convenience, we call such a pair \( (A, \Delta) \) a b-algebra. Admittedly, this is rather nonstandard notation. One is usually concerned with bialgebras, that is, algebras which are endowed with both a comultiplication \( \Delta \) and a counit \( \varepsilon: A \to K \). However, semigroup algebras are not bialgebras in general, and the counit rarely comes into play here. Thus it is useful to have a name for this simpler object.

Now a b-algebra homomorphism \( \theta: A \to B \) is an algebra homomorphism which is compatible with the corresponding comultiplications, and the kernel of such a homomorphism is called a b-ideal. It is easy to see that \( I \) is a b-ideal of \( A \) if and only if \( I \triangleleft A \) with \( \Delta(I) \subseteq I \otimes A + A \otimes I \). Of course, the b-algebra structure can be used to define the tensor product of \( A \)-modules. Specifically, if \( V \) and \( W \) are left \( A \)-modules, then \( A \) acts on \( V \otimes W \) via \( a(v \otimes w) = \Delta(a)(v \otimes w) \) for all \( a \in A, \ v \in V, \ w \in W \). Notice that if \( I \) is a b-ideal of \( A \), then the set of all \( A \)-modules \( V \) with \( \text{ann}_A V \supseteq I \) is closed under tensor product. Conversely, we have

**Proposition 1.** Let \( A \) be a b-algebra and let \( \mathcal{F} \) be a family of \( A \)-modules closed under tensor product. Then
\[
I = \bigcap_{V \in \mathcal{F}} \text{ann}_A V
\]
is a b-ideal of \( A \).

**Proof.** Certainly \( I \) is an ideal of \( A \). Now let \( X = \oplus_{V \in \mathcal{F}} V \) be the direct sum of the modules in \( \mathcal{F} \). Then \( X \) is an \( A \)-module and \( \text{ann}_A X = \bigcap_{V \in \mathcal{F}} \text{ann}_A V = \)
I. Furthermore, since \( X \otimes X = \sum_{V, W \in \mathcal{F}} V \otimes W \) and since each \( V \otimes W \in \mathcal{F} \), it follows that \( I \) annihilates \( X \otimes X \). In other words,
\[
\Delta(I) \subseteq \text{ann}_{\mathcal{A}}(X \otimes X) = I \otimes A + A \otimes I
\]
and \( I \) is a b-ideal of \( A \). \( \Box \)

The assumption that \( \mathcal{F} \) is closed under tensor product can be weakened somewhat in the above. Indeed, suppose that for each \( V, W \in \mathcal{F} \) there exists \( U \in \mathcal{F} \) with \( \text{ann}_A U \subseteq \text{ann}_A V \otimes W \). Then certainly \( I \subseteq \text{ann}_A U \) annihilates \( V \otimes W \), so \( I \) annihilates \( X \otimes X \) and hence \( I \) is a b-ideal of \( A \).

Now if \((A, \Delta, \epsilon)\) is a bialgebra with counit \( \epsilon \), then \( I \) is a bi-ideal of \( A \) if and only if it is a b-ideal with \( \epsilon(I) = 0 \). Furthermore, we can trivially guarantee that the ideal \( I \) of the previous proposition satisfies \( \epsilon(I) = 0 \) by including the principal module \( K_e \) in the set \( \mathcal{F} \). Thus we have

**Proposition 1*. Let \( A \) be a bialgebra and let \( \mathcal{F} \) be a family of \( A \)-modules closed under tensor product. If \( K_e \in \mathcal{F} \), then
\[
I = \bigcap_{V \in \mathcal{F}} \text{ann}_A V
\]
is a bi-ideal of \( A \).

Since the coassociativity of \( \Delta \) guarantees that the tensor product of \( A \)-modules is associative, it makes sense to define the \( n \)-th tensor power of \( V \) by
\[
V^{\otimes n} = V \otimes V \otimes \cdots \otimes V \quad (n \text{ times})
\]
for all \( n \geq 1 \). Here, \( V^{\otimes 1} = V \) and \( V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes (m+n)} \) for all \( m, n \geq 1 \). It is now a simple matter to prove the following result of [Ri].

**Corollary 2.** Let \( A \) be a b-algebra and let \( V \) be an \( A \)-module. If \( \text{ann}_A V \) contains no nonzero b-ideal, then \( \mathcal{F}(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n} \) is a faithful \( A \)-module.

**Proof.** \( \mathcal{F} = \{ V^{\otimes n} \mid n = 1, 2, \ldots \} \) is a set of \( A \)-modules which is clearly closed under tensor product. Thus, by Proposition 1,
\[
I = \bigcap_{n=1}^{\infty} \text{ann}_A V^{\otimes n} = \text{ann}_A \mathcal{F}(V)
\]
is a b-ideal of \( A \). But \( I \subseteq \text{ann}_A V^{\otimes 1} = \text{ann}_A V \), so the hypothesis implies that \( I = 0 \) and hence that \( \mathcal{F}(V) \) is faithful. \( \Box \)

If \( A \) is a bialgebra, then one usually defines \( V^{\otimes 0} \) to equal \( K_e \), since the latter module behaves like the identity element under tensor product. Thus we have

**Corollary 2*. Let \( A \) be a bialgebra and let \( V \) be an \( A \)-module. If \( \text{ann}_A V \) contains no nonzero bi-ideal, then \( \mathcal{F}^*(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n} \) is a faithful \( A \)-module.

Let \( V \) be an \( A \)-module. If \( J \) is an ideal of \( A \) contained in \( \text{ann}_A V \), then we can think of \( V \) as having been lifted from an \( A/J \)-module. In particular, \( V \) is faithful if and only if it is not lifted from any proper homomorphic image of \( A \). Similarly, if \( A \) is a b-algebra, we might say that \( V \) is b-faithful if it is
not lifted from any proper b-algebra homomorphic image of \( A \). In other words, \( V \) is b-faithful if and only if \( \text{ann}_A V \) contains no nonzero b-ideal of \( A \). Thus Corollary 2 asserts that any b-faithful module \( V \) gives rise to the faithful tensor module \( \mathcal{T}(V) \). This is essentially Burnside's Theorem.

Let us look at some examples. To start with, recall that a multiplicative semigroup \( G \) is a set having an associative multiplication and an identity element 1. Semigroups may contain a zero element \( 0 \neq 1 \) satisfying \( 0g = g0 = 0 \) for all \( g \in G \), and as usual we let \( G^* = G \setminus \{0\} \) denote the set of nonzero elements of \( G \). The semigroup algebra \( K[G] \) is then a \( K \)-vector space with basis \( G^* \) and with multiplication inherited from that of \( G \). Notice that the zero element of \( G \), if it exists, is identified with the zero element of \( K[G] \). Furthermore, \( K[G] \) is a b-algebra with \( \Delta \) defined by \( \Delta(g) = g \otimes g \) for all \( g \in G^* \). Given this comultiplication, it is easy to see that the only possible bialgebra structure on \( K[G] \) would have counit \( \epsilon \) given by \( \epsilon(g) = 1 \) for all \( g \in G^* \). But then, \( \epsilon \) is an algebra homomorphism if and only if \( G^* \) is multiplicatively closed, or equivalently if and only if there are no zero divisors in \( G \). In other words, most semigroup algebras are just not bialgebras in this way.

If \( H \) is also a semigroup, then a semigroup homomorphism \( \theta: G \to H \) preserves the multiplication and, by definition, it satisfies \( \theta(1) = 1 \) and \( \theta(0) = 0 \) if \( G \) has a zero element. In particular, it follows that \( \theta \) extends to a \( K \)-algebra homomorphism \( \tilde{\theta}: K[G] \to K[H] \) which is clearly a b-algebra map. Hence \( \ker \tilde{\theta} \) is a b-ideal of \( K[G] \). As is well known, these are the only possible b-ideals. Since the argument is so simple, we briefly sketch it here.

Let \( I \) be a b-ideal of \( K[G] \) and let \( \tilde{\phi} \) be the b-algebra epimorphism defined by \( \tilde{\phi}: K[G] \to K[G]/I = C \). Then \( H = \tilde{\phi}(G) \) is a multiplicative sub-semigroup of \( C \) and, since \( \tilde{\phi} \) is a b-algebra homomorphism, it is easy to see that \( H \) consists of group-like elements. In particular, it follows from [Sw, Proposition 3.2.1(b)] that \( H^* \) is a linearly independent subset of \( C \). Furthermore, since \( G^* \) spans \( K[G] \), we know that \( H^* \) spans \( C \). Thus it is clear that \( C = K[H] \) and that the map \( \phi: K[G] \to K[H] \) is the natural extension of the semigroup epimorphism \( \phi: G \to H \), namely, the restriction of \( \tilde{\phi} \) to \( G \). Since \( I = \ker \phi \), this fact is proved.

By combining the above with Corollary 2, we can quickly obtain Steinberg's generalization of the classical result of Burnside [B, §226]. The original Burnside theorem concerned modules for the complex group algebra \( C[G] \) with \( |G| < \infty \), and the proof used the character theory of finite groups. The argument in [St] is more transparent and, of course, it is more general. But the following proof, due to Rieffel in [Ri], shows precisely why the \( G \)-faithfulness assumption on the \( K[G] \)-module \( V \) is both natural and relevant.

Let \( G \) be a semigroup and let \( V \) be a \( K[G] \)-module. We say that \( G \) acts faithfully on \( V \) if for all distinct \( g_1, g_2 \in G \) we have \( (g_1 - g_2)V \neq 0 \). Of course, if \( G \) is a group, then this condition is equivalent to \( (g-1)V \neq 0 \) for all \( 1 \neq g \in G \).

**Theorem 3.** Let \( G \) be a semigroup and let \( G \) act faithfully on the \( K[G] \)-module \( V \). Then \( K[G] \) acts faithfully on the tensor module \( \mathcal{T}(V) = \oplus_{n=1}^{\infty} V^{\otimes n} \).

**Proof.** Let \( I \) be a b-ideal of \( K[G] \) contained in \( \text{ann}_{K[G]} V \). As we observed, there exists a semigroup epimorphism \( \phi: G \to H \) such that \( I \) is the kernel of the corresponding algebra map \( \phi: K[G] \to K[H] \). If \( I \neq 0 \), then \( \phi \) cannot be
one-to-one on $G$ and hence there exist distinct $g_1, g_2 \in G$ with $\delta(g_1 - g_2) = 0$. In particular, this implies that $g_1 - g_2 \in I$, so $(g_1 - g_2)V = 0$, contradicting the fact that $G$ is faithful on $V$. In other words, the $G$-faithful assumption implies that $\text{ann}_{K[G]} V$ contains no nonzero b-ideal. Corollary 2 now yields the result. \hfill \square

An analogous result holds for enveloping algebras. For simplicity of notation, let us assume that either

1. $K$ is a field of characteristic 0, $L$ is a Lie algebra over $K$, and $U(L)$ is its enveloping algebra, or
2. $K$ has characteristic $p > 0$, $L$ is a restricted Lie algebra over $K$, and $U(L)$ is its restricted enveloping algebra.

In either case, $U(L)$ is a b-algebra, and in fact a Hopf algebra, with comultiplication determined by $\Delta(\ell) = \ell \otimes 1 + 1 \otimes \ell$ for all $\ell \in L$. Furthermore, if $H$ is a second (restricted) Lie algebra and if $\theta: L \to H$ is a (restricted) Lie algebra homomorphism, then $\theta$ extends uniquely to a b-algebra homomorphism $\bar{\theta}: U(L) \to U(H)$. In particular, $\ker \bar{\theta}$ is a b-ideal of $U(L)$. As is well known, the converse is also true, namely, every b-ideal of $U(L)$ arises in this manner. The argument for this is elementary and similar to the one for semigroup rings. A sketch of the proof is as follows.

Let $I$ be a b-ideal of $U(L)$ and let $\phi$ be the b-algebra epimorphism defined by $\phi: U(L) \to U(L)/I = C$. Then $H = \phi(L)$ is a (restricted) Lie subalgebra of $C$ and $H$ generates $C$ as a $K$-algebra. In particular, if $\{ h_i \mid i \in \mathcal{I} \}$ is a basis for $H$, indexed by the ordered set $(\mathcal{I}, \prec)$, then $C$ is spanned by monomials of the form $h_{i_1}^{e_1} h_{i_2}^{e_2} \cdots h_{i_n}^{e_n}$ with $i_1 \prec i_2 \prec \cdots \prec i_n$ and with integers $e_j \geq 0$. Furthermore, when $\text{char} K = p > 0$ and $L$ is restricted, then $e_j < p$ for all $j$. Since $\phi$ is a b-algebra epimorphism, it follows that the elements of $H$ are primitive. Thus, by the work of [Sw, Chapter 13], these straightened monomials are $K$-linearly independent and therefore $C = U(H)$. In other words, the map $\phi: U(L) \to U(H)$ is the natural extension of the (restricted) Lie algebra epimorphism $\phi: L \to H$ where, of course, $\phi$ is the restriction of $\phi$ to $L$. Since $I = \ker \phi$, this fact is proved.

Now let $V$ be a $U(L)$-module. We say that $L$ acts faithfully on $V$ if, for all $0 \neq \ell \in L$, we have $\ell V \neq 0$. The Lie algebra analog of the preceding result is then

**Theorem 4.** Let $U(L)$ be a (restricted) enveloping algebra satisfying (1) or (2) above. If $L$ acts faithfully on the $(U(L))$-module $V$, then $U(L)$ acts faithfully on the tensor module $\mathcal{T}(V) = \oplus_{n=1}^{\infty} V^{\otimes n}$.

As indicated in [M], a theorem of this nature can be used to prove the following interesting result of Harish-Chandra [H, Theorem 1]. Recall that a $K$-algebra $A$ is residually finite if the collection of its ideals $I$ of finite codimension has intersection equal to 0. In other words, these algebras are precisely the subdirect products of finite-dimensional $K$-algebras.

**Corollary 5.** If $L$ is a finite-dimensional Lie algebra over a field $K$ of characteristic 0, then $U(L)$ is residually finite.

**Proof.** By Ado's theorem (see [J, §VI.2]), $A = U(L)$ has a finite $K$-dimensional module $V$ on which $L$ acts faithfully. Thus, the preceding theorem implies
that \( 0 = \text{ann}_A \mathcal{F}(V) = \cap_{n=1}^{\infty} I_n \), where \( I_n = \text{ann}_A V^\otimes n \). But each \( V^\otimes n \) is a finite-dimensional \( A \)-module, so \( I_n = \text{ann}_A V^\otimes n \) is an ideal of \( A \) of finite codimension, and the result follows. \( \square \)

If \( L \) is a finite-dimensional restricted Lie algebra, then its restricted enveloping algebra \( U(L) \) is also finite dimensional. Thus the characteristic \( p > 0 \) analog of the above is trivial. On the other hand, infinite-dimensional analogs in all characteristics are obtained in [M].

In the remainder of this paper we will restrict our attention to finite-dimensional Hopf algebras. To start with, a Hopf algebra \( (A, \Delta, \epsilon, S) \) is a bialgebra with antipode \( S: A \rightarrow A \), and a Hopf ideal is the kernel of a Hopf algebra homomorphism. It is easy to see that \( I \triangleleft A \) is a Hopf ideal if and only if it is a b-ideal with \( \epsilon(I) = 0 \) and \( S(I) \subseteq I \). Similarly, a \( K \)-subalgebra \( B \) of \( A \) is a Hopf subalgebra if and only if it is a b-subalgebra which is closed under the antipode \( S \). Of course, the b-subalgebra condition means that \( \Delta(B) \subseteq B \otimes B \). The following is a special case of a surprising result due to Nichols [N, Theorem 1]. A simple proof of the subalgebra case can also be found in [Ra, Lemma 1].

Lemma 6. If \( A \) is a finite-dimensional Hopf algebra, then any b-subalgebra of \( A \) is a Hopf subalgebra and any b-ideal of \( A \) is a Hopf ideal of \( A \).

Proof. Let \( B \) denote either a b-subalgebra of \( A \) or a b-ideal of \( A \). Furthermore, let \( E = \text{Hom}_K(A, A) \) be the convolution algebra of \( A \) and set

\[
F = \{ f \in E \mid f(B) \subseteq B \}.
\]

Certainly \( F \) is a \( K \)-subspace of \( E \) and, in fact, \( F \) is closed under convolution multiplication. To see the latter, let \( f, g \in F \). If \( B \) is a b-subalgebra of \( A \), then \( \Delta(B) \subseteq B \otimes B \) implies that

\[
(f \ast g)(B) \subseteq f(B)g(B) \subseteq B^2 = B.
\]

On the other hand, if \( B \) is a b-ideal of \( A \), then \( \Delta(B) \subseteq A \otimes B + B \otimes A \) implies that

\[
(f \ast g)(B) \subseteq f(A)g(B) + f(B)g(A) \subseteq AB + BA = B
\]

since \( B \triangleleft A \).

Now observe that the identity map \( \text{Id} \) is contained in \( F \). Thus, by the above, \( F \) contains the convolution powers \( \text{Id}^n \) of \( \text{Id} \) for all \( n > 0 \). Furthermore, since \( A \) is finite dimensional, \( E \) is also finite dimensional and hence the map \( \text{Id} \) is algebraic over \( K \). In particular, for some \( m \geq 1 \), we can write \( \text{Id}^m \) as a finite \( K \)-linear combination of the powers \( \text{Id}^i \) with \( i > m \). But \( \text{Id} \) has convolution inverse \( S \), so by multiplying the expression for \( \text{Id}^m \) by \( S^m \) and by \( S^{m+1} \) in turn, we deduce first that \( \epsilon = \text{Id}^{\otimes 0} \in F \) and then that \( S = \text{Id}^{(-1)} \in F \). In other words, \( \epsilon(B) \subseteq B \) and \( S(B) \subseteq B \).

Finally, if \( B \) is a b-subalgebra of \( A \), then \( S(B) \subseteq B \) implies that \( B \) is a Hopf subalgebra. On the other hand, if \( B \) is a b-ideal of \( A \), then \( \epsilon(B) \subseteq B \cap K = 0 \). Thus, since \( S(B) \subseteq B \), we conclude that \( B \) is a Hopf ideal of \( A \). \( \square \)

The preceding result is false in general for infinite-dimensional Hopf algebras. Some rather complicated counterexamples appear in [N].
**Theorem 7.** Let $A$ be a finite-dimensional Hopf algebra.

(i) If $\mathcal{F}$ is a family of $A$-modules which is closed under tensor product, then
$$\bigcap_{V \in \mathcal{F}} \text{ann}_A V$$
is a Hopf ideal of $A$.

(ii) Suppose $V$ is an $A$-module whose annihilator contains no nonzero Hopf ideal of $A$. Then $\mathcal{F}(V) = \oplus_{n=1}^{\infty} V^{\otimes n}$ is a faithful $A$-module.

This follows immediately from Proposition 1, Corollary 2, and Lemma 6. We can now obtain some consequences of interest. First, recall that an $A$-module $V$ is semisimple if it is a direct sum of simple modules.

**Corollary 8.** If $A$ is a finite-dimensional Hopf algebra, then the set of semisimple $A$-modules is closed under tensor product if and only if the Jacobson radical $J(A)$ is a Hopf ideal of $A$.

**Proof.** Let $\mathcal{F}$ be the set of all semisimple $A$-modules. If $\mathcal{F}$ is closed under $\otimes$, then Theorem 7(i) implies $J(A) = \bigcap_{V \in \mathcal{F}} \text{ann}_A V$ is a Hopf ideal of $A$. Conversely, if $J(A)$ is a Hopf ideal, then $\mathcal{F}$ consists of all the modules for the Hopf algebra $A/J(A)$ and therefore $\mathcal{F}$ is surely closed under tensor product.

In a similar manner, we prove

**Corollary 9.** Let $A$ be a finite-dimensional semisimple Hopf algebra and let $\mathcal{F}$ be a family of simple $A$-modules. Suppose that, for all $V, W \in \mathcal{F}$, every irreducible submodule of $V \otimes W$ is contained in $\mathcal{F}$. Then $I = \bigcap_{V \in \mathcal{F}} \text{ann}_A V$ is a Hopf ideal of $A$ and $\mathcal{F}$ is the set of all simple $A/I$-modules.

**Proof.** Let $\mathcal{F}$ be the set of all finite direct sums (with multiplicities) of elements of $\mathcal{F}$. Since $A$ is semisimple, the hypothesis implies that $\mathcal{F}$ is closed under tensor product. Hence, by Theorem 7(i), $I = \bigcap_{V \in \mathcal{F}} \text{ann}_A V = \bigcap_{W \in \mathcal{F}} \text{ann}_A W$ is a Hopf ideal of $A$. Furthermore, since $A/I$ is semisimple, it follows that $\mathcal{F}$ must be the set of all simple $A/I$-modules.

Our final consequence uses the fact that any finite-dimensional Hopf algebra $A$ is a Frobenius algebra [LS, §5] and hence that every simple $A$-module is isomorphic to a minimal left ideal of $A$.

**Corollary 10.** Let $A$ be a finite-dimensional Hopf algebra and let $V$ be an $A$-module whose annihilator contains no nonzero Hopf ideal of $A$. Then every simple $A$-module is isomorphic to a submodule of $V^{\otimes n}$ for some $n \geq 1$.

**Proof.** It follows from Theorem 7(ii) that $\mathcal{F}(V) = \oplus_{n=1}^{\infty} V^{\otimes n}$ is a faithful $A$-module. Now let $W$ be a simple $A$-module, so that $W$ is isomorphic to a minimal left ideal $L \subseteq A$. Since $L \neq 0$, we have $L \mathcal{F}(V) = 0$ and hence $LV^{\otimes n} \neq 0$ for some $n \geq 1$. In particular, there exists $u \in V^{\otimes n}$ with $Lu \neq 0$. But then the minimality of $L$ implies that $W \cong L \cong Lu \subseteq V^{\otimes n}$, as required.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, MADISON, WISCONSIN 53706
E-mail address: passman@math.wisc.edu

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK 13244
E-mail address: dpquinn@mailbox.syr.edu